Name:____________

2014

Mathematics Extension 2
Trial HSC Exam

General Instructions
- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen. Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11-16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I pages 2 – 5

10 marks
- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II pages 6 – 16

90 marks
- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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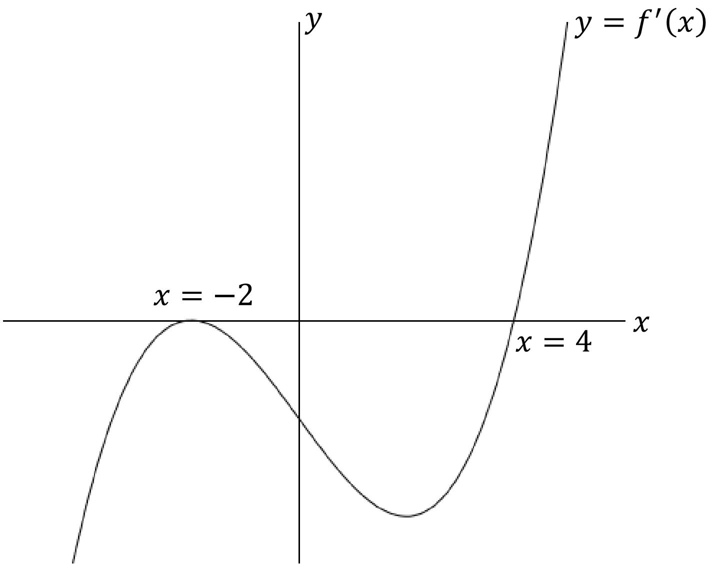
1 Which of the following is equivalent to

$$\int x\sec^2(x^2)\,dx$$

(A) $2\tan(x^2) + C$
(B) $\frac{1}{2}\tan(x^2) + C$
(C) $\frac{1}{3}\tan(x^2) + C$
(D) $3\tan(x^2) + C$

2 From the graph of $y = f'(x)$ drawn, which could be the equation of $y = f(x)$



(A) $(x-6)(x-2)^3$
(B) $(x+6)(x-2)^3$
(C) $(x-6)(x+2)^3$
(D) $(x+6)(x+2)^3$

3 The $\sqrt{-3+4i}$ is

(A) $2+i$
(B) $1+2i$
(C) $2-i$
(D) $1-2i$

4 A conic section has foci $S=(3,0)$ and $S'=(-3,0)$ and vertices $(2,0)$ and $(-2,0)$.
The equation of the conic is

(A) $\frac{x^2}{5}+\frac{y}{4}=1$
(B) $\frac{x^2}{4}+\frac{y^2}{5}=1$
(C) $\frac{x^2}{4}-\frac{y^2}{5}=1$
(D) $\frac{x^2}{5}-\frac{y^2}{4}=1$

5 For the function $g(x)=\tan^{-1}(e^x)$ the range is

(A) $0\le y\le\frac{\pi}{2}$
(B) $0\le y<\frac{\pi}{2}$
(C) $0<y\le\frac{\pi}{2}$
(D) $0<y<\frac{\pi}{2}$

6 If $e^x+e^y=1$, which of the following is an expression for $\frac{dy}{dx}$?

(A) $-e^{x-y}$
(B) e^{x-y}
(C) e^{y-x}
(D) $-e^{y-x}$

7 The polynomial $P(x) = x^4 + ax^2 + bx + 28$ has a double root at $x = 2$.
What are the values of a and b?

(A) $a = -11$ and $b = -12$
(B) $a = -5$ and $b = -12$
(C) $a = -11$ and $b = 12$
(D) $a = -5$ and $b = 12$

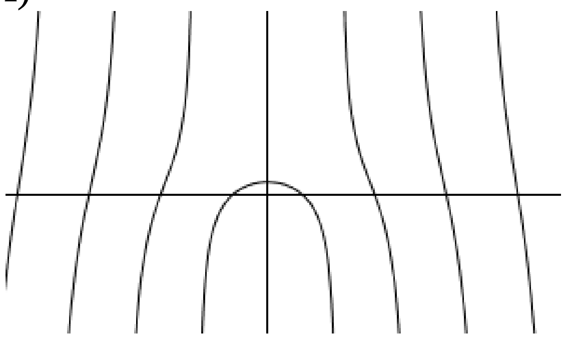
8 If ω is a non-real sixth root of -1 and ϕ is a non-real fifth root of 1 consider the following two statements:

(I) $1 - \omega + \omega^2 - \omega^3 + \omega^4 - \omega^5 = 0$
(II) $1 + \phi + \phi^2 + \phi^3 + \phi^4 = 0$

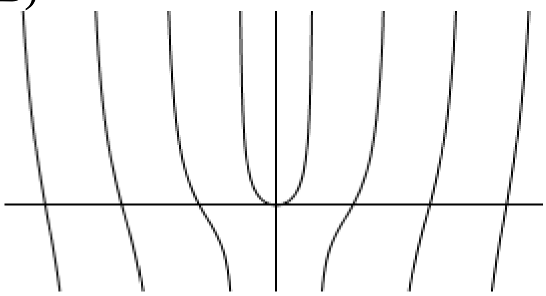
(A) Both (I) and (II) are correct
(B) Only (I) is correct
(C) Only (II) is correct
(D) Neither is correct

9 Which of the following best represents the graph of $g(x) = x \tan x$

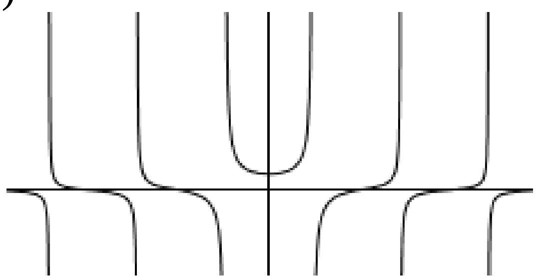
(A)



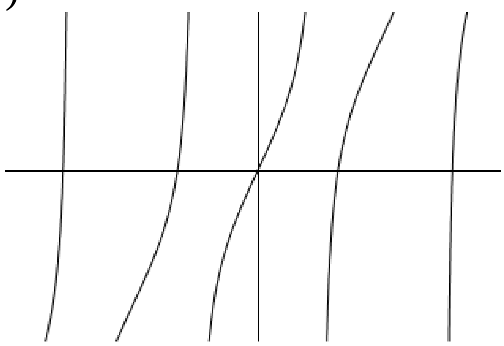
(B)



(C)



(D)



10 Without evaluating the integrals, which one of the following integrals is greater than zero?

(A) $\int_{-1}^{1} -e^{x}\, dx$

(B) $\int_{-1}^{1} \frac{\sin^{-1} x}{x^2+1}\, dx$

(C) $\int_{-1}^{1} \frac{\tan^{-1} x}{\cos x}\, dx$

(D) $\int_{-1}^{1} e^{-x^2}\, dx$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet

(a) For the complex numbers $z = 1 + i$ and $w = 2 - 3i$ find:

(i) $\bar{z} - w$ **1**

(ii) zw **1**

(iii) Write z in modulus-argument form **2**

(b) Show that $z = \sqrt{2}\ cis\left(\frac{\pi}{3}\right)$ is a solution of the equation **2**

$$z^8 - 8z^2 = 0$$

(c) Find all real roots of the polynomial **3**

$$P(x) = x^4 - x^3 - 4x^2 - 2x - 12,$$

given that one root is $\sqrt{2}\ i$.

Question 11 continues on page 7

(d) Sketch the locus of z where the following conditions hold simultaneously **3**

$0 \leq \arg(z - i) \leq \frac{2\pi}{3}$ and $|z - i| \leq 2$

(e) In an Argand diagram the points P, Q and R represent complex numbers z_1, z_2 and $z_2 + i(z_2 - z_1)$ respectively.

(i) Show that PQR is a right-angled isosceles triangle **2**

(ii) Find in terms of z_1 and z_2 the complex number represented by the point S such that $PQRS$ is a square. **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) By completing the square find **2**

$$\int \frac{1}{\sqrt{2-(x^2+4x)}}\,dx$$

(b) Find **2**

$$\int \frac{\sin^3 x}{\cos^2 x}\,dx$$

(c) Use the substitution $t = \tan\left(\frac{x}{2}\right)$ to evaluate **4**

$$\int_0^{\frac{\pi}{2}} \frac{dx}{\cos x + \sin x + 1}$$

(d) (i) Find the values of $A,\ B,\ C$ and D such that **2**

$$\frac{5x^3-3x^2+2x-1}{x^4+x^2} = \frac{A}{x} + \frac{B}{x^2} + \frac{Cx+D}{x^2+1}$$

(ii) Hence find **2**

$$\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}\,dx$$

(e) Evaluate **3**

$$\int_1^e \frac{\ln x}{\sqrt{x}}\,dx$$

Question 13 (15 marks) Use a SEPARATE writing booklet.

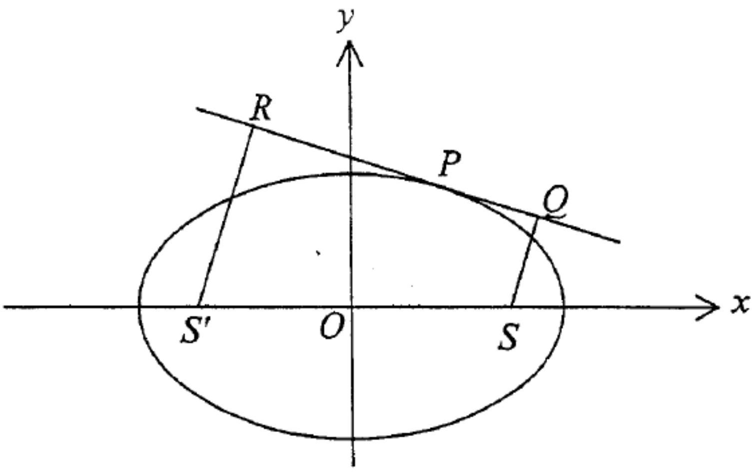
(a) (i) Use the substitution $x = t - y$ where t is a constant to show **1**

$$\int_0^t f(x)dx = \int_0^t f(t-x)dx$$

(ii) Hence, evaluate **2**

$$\int_0^1 x(1-x)^{2014}dx$$

(b)



(i) Prove that the equation of the tangent to the ellipse **3**

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

at the point $P(a\cos\theta, b\sin\theta)$ is

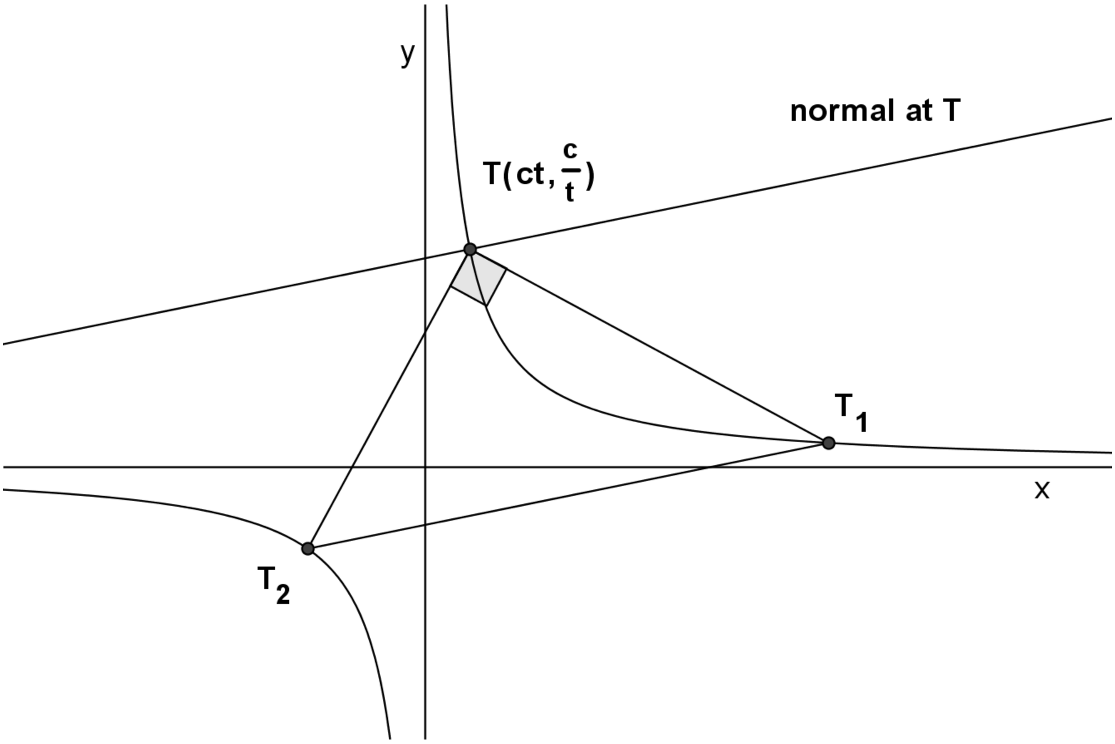
$$(b\cos\theta)x + (a\sin\theta)y - ab = 0$$

(ii) Q and R are the feet of the perpendiculars to the tangent from the foci S and S' respectively **3**

Prove that $SQ \times S'R = b^2$

Question 13 continues on page 10

(c) As shown in the diagram below T_1 and T_2 are two points on the rectangular hyperbola $xy = c^2$ with parameters t_1 and t_2 respectively and T is a third point on it with parameter t such that $\angle T_1TT_2$ is a right angle



(i) Show that gradient of T_1T is $-\frac{1}{t_1t}$ and deduce that since $\angle T_1TT_2$ is a right angle then $t^2 = -\frac{1}{t_1t_2}$ **3**

(ii) Write down the gradient of T_1T_2 **1**

(iii) Hence, prove that T_1T_2 is parallel to the normal at T **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

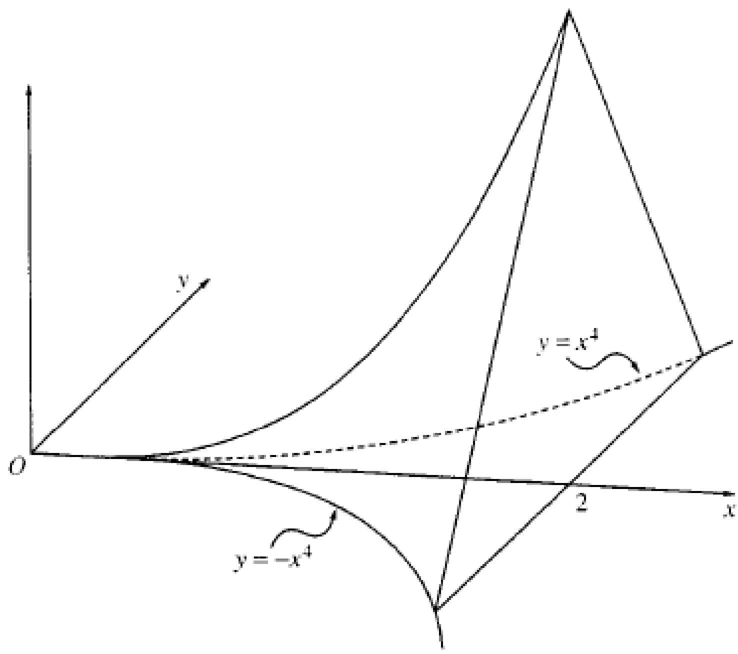
(a) α, β and γ are the roots of the equation **2**

$$x^3 - 6x^2 + 12x - 35 = 0$$

Form a cubic equation whose roots are $\alpha - 2$, $\beta - 2$ and $\gamma - 2$

(b) By taking slices perpendicular to the axis of rotation find the volume of the solid generated by rotating the region bounded by the curve $y = (x-2)^2$ and the line $y = x$ about the x −axis **3**

(c) The base of a solid is the region in the xy plane enclosed by the curves $y = x^4$, $y = -x^4$ and the line $x = 2$. Each cross-section perpendicular to the x −axis is an equilateral triangle

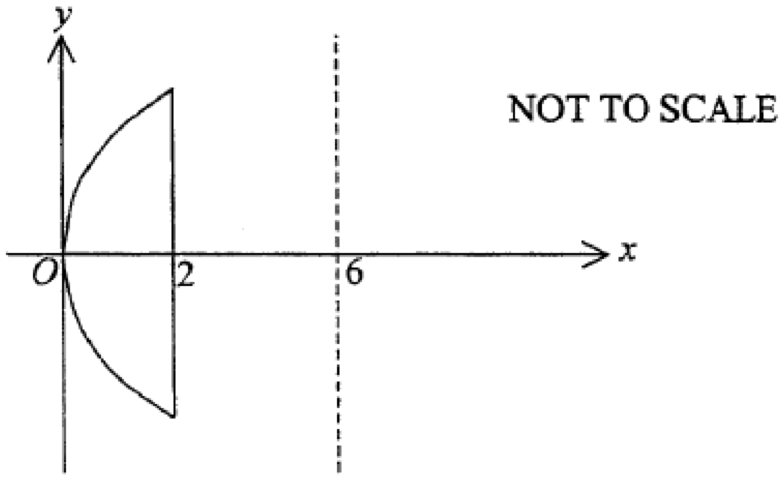


(i) Show that the area of the triangular cross-section at $x = h$ is $\sqrt{3}\, h^8$ **2**

(ii) Hence, find the volume of the solid **3**

Question 14 continues on page 12

(d) **5**

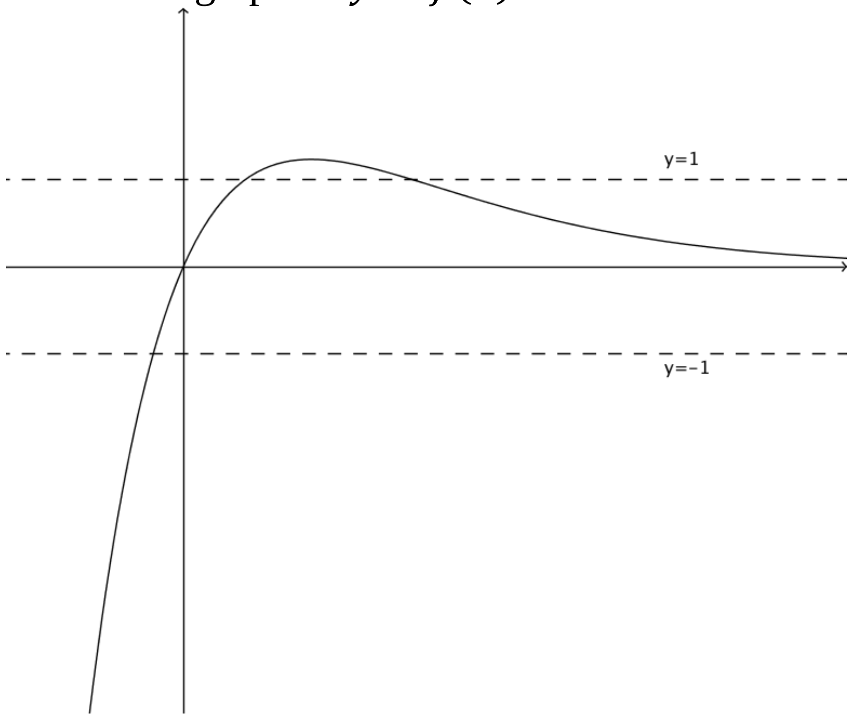


The region bounded by the parabola $y^2 = 4x$ and the line $x = 2$ is rotated about the line $x = 6$.
Using the method of cylindrical shells, find the volume of the solid formed.

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) This is the graph of $y = f(x)$:



On the separate graph answer sheet, sketch:

(i) $y = \left(f(x)\right)^2$ **1**

(ii) $y = \sqrt{f(x)}$ **1**

(iii) $y = \frac{1}{f(x)}$ **1**

Question 15 continues on page 14

(b) A particle of mass m is projected vertically upwards under gravity, the air resistance to the motion being $\frac{mgv^2}{k^2}$, where the speed is v, and k is a constant

(i) Show that during the upward motion of the ball **2**

$$\ddot{x} = -\frac{g}{k^2}(k^2 + v^2)$$

where x is the upward displacement.

(ii) Hence, show that the greatest height reached is **3**

$$\frac{k^2}{2g}\ln\left(1 + \frac{u^2}{k^2}\right)$$

where u is the speed of projection

(c) An object is to undergo vertical motion on a bungee cord in a vacuum so that air resistance can be neglected. The only forces it experiences are gravity, mg and an elastic force, $-kmx$, where x is the particle's displacement from the origin. Initially, the object is at its lowest point given by $x = -a$. All constants are positive.

(i) Using a force diagram show that **1**

$$\ddot{x} = -g - kx$$

(ii) By integration show that **3**

$$v^2 = k\left(\left(a - \frac{g}{k}\right)^2 - \left(x + \frac{g}{k}\right)^2\right)$$

(iii) Show that the motion is described by **3**

$$x = \left(\frac{g}{k} - a\right)\cos\left(\sqrt{k}t\right) - \frac{g}{k}$$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Let **2**

$$I_n = \int_0^1 x^n e^{-x} dx$$

Prove that for $n \geq 1$

$$I_n = nI_{n-1} - \frac{1}{e}$$

(b) Use the binomial theorem to find the term independent of x in the expansion **3**

$$\left((x+1) + x^{-1}\right)^4$$

(c) Find the cartesian equation of the locus of w where $w = \frac{z}{z+2}$ is purely imaginary. **2**

Question 16 continues on page 16

(d) (i) Use de Moivre's theorem to show that **2**

$$\cos 3\theta = 4\cos^3\theta - 3\cos\theta$$

(ii) Deduce that $8x^3 - 6x - 1 = 0$ has solutions $x = \cos\theta$, where $\cos 3\theta = \frac{1}{2}$ **2**

(iii) Find the roots of $8x^3 - 6x - 1 = 0$ in the form $\cos\theta$ **2**

(iv) Hence, evaluate $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$ **2**

End of paper

2014 Ext 2 Trial

1) $\int x\sec^2(x^2)\,dx = \frac{1}{2}\int 2x\sec^2(x^2)\,dx$
$= \frac{1}{2}\tan x^2 + C$ B

2) $f'(x)$ has a double root @ $x=-2$ ~~&~~ so f has a triple root
Since f is increasing to right of stat pt @ $x=4$
f must be C

3) $\sqrt{-3+4i} = \sqrt{1+4i-4}$
$= \sqrt{1+4i+4i^2}$
$= \sqrt{(1+2i)^2}$
$= 1+2i$ B

4) Must be ~~el~~ hyperbola.
$S=(ae,0)$ $\therefore e=\frac{3}{2}$
$b^2 = 2^2\left(\left(\frac{3}{2}\right)^2-1\right)$
$=5$ $\frac{x^2}{4}-\frac{y^2}{5}=1$ C

5) $0<y<\pi/2$ D

6) $e^x + e^y \times \frac{dy}{dx} = 0$
$\frac{dy}{dx} = -\frac{e^x}{e^y}$
$= -e^{x-y}$ A

7) $P(2)=0=16+4a+2b+28$
$P'(x)=4x^3+2ax+b$
$P'(2)=32+4a+b=0$
$b=-12$
$a=-5$ B

8) $\omega^6 + 1 = 0$

$\therefore (\omega+1)(\omega^5 - \omega^4 + \omega^3 - \omega^2 + \omega - 1)$ doesn't work for even power

$\phi^5 - 1 = 0$

$\therefore (\phi-1)(\phi^4 + \phi^3 + \phi^2 + \phi^1 + 1) = 0$

$\therefore$ Only II

C

9) $g(x) = x\tan x$ passes through origin

D

10) D is even and f(x) is always true

$\therefore$ D

Question 11

a) i) $\bar{z} - w = 1 - i - (2 - 3i)$
$= 1 - i - 2 + 3i$
$= -1 + 2i$

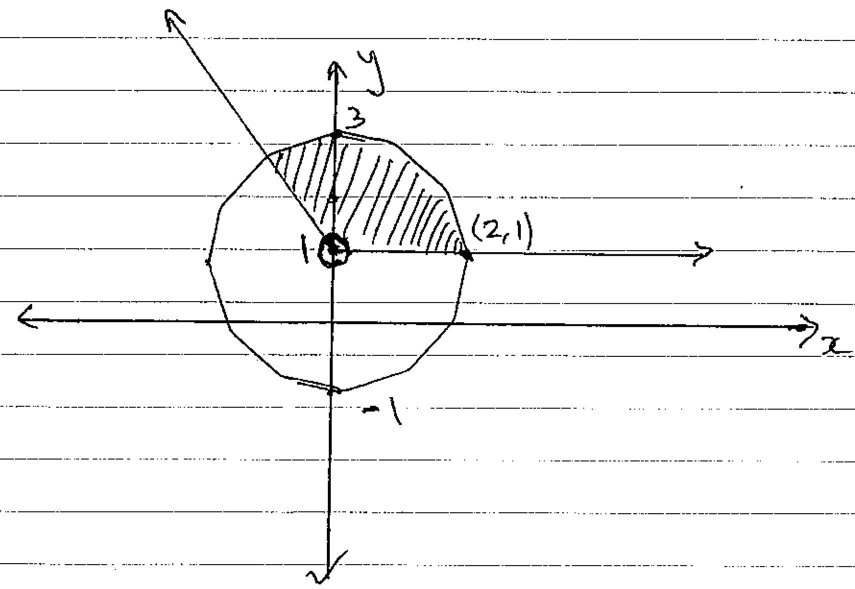
ii) $zw = (1+i)(2-3i)$
$= 2 - 3i + 2i - 3i^2$
$= 5 - i$

iii) $|z| = \sqrt{2}$ $\quad \arg(z) = \pi/4$
$z = \sqrt{2} \text{ cis } \left(\frac{\pi}{4}\right)$

b) $LHS = \left(\sqrt{2} \text{ cis}\left(\frac{\pi}{3}\right)\right)^8 - 8\left(\sqrt{2} \text{ cis}\left(\frac{\pi}{3}\right)\right)^2$
$= 16 \text{ cis } \frac{8\pi}{3} - 16 \text{ cis } \frac{2\pi}{3}$
$= 16 \text{ cis } \frac{2\pi}{3} - 16 \text{ cis } \frac{2\pi}{3}$
$= 0$
$= RHS$

c) Since one root is $\sqrt{2}i$ another is $-\sqrt{2}i$
let other two roots be α, β
$\alpha + \beta + \sqrt{2}i + -\sqrt{2}i = 1$
$\alpha + \beta = 1$
$\alpha\beta \times \sqrt{2}i \times -\sqrt{2}i = -12$
$\alpha\beta = -6$
$\therefore \alpha = 3$ and $\beta = -2$ are the other roots
$\therefore$ real roots are 3 and -2

d)



e) i) $\overrightarrow{PQ} = z_2 - z_1$

$\overrightarrow{QR} = z_2 + i(z_2 - z_1) - z_2$

$= i(z_2 - z_1)$

Since multiplication by i rotates through $90°$
& doesnt change modulus
PQ is perpendicular to QR & equal in length
$\therefore$ PQR is right $\triangle$ isosceles

ii)

$\cancel{\overrightarrow{O}= \overrightarrow{PO} + \overrightarrow{QR}}$

$\overrightarrow{OS} = \overrightarrow{OP} + \overrightarrow{QR}$

$= z_1 + z_2 - z_2 + i(z_2 - z_1)$

$= z_1 + i(z_2 - z_1)$

Question 12

a) $\displaystyle\int \frac{dx}{\sqrt{6-(x^2+4x+4)}} = \int \frac{dx}{\sqrt{6-(x+2)^2}}$

$\displaystyle = \sin^{-1}\left(\frac{x+2}{\sqrt{6}}\right) + C$

b) $\displaystyle\int \frac{\sin^2 x}{\cos^2 x} \times \sin x \, dx = \int \frac{(1-\cos^2 x)\sin x}{\cos^2 x} dx$

$\displaystyle = \int \frac{\sin x}{\cos^2 x} - \frac{\cos^2 x \sin x}{\cos^2 x} dx$

$\displaystyle = \int \sec x \tan x - \sin x \; dx$

by Std Integrals $= \sec x + \cos x + C$

c) $\displaystyle\int_0^1 \frac{2dt}{(1+t^2)\times\left(\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}+1\right)}$

$\displaystyle = \int_0^1 \frac{2dt}{1-t^2+2t+1+t^2}$

$\displaystyle = \int_0^1 \frac{2dt}{2+2t}$

$= \left[\ln(2+2t)\right]_0^1$

$= \ln 4 - \ln 2$

$= \ln 2$

$t = \tan\left(\frac{x}{2}\right)$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\left(\frac{x}{2}\right)$

$\frac{dt}{dx} = \frac{1}{2}\left(1+\tan^2\left(\frac{x}{2}\right)\right)$

$\frac{2dt}{1+t^2} = dx$

when $x = \frac{\pi}{2}$ $t = 1$

$x = 0$ $t = 0$

d)i) $5x^3 - 3x^2 + 2x - 1 = Ax(x^2+1) + B(x^2+1) + (Cx+D)x^2$

let $x = 0$ $\quad -1 = B$

$= Ax^3 + Ax - x^2 - 1 + Cx^3 + Dx^2$

$= (A+C)x^3 + (D-1)x^2 + Ax - 1$

by equating coeffs. $A = 2$

$D - 1 = -3 \Rightarrow D = -2$

$C = 3$

ii) $\displaystyle\int \frac{5x^3-3x^2+2x-1}{x^4+x^2}\,dx = \int \frac{2}{x} - \frac{1}{x^2} + \frac{3x-2}{x^2+1}\,dx$

$\displaystyle = \int \frac{2}{x} - \frac{1}{x^2} + \frac{3x}{x^2+1} - \frac{2}{x^2+1}\,dx$

$\displaystyle = 2\ln x + \cancel{2x} + \frac{3}{2}\ln(x^2+1) - 2\tan^{-1}x + C$

e) $\displaystyle\int_1^e \frac{\ln x}{\sqrt{x}}\,dx \cancel{= 2\sqrt{x}\ln x\Big]_1^e}$ $\qquad u = \ln x \qquad v' = x^{-1/2}$

$\qquad u' = \frac{1}{x} \qquad v = 2x^{1/2}$

$\displaystyle = 2\sqrt{x}\ln x\Big]_1^e - \int_1^e 2x^{-1/2}\,dx$

$\displaystyle = 2\sqrt{e}\ln e - 2\ln 1 - 2\Big[2x^{1/2}\Big]_1^e$

$= 2\sqrt{e} - 2(2\sqrt{e} - 2)$

$= 2\sqrt{e} - 4\sqrt{e} + 4$

$= 4 - 2\sqrt{e}$

Question 13

a i)
$$\int_0^t f(x)\,dx = \int_t^0 f(t-y)\times -dy \qquad \begin{aligned} x &= t-y \\ \frac{dx}{dy} &= -1 \\ dx &= -dy \end{aligned}$$

$$= -\int_t^0 f(t-y)\,dy \qquad \text{when } x = t \quad y = 0, \quad x = 0 \quad y = t$$

$$= \int_0^t f(t-y)\,dy$$

since y is just a dummy variable

$$= \int_0^t f(t-x)\,dx$$

ii)
$$\int_0^1 x(1-x)^{2014}\,dx = \int_0^1 (1-x)\,x^{2014}\,dx$$

$$= \int_0^1 x^{2014} - x^{2015}\,dx$$

$$= \left[\frac{1}{2015}x^{2015} - \frac{1}{2016}x^{2016}\right]_0^1$$

$$= \frac{1}{2015}\times 1 - \frac{1}{2016}\times 1 - (0)$$

$$= \frac{1}{4\ 062\ 240}$$

b)i)
$$x = a\cos\theta \qquad\qquad y = b\sin\theta$$

$$\frac{dx}{d\theta} = -a\sin\theta \qquad\qquad \frac{dy}{d\theta} = b\cos\theta$$

$$\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx}$$

$$= \frac{b\cos\theta}{-a\sin\theta}$$

$$y - b\sin\theta = \frac{b\cos\theta}{-a\sin\theta}(x - a\cos\theta)$$

$$-ay\sin\theta + ab\sin^2\theta = b\cos\theta\, x - ab\cos^2\theta$$

$$0 = (b\cos\theta)x + (a\sin\theta)y - ab(\sin^2\theta + \cos^2\theta)$$

$$0 = (b\cos\theta)x + (a\sin\theta)y - ab$$

ii) $S = (ae, 0)$ $\qquad S' = (-ae, 0)$

$$QS = \left| \frac{abe\cos\theta - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \right| \qquad RS' = \left| \frac{-abe\cos\theta - ab}{\sqrt{b^2\cos^2\theta + a^2\sin^2\theta}} \right|$$

$$SQ \times S'R = \frac{|ab(e\cos\theta - 1)| \times |-ab(e\cos\theta + 1)|}{b^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{|-a^2b^2(e^2\cos^2\theta - 1)|}{b^2\cos^2\theta + a^2\sin^2\theta}$$

for an ellipse

$b^2 = a^2(1-e^2)$

$$= \frac{|-a^2b^2(e^2\cos^2\theta - 1)|}{a^2(1-e^2)\cos^2\theta + a^2\sin^2\theta}$$

$$= \frac{|-a^2b^2(e^2\cos^2\theta - 1)|}{a^2\cos^2\theta - a^2e^2\cos^2\theta + a^2\sin^2\theta}$$

$$= \left| \frac{-a^2b^2(e^2\cos^2\theta - 1)}{a^2(\cos^2\theta + \sin^2\theta) - a^2e^2\cos^2\theta} \right|$$

$$= \left| \frac{a^2b^2(1 - e^2\cos^2\theta)}{a^2(1 - e^2\cos^2\theta)} \right|$$

$$= b^2$$

c) i) $T_1 = \left(ct_1, \frac{c}{t_1}\right)$ $\qquad m_{T_1T} = \dfrac{\frac{c}{t} - \frac{c}{t_1}}{ct - ct_1}$

$$= \frac{\frac{ct_1 - ct}{t_1 t}}{c(t - t_1)}$$

$$= \frac{-c(t - t_1)}{t_1 t} \div \frac{c(t - t_1)}{1}$$

$$= -\frac{1}{t_1 t}$$

Slope for $TT_2 = -\frac{1}{t_2 t}$

Since perpendicular

$$-\frac{1}{t_2 t} \times -\frac{1}{t_1 t} = -1$$

$$\frac{1}{t_1 t_2 t^2} = -1$$

$$t^2 = -\frac{1}{t_1 t_2}$$

ii) $M_{T_1T_2} = -\frac{1}{t_1t_2}$

iii) $y = \frac{c^2}{x}$

$\frac{dy}{dx} = -\frac{c^2}{x^2}$

@ T slope is $-\frac{c^2}{(ct)^2} = -\frac{1}{t^2}$

$\therefore$ Normal @ T has slope t^2

From i) $t^2 = -\frac{1}{t_1t_2}$

$\therefore$ normal parallel to T_2T_1

Question 14

a) let $x = \alpha - 2$

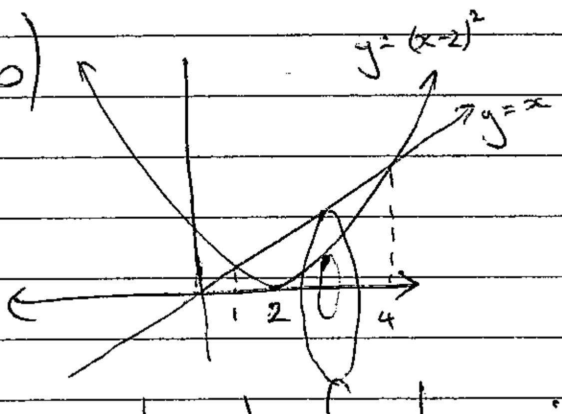
$$\alpha = x + 2$$

$$(x+2)^3 - 6(x+2)^2 + 12(x+2) - 35 = 0$$

$$x^3 + 6x^2 + 12x + 8 - 6x^2 - 24x - 24 + 12x + 24 - 35 = 0$$

$$x^3 - 27 = 0$$

b)



$$(x-2)^2 = x$$

$$x^2 - 4x + 4 = x$$

$$x^2 - 5x + 4 = 0$$

$$(x-4)(x-1) = 0$$

$$x = 1 \text{ or } 4$$

an element of volume is

$$\Delta V = \pi(y_1^2 - y_2^2)\,\delta x \qquad y_1 = x \qquad y_2 = (x-2)^2$$

$$= \pi(x^2 - (x-2)^4)\,\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=1}^{4} \pi\left(\cancel{x^2 - x^2 + 4x - 4}\right)\delta x$$

$$= \pi \int_1^4 \cancel{4x - 4\,dx}$$

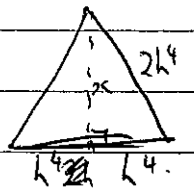
$$= \pi \left[2x^2 - 4x\right]_1^4$$

$$= \pi(32 - 16 - (2 - 4))$$

$$=$$

c) i) @ $x = h$ $\quad y = h^4$

length of all sides $2h^4$

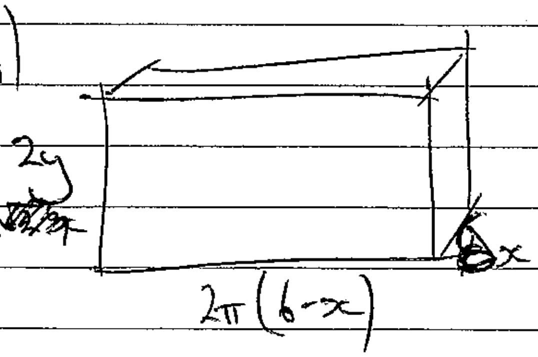


$$(2h^4)^2 = x^2 + (h^4)^2$$
$$4h^8 = x^2 + h^8$$
$$x^2 = 3h^8$$
$$x = \sqrt{3}\, h^4$$

$$\text{Area} = \frac{1}{2} \times 2h^4 \times \sqrt{3}\, h^4$$
$$= \sqrt{3}\, h^8$$

ii) $$V = \int_0^2 \sqrt{3}\, x^8 \, dx$$
$$= \sqrt{3} \left[\frac{x^9}{9} \right]_0^2$$
$$= \frac{512\sqrt{3}}{9}$$

d) i)



An element of volume is

$$\delta V = 2\sqrt{x} \times 2\pi(6-x) \times \delta x$$
$$= 8\pi\sqrt{x}\,(6-x)\,\delta x$$

ii) $$V = \lim_{\delta x \to 0} \sum_{x=0}^{2} 8\pi\sqrt{x}\,(6-x)\,\delta x$$
$$= 8\pi \int_0^2 \sqrt{x}\,(6-x)\, dx$$

iii) $V = 8\pi \int_0^2 6x^{1/2} - x^{3/2}\, dx$

$= 8\pi \left[4x^{3/2} - \frac{2}{5}x^{5/2} \right]_0^2$

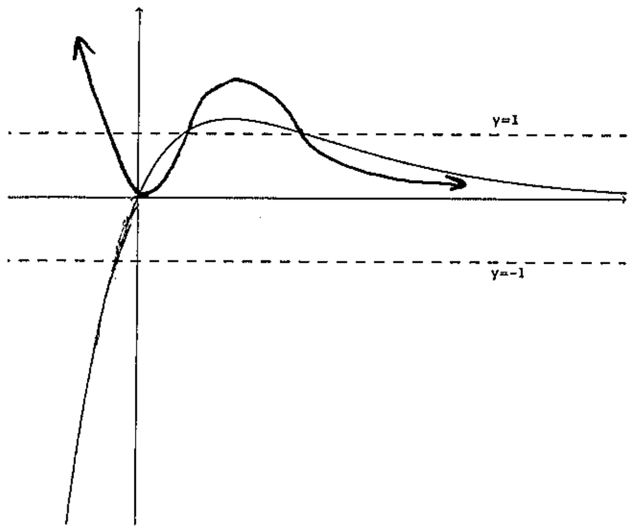
$= 8\pi \left(8\sqrt{2} - \frac{8\sqrt{2}}{5} - (0) \right)$

$= \frac{256\sqrt{2}\,\pi}{5}$

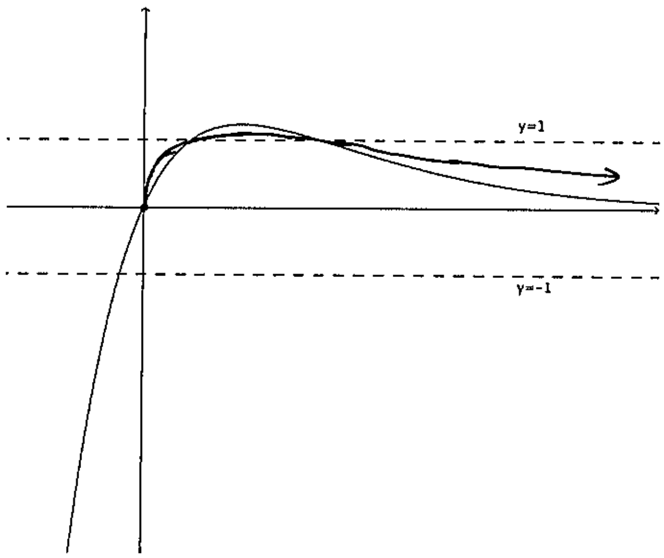
Question 15 a **Answer Sheet**

Insert into question 15 answer booklet

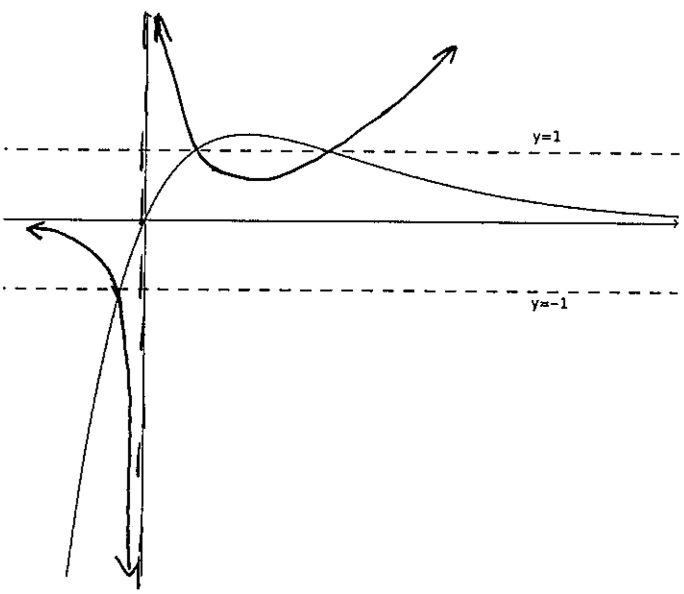
(i) $y = \left(f(x)\right)^2$



(ii) $y = \sqrt{f(x)}$



(iii) $y = \frac{1}{f(x)}$



Qn 15

b) i)

$$F = -mg - \frac{mgv^2}{k^2}$$

$-\frac{mgv^2}{k^2} \downarrow \ominus \downarrow -mg$

$$m\ddot{x} = -mg\left(\frac{k^2+v^2}{k^2}\right)$$

$$= -\frac{mg}{k^2}\left(k^2+v^2\right)$$

ii)

$$v\frac{dv}{dx} = -\frac{mg}{k^2}\left(k^2+v^2\right)$$

$$\frac{1}{2}\times\frac{2v}{k^2+v^2}\,dv = -\frac{mg}{k^2}\,dx$$

$$\frac{1}{2}\ln\left(k^2+v^2\right) = -\frac{mg}{k^2}x + C$$

@ $x=0$ $v=u$

$$\frac{1}{2}\ln\left(k^2+u^2\right) = C$$

max height when $v=0$

$$\frac{1}{2}\ln\left(k^2\right) = -\frac{g}{k^2}x + \frac{1}{2}\ln\left(k^2+u^4\right)$$

$$\frac{g}{k^2}x = \frac{1}{2}\left(\ln\left(k^2+u^2\right) - \ln k^2\right)$$

$$x = \frac{k^2}{2g}\ln\left(\frac{k^2+u^2}{k^2}\right)$$

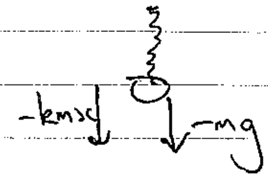
$$= \frac{k^2}{2g}\ln\left(1+\frac{u^2}{k^2}\right)$$

e) i) $F = -mg - kmx$

$m\ddot{x} = -mg - kmx$

$m\ddot{x} = m(-g - kx)$

$\ddot{x} = -g - kx$



ii) $v\frac{dv}{dx} = -g - kx$

$\frac{1}{2}v^2 = -gx - \frac{k}{2}x^2 + C$

@ $x = -a$ $\quad v = 0$ (initially stationary at limit of motion)

$0 = ag - \frac{a^2k}{2} + C$

$C = \frac{a^2k}{2} - ag$

$$\frac{1}{2}v^2 = -gx - \frac{k}{2}x^2 + \frac{a^2k}{2} - ag$$

$$v^2 = a^2k - 2ag \qquad -2gx - kx^2$$

$$= k\left(a^2 - 2a\frac{g}{k} - \frac{2gx}{k} - x^2\right)$$

$$= k\left(a^2 - \frac{2ag}{k} + \frac{g^2}{k^2} - \frac{g^2}{k^2} - \frac{2gx}{k} - x^2\right)$$

$$= k\left(\left(a - \frac{g}{k}\right)^2 - \left(\frac{g^2}{k^2} + \frac{2gx}{k} + x^2\right)\right)$$

$$= k\left(\left(a - \frac{g}{k}\right)^2 - \left(x + \frac{g}{k}\right)^2\right)$$

iii) $v = \pm\sqrt{k}\sqrt{\left(a - \frac{g}{k}\right)^2 - \left(x + \frac{g}{k}\right)^2}$

$$\frac{-dx}{\sqrt{\left(a - \frac{g}{k}\right)^2 - \left(x + \frac{g}{k}\right)^2}} = -\pm\sqrt{k}\,dt$$

$$\cos^{-1}\left(\frac{x + \frac{g}{k}}{a - \frac{g}{k}}\right) = \mp\sqrt{k}\,t + C$$

@ $t = 0$ $\quad x = -a$

$$\cos^{-1}\left(\frac{-a + \frac{g}{k}}{a - \frac{g}{k}}\right) = \mp\sqrt{k} \times 0 + C$$

$$\cos^{-1}(-1) = C$$

$$C = \pi$$

$$\frac{x + \frac{g}{k}}{a - \frac{g}{k}} = \cos\left(\pi \mp \sqrt{k}\,t\right)$$

Since $\cos(\pi+\alpha) = \cos(\pi-\alpha)$ we can take either without any concern.

choosing the $-ve$

$$\frac{x+\frac{g}{k}}{a-\frac{g}{k}} = \cos\left(\pi - \sqrt{k}\,t\right)$$

Since $\cos(\pi-\alpha) = -\cos\alpha$ for all α

$$\frac{x+\frac{g}{k}}{a-\frac{g}{k}} = -\cos\sqrt{k}\,t$$

$$x + \frac{g}{k} = \left(\frac{g}{k} - a\right)\cos\sqrt{k}\,t$$

$$x = \left(\frac{g}{k} - a\right)\cos\sqrt{k}\,t - \frac{g}{k}$$

Question 16

a) $I_n = \int_0^1 x^n e^{-x}\,dx$ $\quad u = x^n \quad v' = e^{-x}$ $\quad u' = nx^{n-1} \quad v = -e^{-x}$

$$= \left[x^n \times -e^{-x}\right]_0^1 - \int_0^1 nx^{n-1} \times -e^{-x} \times dx$$

$$= 1^n \times -e^{-1} - (0^n \times -e^0) + n\int_0^1 x^{n-1} e^{-x}\,dx$$

$$= n\int_0^1 x^{n-1} e^{-x}\,dx - \frac{1}{e}$$

$$= n I_{n-1} - \frac{1}{e}$$

b) $$\left((x+1) + x^{-1}\right)^4 = \sum_{r=0}^{4} {}^4C_r (x+1)^{4-r} (x^{-1})^r$$

Using binomial theorem on $(x+1)^{4-r} = \sum_{k=0}^{4-r} {}^{4-r}C_k\, x^k \times 1^{4-r-k}$

$$\left((x+1) + x^{-1}\right)^4 = \sum_{r=0}^{4} {}^4C_r\, x^{-r} \times \sum_{k=0}^{4-r} {}^{4-r}C_k\, x^k$$

for term independent of x

$$x^{-r} \times x^k = x^0$$

$$-r + k = 0$$

$$r = k$$

$\therefore$ const term is

$$\sum_{r=0}^{4} {}^4C_r \times \sum_{r=0}^{4-r} {}^{4-r}C_r = {}^4C_0 \times {}^4C_0 + {}^4C_1 \times {}^3C_1 + {}^4C_2 \times {}^2C_2$$

$$= 19$$

c) let $w = x + iy$

to be purely imaginary real part $= 0$

$$x = 0$$

but 0 is not an imaginary number so locus is

$x = 0$ except $(0,0)$

d) i) consider

$$\text{cis}\,3\theta = (\text{cis}\,\theta)^3$$
$$= (\cos\theta + i\sin\theta)^3$$
$$= \cos^3\theta + 3i\cos^2\theta\sin\theta + 3i^2\cos\theta\sin^2\theta + i^3\sin^3\theta$$
$$= \cos^3\theta + 3i\cos^2\theta\sin\theta - 3\cos\theta\sin^2\theta - i\sin^3\theta$$

equating real parts

$$\cos 3\theta = \cos^3\theta - 3\cos\theta\sin^2\theta$$
$$= \cos^3\theta - 3\cos\theta(1-\cos^2\theta)$$
$$= \cos^3\theta - 3\cos\theta + 3\cos^3\theta$$
$$= 4\cos^3\theta - 3\cos\theta$$

ii) when $\cos 3\theta = \frac{1}{2}$

$$4\cos^3\theta - 3\cos\theta = \frac{1}{2}$$
$$8\cos^3\theta - 6\cos\theta = 1$$
$$8\cos^3\theta - 6\cos\theta - 1 = 0$$

which is $8x^3 - 6x - 1 = 0$ when $x = \cos\theta$

$\therefore$ the solutions are given by $x = \cos\theta$

iii) the roots of $8x^3 - 6x - 1 = 0$

are given by the solutions of $\cos 3\theta = \frac{1}{2}$

$$3\theta = 2\pi n \pm \frac{\pi}{3}$$
$$\theta = \frac{2n\pi}{3} \pm \frac{\pi}{9}$$

for $n = 0$ $\quad \theta = \frac{\pi}{9}$ ~~or~~ taking +ve

$n = 1$ $\quad \theta = \frac{7\pi}{9}$

$n = 2$ $\quad \theta = \frac{13\pi}{9}$

only need 3 roots since degree 3 polynomial

$\therefore$ roots are

$$\cos\frac{\pi}{9},\ \cos\frac{13\pi}{9},\ \cos\frac{7\pi}{9}$$

iv) ~~$\cos\frac{7\pi}{9} = \cos\frac{-2\pi}{9}$ subtracting 2π~~

~~$= \cos\frac{2\pi}{9}$ cos is even function~~

~~$\therefore \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$ is the product of the roots.~~

~~$\therefore \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$~~

iv) roots are $\cos\frac{\pi}{9}$, $\cos\frac{7\pi}{9}$ and $\cos\frac{13\pi}{9}$

consider $\cos\frac{7\pi}{9}$.

$$\cos\frac{7\pi}{9} = \cos\left(\pi - \frac{2\pi}{9}\right)$$
$$= -\cos\frac{2\pi}{9}$$

consider $\cos\frac{13\pi}{9} = \cos\left(-\frac{5\pi}{9}\right)$
$$= \cos\left(-\pi + \frac{4\pi}{9}\right)$$
$$= -\cos\frac{4\pi}{9}$$

$\therefore$ roots are $\cos\pi/9$, $-\cos\frac{2\pi}{9}$ and $-\cos\frac{4\pi}{9}$

$\therefore$ product of roots is
$$\cos\frac{\pi}{9} \times -\cos\frac{2\pi}{9} \times -\cos\frac{4\pi}{9}$$
$$= \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$$

product of roots is $\frac{1}{8}$

$$\therefore \cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9} = \frac{1}{8}$$