Penrith Selective High School

2014

Higher School Certificate
Examination

Mathematics Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11-14, show relevant mathematical reasoning and/or calculations
- All diagrams are not to scale

Total Marks – 70

Section I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 6–12

60 marks

- Attempt Questions 11–14
- Allow about 1 hour 45 minutes for this section

Student Number: ____

Students are advised that this is a trial examination only and cannot in any way guarantee the content or format of the 2014 Higher School Certificate Examination.

Section I:

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

Q1. Evaluate $\lim_{x \to 0} \frac{5x}{\sin \frac{\pi x}{2}}$

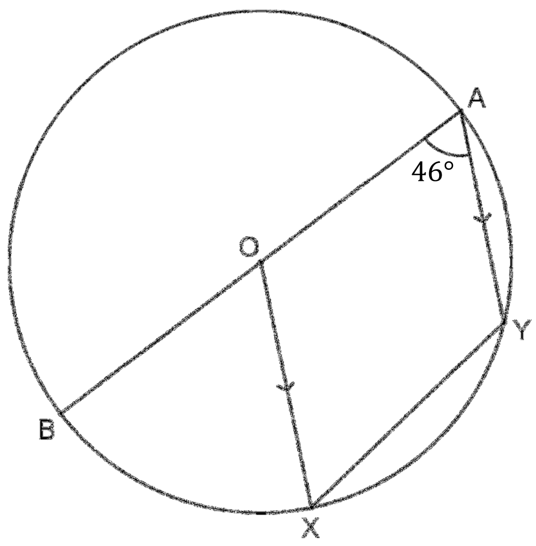
(A) $\frac{5}{2\pi}$

(B) $\frac{\pi}{10}$

(C) $\frac{2}{5\pi}$

(D) $\frac{10}{\pi}$

Q2. AB is the diameter of the circle and O is the centre. $AY \parallel OX, \angle OAY = 46°$. What is the size of $\angle XYA$?



NOT TO SCALE

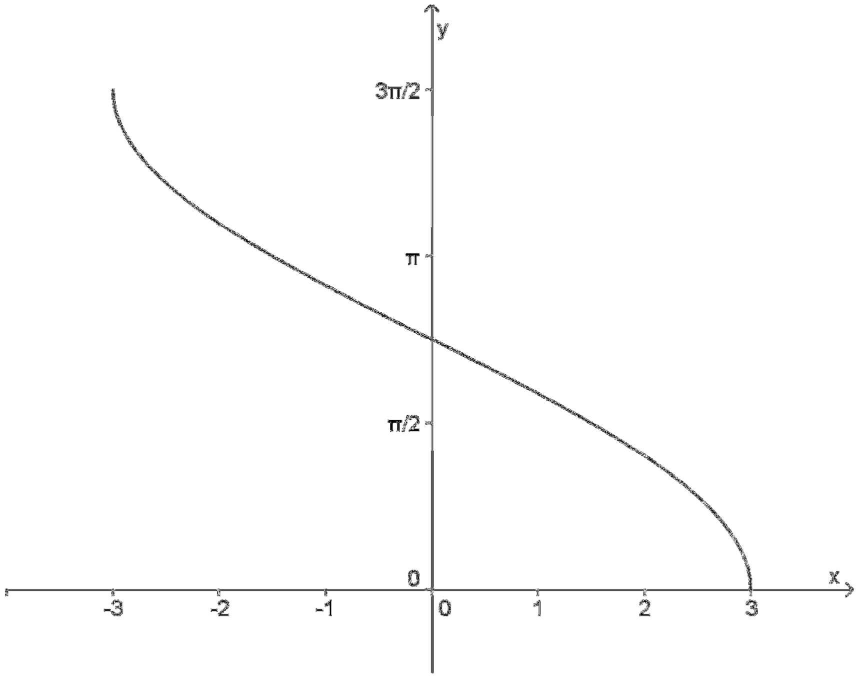
(A) $100°$

(B) $113°$

(C) $134°$

(D) $146°$

Q3. Which function best describes the following graph?



(A) $y = \frac{2}{3}\cos^{-1} 3x$

(B) $y = \frac{3}{2}\cos^{-1} 3x$

(C) $y = \frac{2}{3}\cos^{-1} \frac{x}{3}$

(D) $y = \frac{3}{2}\cos^{-1} \frac{x}{3}$

Q4. How many ways can you arrange the letters in the word INDEPENDENT so that the D's are separated?

(A) $\frac{11!}{2! \times 3! \times 3!} - \frac{9!}{3! \times 3!}$

(B) $\frac{11!}{2! \times 3! \times 3!} - \frac{10!}{3! \times 3!}$

(C) $\frac{11!}{2! \times 3! \times 3!} - \frac{11!}{3! \times 3!}$

(D) $\frac{11!}{2! \times 3! \times 3!}$

Q5. When the polynomial $P(x)$ is divided by $x^2 + x - 2$, the remainder is $4x - 1$. The remainder when $P(x)$ is divided by $x + 2$ is:

(A) $4x^2 + 7x - 2$

(B) $4x - 1$

(C) -9

(D) -7

Q6. The general solution to $\sin 2\theta = \sqrt{3}\sin\theta$ is:

(A) πn or $2\pi n \pm \frac{\pi}{6}$, n is an integer

(B) πn or $2\pi n \pm \frac{\pi}{3}$, n is an integer

(C) $2\pi n$ or $\pi n \pm \frac{\pi}{6}$, n is an integer

(D) $2\pi n$ or $\pi n \pm \frac{\pi}{3}$, n is an integer

Q7. Let $x = 0.8$ be a first approximation to the root of the equation $3ln x + x = \cos 2x$. What is the second approximation to the root using Newton's method to 3 significant figures?

(A) 0.742

(B) 0.776

(C) 0.981

(D) 0.985

Q8. An inverse function exists for the monotonic increasing part of this function

$$f(x) = \frac{4}{\sqrt{4 - x^2}}.$$

For what x values is this?

(A) $-2 \leq x \leq 2$

(B) $x \leq -2$

(C) $x \geq 2$

(D) $0 < x < 2$

Q9. The points A, B and P are collinear. If P divides the interval AB internally in the ratio 2 : 3, then B divides AP:

(A) 3 : 2 internally

(B) 2 : 5 internally

(C) 5 : 3 externally

(D) 5 : 1 externally

Q10. The motion of a particle moving along the x-axis executes simple harmonic motion. The maximum speed of the particle is 3 metres per second and the period of motion is $\frac{\pi}{3}$ seconds. Which of the following could be the displacement equation for this particle?

(A) $x = \frac{1}{2}\sin 6t$

(B) $x = 3\cos\frac{\pi}{2}t$

(C) $x = 3\cos\pi t$

(D) $x = \frac{1}{2} + \sin 6t$

Section II

60 Marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a) Differentiate $\tan^{-1}(4+x^3)$ **1**

b) Find $\int \frac{2}{\sqrt{25-2x^2}}\,dx$ **2**

c) From a group of 4 teachers and 10 students a committee of 3 is to be chosen.

i) If the committee only contain students, how many different committees are possible? **1**

ii) What is the probability that at least one teacher is chosen to be on the committee? **1**

d) Use the substitution $u = e^{4x} + 9$ to evaluate **2**

$$\int_0^{\ln 2} \frac{3e^{4x}}{\sqrt{e^{4x}+9}}\,dx$$

Leave your answer in exact form.

Question 11 continues on page 7

e) The two functions $y = 2\ln(x+1)$ and $y = \ln(x^2+3)$ meet at the point T.

i) Find the coordinates of T. **2**

ii) Find the acute angle between the tangents to the curve at T to the nearest minute. **2**

f) i) Find the quotient and the remainder obtained by dividing $P(x) = x^3 - mx^2 - mx + 9$ by $A(x) = x - 3$. **2**

ii) Hence, or otherwise, find a value of the constant m such that $P(x)$ is divisible by $A(x)$. **1**

iii) Find all the zeroes of $P(x)$ for this value of m. **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) The series $1+\frac{4}{\sqrt{3}}\sin x\cos x+\frac{16}{3}\sin^2 x\cos^2 x+\frac{64}{3\sqrt{3}}\sin^3 x\cos^3 x+\ \ldots\ldots\ldots$

has a limiting sum.

i) Show that $-\frac{\sqrt{3}}{2}<\sin 2x<\frac{\sqrt{3}}{2}$ **1**

ii) Hence, or otherwise, find the values of x, where $\frac{\pi}{4}\leq x\leq\frac{3\pi}{4}$ **2**

b) Find the term independent of x in the expansion of $\left(x^2-\frac{1}{2x^5}\right)^{14}$ **2**

c) Find the exact volume of the solid formed if the curve $y=\cos x+1$ is rotated about the x-axis from $x=0$ to $x=\frac{\pi}{2}$ **3**

d) The velocity, $v\ cm/s$, of a particle moving along the x-axis is given by $v^2=56-20x-4x^2$.

i) Show that this particle is undergoing simple harmonic motion. **2**

ii) Find the period, the amplitude and the centre of the motion. **2**

e) The function $(k-1)x^3+(k+4)x^2+(k-6)x-k=0$ has roots α,β and γ and their product is $\frac{4}{5}$

i) Find the sum of its roots. **1**

ii) Hence, or otherwise, find the value of $\alpha^2+\beta^2+\gamma^2$. **2**

End of Question 12

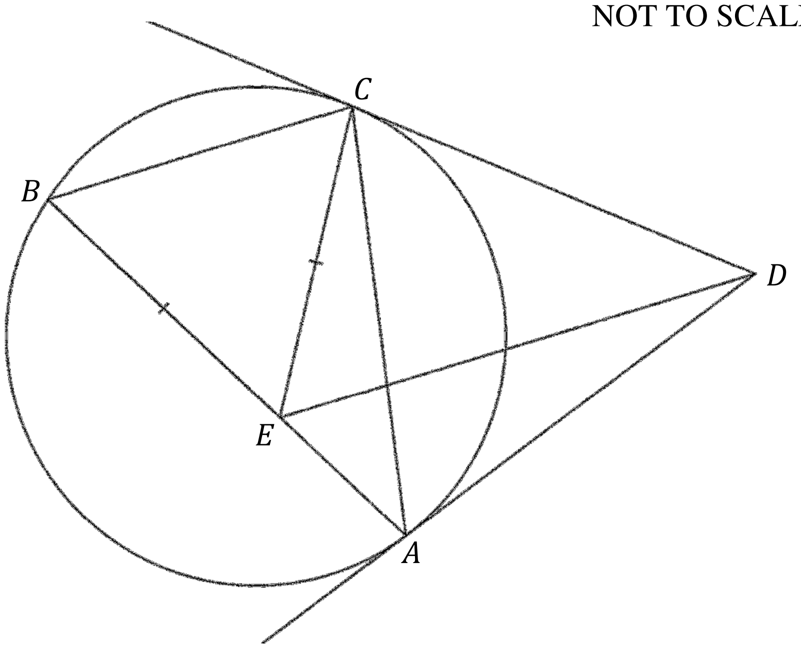
Question 13 (15 marks) Use a SEPARATE writing booklet.

a) Prove by mathematical induction for all positive integers n that, **3**

$a^{3n-1} - 1$ is divisible by $a - 1$ where a is an integer.

b) ΔABC is inscribed inside a circle. Two tangents from the point D intersect with the circle at points A and C. The point E is on the line AB and it is joined to point C, such that $BE = CE$. The point E is also joined by a line to the point D. This information is in the diagram below.

NOT TO SCALE



Copy or trace the diagram into your writing booklet.

i) Prove that $\angle BCE = \angle ACD$. **1**

ii) Prove that $ADCE$ is a cyclic quadrilateral. **2**

iii) Hence, or otherwise, prove that $BC \parallel ED$. **1**

Question 13 continues on page 10

c) By expanding $[x + (1 - x)]^n$, for all real numbers x and all positive integers n,

i) Show that $\binom{n}{0} x^n + \binom{n}{1} x^{n-1}(1 - x) + \ldots\ldots + \binom{n}{n} (1 - x)^n = 1$ **1**

ii) Deduce that $\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \cdots + \binom{n}{n} = 2^n$ **2**

d) $P(2ap, ap^2), Q(2aq, aq^2)$ are the end points of a focal chord of the parabola $x^2 = 4ay$.

i) Show that the distance between P and Q is $2a + a\left(p^2 + \frac{1}{p^2}\right)$ **3**

ii) A circle is drawn with PQ is its diameter. Prove that the directrix of the parabola $x^2 = 4ay$ is a tangent to this circle. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

a) Kirk is standing on a balcony and spots Spock walking along a street below. Kirk decides to launch a flour bomb down onto Spock when he stops to buy a pastry. The balcony is 70 metres above the street. Kirk launches the flour bomb at a velocity of 50 metres per second at an angle of θ to the horizontal.

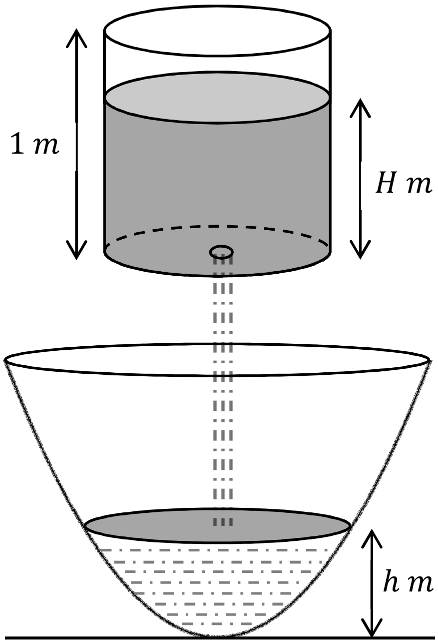
i) Derive the equations for the horizontal and vertical displacements after time t seconds. Assume that gravity is 9.8 ms^{-2}. **2**

ii) If Spock is 1.88 metres tall and standing approximately 300 metres horizontally from where Kirk is standing on the balcony above, determine the possible values of θ to the nearest minute which allow Kirk to hit Spock with the flour bomb. **3**

iii) If it takes 8 seconds for Kirk to hide himself to avoid being seen by Spock, which value of θ should Kirk use? Justify your answer. **2**

Question 14 continues on page 12

b) A cylindrical can, with radius $0.5\ m$ and height $1\ m$ is held above a bowl as shown in the diagram below. Initially the can was full and the bowl was empty. Water has dripped from the bottom of the can into the bowl. After t minutes, the height of the water in the can is H metres and the height of the water in the bowl is h metres.



NOT TO SCALE

If the flow rate of the water is $0.25\ m^3/min$ and the volume of water in the bowl is given by πh^2 cubic metres.

i) Show that $H = 1 - \frac{t}{\pi}$ metres. **2**

ii) Show that $h = \sqrt{\frac{t}{4\pi}}$ metres. **2**

iii) At the moment of time $t = t_1$ minutes, the height of the water in the two containers is equal. Prove that t_1 satisfies the quadratic equation $4{t_1}^2 - 9\pi t_1 + 4\pi^2 = 0$. **2**

iv) Find the value of t_1 minutes in terms of π and hence explain why t_1 only has one value. **2**

End of Paper

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax\,\tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Student Number:____

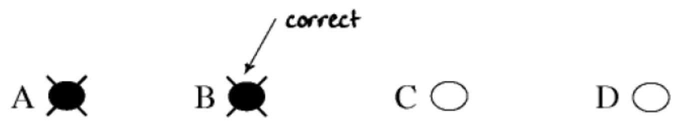
Multiple Choice Answer Sheet

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 A ○ (B) 6 B ● (C) 8 C ○ (D) 9 D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word 'correct' and drawing an arrow as follows.



Start Here ➔

1.	A ○	B ○	C ○	D ○
2.	A ○	B ○	C ○	D ○
3.	A ○	B ○	C ○	D ○
4.	A ○	B ○	C ○	D ○
5.	A ○	B ○	C ○	D ○
6.	A ○	B ○	C ○	D ○
7.	A ○	B ○	C ○	D ○
8.	A ○	B ○	C ○	D ○
9.	A ○	B ○	C ○	D ○
10.	A ○	B ○	C ○	D ○

2014 Mathematics Extension 1 Trial Solutions

Markers:

Q11. Katyal

Q12. Chirgwin

Q13. Young

Q14. Clarke

Section 1:

Q1. D	Q2. B	Q3. D	Q4. B	Q5. C
Q6. A	Q7. B	Q8. D	Q9. C	Q10. A

Exam Extn 1	MATHEMATICS : Question...11	
Suggested Solutions		Marker's Comments

a) $\frac{d}{dx}\tan^{-1}(4+x^3) = \frac{3x^2}{1+(4+x^3)^2}$ ✓

b) $\int \frac{2}{\sqrt{25-2x^2}}\,dx = \sqrt{2}\sin^{-1}\frac{\sqrt{2}x}{5} + c$ ✓✓

Many students made an error by putting wrong coeff. in front of $\sin^{-1}(\ )$

c) (i) ${}^{10}C_0 \times {}^{10}C_3 = 120$ ways ✓

(ii) ${}^{14}C_3 - {}^{10}C_3 = 244$

probability $= \frac{244}{364} = \frac{61}{91}$ ✓

d) $u = e^{4x}+9$, $\frac{du}{dx} = 4e^{4x}$

$\int \frac{3}{4}\frac{du}{u^{1/2}} = \frac{3}{4}\left[\frac{u^{1/2}}{1/2}\right]_{10}^{25}$ ✓

$= \frac{3}{2}(5-\sqrt{10})$ ✓

e) $y = 2\ln(x+1)$, $y = \ln(x^2+3)$

$(x+1)^2 = x^2+3$ ✓ ✓

(i) $2x = 2$

$x = 1$, $y = 2\ln 2$ $T(1, \ln 4)$ OR $T(1, 2\ln 2)$

(ii) for $y = 2\ln(x+1)$

$\frac{dy}{dx} = \frac{2}{x+1}\Big|_{x=1} = 1$ ✓

for $y = \ln(x^2+3)$

$\frac{dy}{dx} = \frac{2x}{x^2+3}\Big|_{x=1}$ ✓

$= \frac{1}{2}$

$\tan\theta = \left|\frac{m_1-m_2}{1+m_1m_2}\right|$

$= \frac{1}{3}$

$\theta = 18°26'$

Exam	MATHEMATICS : Question...11	
	Suggested Solutions	Marker's Comments

(f)

(i)

$$\begin{array}{r} x^2 + (3-m)x + (9-4m) \\ x-3 \overline{\big)\, x^3 - mx^2 - mx + 9} \\ \underline{x^3 - 3x^2} \\ (3-m)x^2 - mx \\ \underline{(3-m)x^2 - 3(3-m)x} \\ (9-4m)x + 9 \\ \underline{(9-4m)x - 27 + 12m} \\ 36 - 12m \end{array}$$

Quotient : $x^2 + (3-m)x + (9-4m)$ ✓

Remainder : $36 - 12m$ ✓

(ii) Remainder $= 0$ $\therefore$ $m = 3$ ✓

(iii) $x^3 - 3x^2 - 3x + 9 = (x-3)(x^2-3)$

$= (x-3)(x+\sqrt{3})(x-\sqrt{3})$

the roots are $3, \pm\sqrt{3}$. ✓

Exam Maths Ext 1	MATHEMATICS	: Question...12

Suggested Solutions	Marker's Comments

a) i) $r = \frac{4}{\sqrt{3}} \sin x \cos x$

$-1 < r < 1$

$-1 < \frac{4}{\sqrt{3}} \sin x \cos x < 1$

$-\frac{\sqrt{3}}{4} < \sin x \cos x < \frac{\sqrt{3}}{4}$

$-\frac{\sqrt{3}}{2} < 2 \sin x \cos x < \frac{\sqrt{3}}{2}$

$\therefore -\frac{\sqrt{3}}{2} < \sin 2x < \frac{\sqrt{3}}{2}$

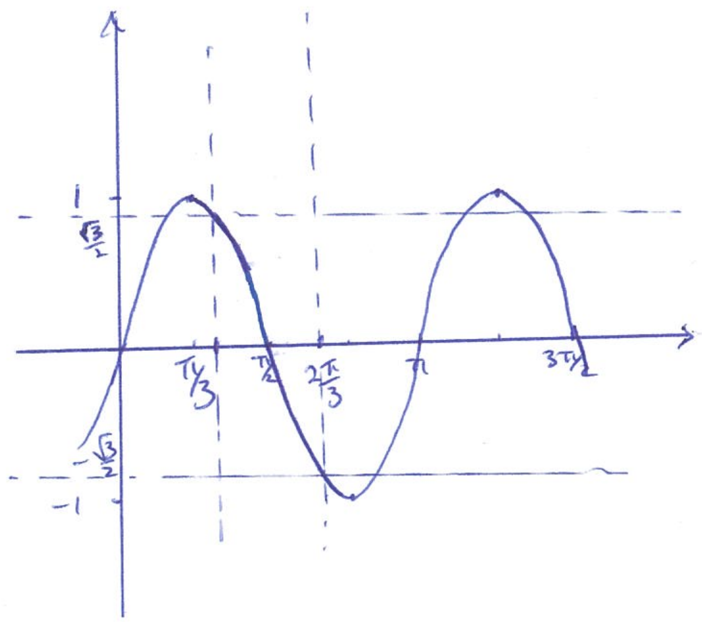
ii) $\frac{\pi}{4} \leq x < \frac{3\pi}{4} \rightarrow \frac{\pi}{2} \leq 2x < \frac{3\pi}{2}$

$-\frac{\sqrt{3}}{2} \leq \sin 2x < \frac{\sqrt{3}}{2}$

$-\frac{\pi}{3} < 2x < \frac{\pi}{3}$ (Quadrant 1 & 4)

$\frac{4\pi}{3} > 2x > \frac{2\pi}{3}$ (Quadrant 2 & 3)

$\therefore \frac{\pi}{3} < x < \frac{2\pi}{3}$



Exam Maths Ext 1	MATHEMATICS : Question 12	
	Suggested Solutions	Marker's Comments

b) $\left(x^2 - \frac{1}{2x^5}\right)^{14}$

$^{14}C_k (x^2)^{14-k} \left(\frac{1}{2}x^{-5}\right)^k$

$= {}^{14}C_k\, x^{28-2k} \times \left(\frac{1}{2}\right)^k \times x^{-5k}$

$= {}^{14}C_k \left(\frac{1}{2}\right)^k x^{28-7k}$

$28 - 7k = 0$

$k = 4$

$\therefore$ term independent of x is

${}^{14}C_4 \left(\frac{1}{2}\right)^4 = 62\frac{9}{16}$

c) $V = \pi \int_0^{\pi/2} y^2\, dx$

$= \pi \int_0^{\pi/2} (\cos^2 x + 2\cos x + 1)\, dx$

$= \pi \int_0^{\pi/2} \left(\frac{1}{2}\cos 2x + \frac{1}{2} + 2\cos x + 1\right) dx$

$= \pi \int_0^{\pi/2} \left(\frac{1}{2}\cos 2x + 2\cos x + \frac{3}{2}\right) dx$

$= \pi \left[\frac{1}{4}\sin 2x + 2\sin x + \frac{3}{2}x\right]_0^{\pi/2}$

$= \pi \left[\frac{1}{4}\sin \pi + 2\sin\frac{\pi}{2} + \frac{3}{2}\times\frac{\pi}{2}\right]$

$= 2\pi + \frac{3\pi^2}{4}$ units^3

Marker's Comments:

* incorrect expansion of y^2.

* incorrect integration of $\cos^2 x$

Exam Maths Ext	MATHEMATICS	: Question 12

Suggested Solutions	Marker's Comments

d) i) $v^2 = 56 - 20x - 4x^2$

$\frac{1}{2}v^2 = 28 - 10x - 2x^2$

$\ddot{x} = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = -10 - 4x$

$\ddot{x} = -4\left(x + \frac{5}{2}\right)$

It is in simple harmonic motion as it is in the form of $\ddot{x} = -n^2(x - x_0)$.

ii) $n = 2$, period $= \frac{2\pi}{n} = \pi$,

The particle oscillates between two end points where it stops, $v = 0$

$56 - 20x - 4x^2 = 0$

$x^2 + 5x - 14 = 0$

$(x+7)(x-2) = 0$

$x = -7,\ x = 2$

$\therefore$ amplitude is 4.5

Centre of motion is -2.5.

e) i) $(k-1)x^3 + (k+4)x^2 + (k-6)x - k = 0$

$\alpha\beta\gamma = -\frac{d}{a}$

$\frac{k}{k-1} = \frac{4}{5}$

$5k = 4k - 4$

$k = -4$

$\alpha + \beta + \gamma = -\frac{k+4}{k-1}$

$\therefore \alpha + \beta + \gamma = 0$

ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)$

$= 0 - 2 \times \frac{k-6}{k-1}$

$= -4$

Exam	MATHEMATICS	: Question…… 13a

Suggested Solutions	Marker's Comments
13a	Must factorise a^2-1 to show that it is divisible by $a-1$

Prove true for $n=1$

$a^2-1=(a+1)(a-1)$ which is divisible by $a-1$

Assume true for $n=k$

$a^{3k-1}-1=m(a-1)$ where m is an integer

Prove true for $n=k+1$

$a^{3(k+1)-1}-1$

$=a^{3k+3-1}-1$

$=a^{3k-1+3}-1$

$=a^3.a^{3k-1}-a^3+a^3-1$

$=a^3(a^{3k-1}-1)+a^3-1$

$=a^3(m(a-1))+a^3-1$

$=a^3(m(a-1))+(a-1)(a^2+a+1)$

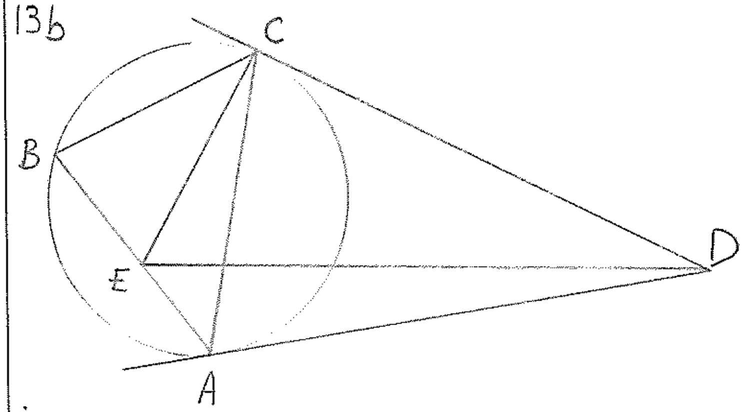
$=(a-1)(a^3m+a^2+a+1)$

which is divisible by $a-1$

$\therefore$ true for $n=k+1$

Since true for $n=1$ and true for $n=k$ and true for $n=k+1$ then true for $n=2$ and so on.

13b



i. $\angle ACD=\angle ABC$ (angle between a tangent and a chord equals the angle at the circumference in the alternate segment)

ΔCBE is isosceles since $BE=CE$

$\angle EBC=\angle ECB$ (equal angles in isosceles Δ)

$\therefore \angle ACD=\angle BCE$

Marker's Comments: Some students did not write the last line

Exam MATHEMATICS : Question……

Suggested Solutions	Marker's Comments

ii. Let $\angle ABC = \angle BCE = \angle ACD = x$ (from i)

$\angle BEC = 180 - 2x$ (angle sum ΔBCE)

$\angle CEA = 2x$ (adjacent supplementary angles)

$AD = CD$ (tangents to a circle from an external point are equal)

$\therefore \Delta ACD$ is isosceles

$\therefore \angle CAD = x$ (equal angles in isosceles Δ)

$\angle CDA = 180 - 2x$ ($\angle$ sum ΔDAC)

$\angle ADC + \angle AEC = 180$

$\therefore ADCE$ is a cylic quadrilateral since opposite angles are supplementary.

Marker's comment: make sure reasoning includes full description of properties, not just "external angle theorem"

iii. $\angle ACD = \angle AED = x$

(chord subtends equal angles to circumference in the same segment)

$\therefore BC \parallel ED$ since corresponding angles are equal

c i. $\binom{n}{0}x^n(1-x)^0 + \binom{n}{1}x^{n-1}(1-x) + \binom{n}{2}x^{n-2}(1-x)^2 + \ldots + \binom{n}{n}x^0(1-x)^n = (x+(1-x))^n$

$\binom{n}{0}x^n + \binom{n}{1}x^{n-1}(1-x) + \binom{n}{2}x^{n-2}(1-x)^2 + \ldots + \binom{n}{n}(1-x)^n = 1$

ii. sub $x = \frac{1}{2}$ (from part (i))

$\binom{n}{0}\left(\frac{1}{2}\right)^n + \binom{n}{1}\left(\frac{1}{2}\right)^{n-1}\left(\frac{1}{2}\right) + \binom{n}{2}\left(\frac{1}{2}\right)^{n-2}\left(\frac{1}{2}\right)^2 + \ldots + \binom{n}{n}\left(\frac{1}{2}\right)^n = 1$

$\binom{n}{0}2^{-n} + \binom{n}{1}2^{-n} + \binom{n}{2}2^{-n} + \ldots + \binom{n}{n}2^{-n} = 1$

$\left[\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n}\right]2^{-n} = 1$

$\binom{n}{0} + \binom{n}{1} + \binom{n}{2} + \ldots + \binom{n}{n} = 2^n$

Marker's comment: use part (i) to solve part (ii)

Exam	MATHEMATICS : Question......	
	Suggested Solutions	**Marker's Comments**

di. $PQ = \sqrt{(2ap-2aq)^2+(ap^2-aq^2)}$ focal chord $q = -\frac{1}{p}$

$$PQ = \sqrt{\left(2ap+\frac{2a}{p}\right)^2+\left(ap^2-\frac{a}{p^2}\right)^2}$$

$$= \sqrt{4a^2\left(p+\frac{1}{p}\right)^2+a^2\left(p^2-\frac{1}{p^2}\right)^2}$$

$$= a\sqrt{4\left(p+\frac{1}{p}\right)^2+\left(p+\frac{1}{p}\right)^2\left(p-\frac{1}{p}\right)^2}$$

$$= a\sqrt{\left(p+\frac{1}{p}\right)^2\left(4+\left(p-\frac{1}{p}\right)^2\right)}$$

$$= a\sqrt{\left(p+\frac{1}{p}\right)^2\left(p^2+2+\frac{1}{p^2}\right)}$$

$$= a\sqrt{\left(p+\frac{1}{p}\right)^2\left(p+\frac{1}{p}\right)^2}$$

$$= a\left(p+\frac{1}{p}\right)^2$$

$$= a\left(p^2+2+\frac{1}{p^2}\right)$$

$$= 2a+a\left(p^2+\frac{1}{p^2}\right)$$

ii centre of circle is midpt of PQ

$$\left(ap+aq,\ \frac{ap^2+aq^2}{2}\right)$$

Radius $= \frac{1}{2}PQ = \dfrac{2a+a\left(p^2+\frac{1}{p^2}\right)}{2}$

$$= a+\frac{a}{2}\left(p^2+\frac{1}{p^2}\right)$$

Distance from directrix to centre is $a+\dfrac{ap^2+aq^2}{2}$

$q=-\frac{1}{p}$ $\quad a+\dfrac{ap^2+\frac{a}{p^2}}{2}$

$$= \frac{2a+ap^2+\frac{a}{p^2}}{2}$$

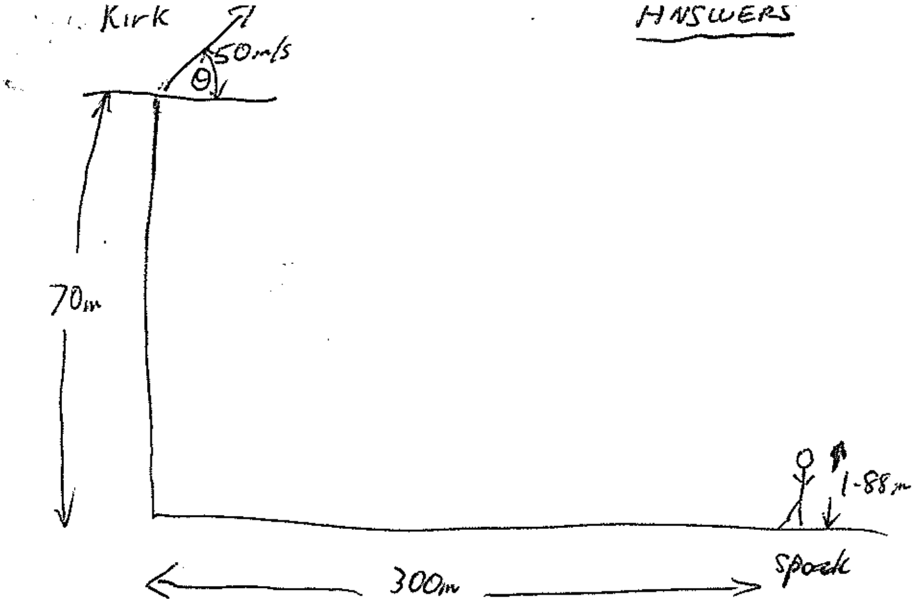
$$= a+\frac{a}{2}\left(p^2+\frac{1}{p^2}\right)$$

$$= \text{radius}$$

∴ directrix is tangent to the circle

Marker's Comments: Some students used $pq = 1$ instead of $pq = -1$ for focal chord

ANSWERS



Q14 (a) i) Vertical Equation

$\frac{d^2y}{dt^2} = -g$

$\frac{dy}{dt} = -gt + 50\sin\theta$ Since $50\sin\theta$ is initial velocity when $t=0$ (1)

$y = 50\sin\theta \times t - \frac{1}{2}\times 9{\cdot}8 \times t^2 + c$

But when $t=0$ $y=70$ ∴ $c=70$

$y = 50\sin\theta \times t - 4{\cdot}9 \times t^2 + 70$

a) ii) Horizonal Equations

$\frac{dx}{dt} = 50\cos\theta$

$x = 50\cos\theta \times t + c$

But when $t=0$ $x=0$ ∴ $c=0$

$x = 50\cos\theta \times t$ (1)

ii) $x = 50\cos\theta \times t$

$t = \frac{300}{50\cos\theta}$ since

$t = \frac{6}{\cos\theta}$ (i)

$+1{\cdot}88 = 50\sin\theta \times t - 4{\cdot}9 \times t^2 + 70$

sub (i) into (ii)

$1{\cdot}88 = 50\sin\theta \times \frac{6}{\cos\theta} - 4{\cdot}9 \times \left(\frac{6}{\cos\theta}\right)^2 + 70$ (1)

$-68{\cdot}12 = 300\tan\theta - 176{\cdot}4(1+\tan^2\theta)$

$-68{\cdot}12 = 300\tan\theta - 176{\cdot}4 - 176{\cdot}4\tan^2\theta$

$-176{\cdot}4\tan^2\theta + 300\tan\theta - 108{\cdot}28 = 0$ (1)

Let $a = \tan\theta$

$-176{\cdot}4a^2 + 300a - 108{\cdot}28 = 0$

$a = \frac{-300 \pm \sqrt{300^2 - 4\times -176{\cdot}4 \times -108{\cdot}28}}{2\times -176{\cdot}4}$

$a = \frac{-300 \pm \sqrt{13{,}597{\cdot}632}}{-352{\cdot}8}$ (1)

$= 0{\cdot}519816086$ or $1{\cdot}180864186$

$\tan\theta = 0{\cdot}519816086$ $\tan\theta = 1{\cdot}180864186$

$\theta = 27°28'$ $\theta = 49°44$

iii) Sub $\theta = 27°28'$ into (i)

$t = \frac{6}{\cos 27°28'}$

$= 6{\cdot}76$ sec (1)

or

$t = \frac{6}{\cos 49°44}$

$= 9{\cdot}28$ sec

∴ Take $\theta = 49°44'$ because it allows Kirk the necessary time to hide before spooks looks up after being hit with the bomb. (1)

b) i) The volume of water thats left = Rate of flow × time

ie $(1-H)\times\pi\times(0.5)^2 = 0.25\, m^3/min \times t$ } (1)

$(1-H)\times\pi\times(0.25) = 0.25\times t$ $\quad \frac{\pi}{4}-\frac{H\pi}{4}=\frac{t}{4}$

$\pi - H\pi = t$

$\pi - \pi H = t$

$-\pi H = t - \pi$

$H = \frac{\pi - t}{\pi}$

$= 1 - \frac{t}{\pi}$ } (1)

Vol of water flowing from the can in terms of t

$V(t) = \pi r^2 (H)$

$\frac{dV}{dt} = \frac{\pi}{4}\frac{dH}{dt}$

$-\frac{1}{4} = \frac{\pi}{4}\frac{dH}{dt}$

$-1 = \pi\frac{dH}{dt}$

$\frac{dH}{dt} = -\frac{1}{\pi}$

$H(t) = -\frac{1}{\pi}t + c$

$1 = c$ since $H=1$ when

$H(t) = 1 - \frac{t}{\pi}$

ii) Again Vol of water in bowl = Rate of flow × time

$(V=)\,\pi h^2 = 0.25\times t$ } (1)

$4\pi h^2 = t$

$h^2 = \frac{t}{4\pi}$

$h = \sqrt{\frac{t}{4\pi}}$ } (1)

iii) $H = h$ ∴

$1 - \frac{t_1}{\pi} = \sqrt{\frac{t_1}{4\pi}}$

$1 - \frac{t_1}{\pi} - \frac{t_1}{\pi} + \frac{t_1^2}{\pi^2} = \frac{t_1}{4\pi}$ } (1)

$1 - \frac{2t_1}{\pi} + \frac{t_1^2}{\pi^2} = \frac{t_1}{4\pi}$

$4\pi^2 - 8\pi t_1 + 4t_1^2 = \pi t_1$

$4t_1^2 - 9\pi t_1 + 4\pi^2 = 0$ } (1)

(iv) $t = \frac{9\pi \pm\sqrt{(-9\pi)^2 - 4\times4\times4\pi^2}}{2\times4}$

$= \frac{9\pi \pm\sqrt{17\pi^2}}{8}$

$= \frac{9\pi \pm \sqrt{17}\times\pi}{8}$

$= \frac{\pi(9\pm\sqrt{17})}{8}$

~~= 0.6~~ } (1)

iv) Continued

Amount of Water in can

$= \pi r^2 \times H$

$= \pi r^2$

$= \pi\times 0.5^2$

$= \frac{\pi}{4}\, m^3$

∴ $V = \frac{\pi}{4} > 0.25\times t$

$\pi > t$

$t < \pi$

Now $t = \frac{\pi(9\pm\sqrt{17})}{8}$

$= \pi\times 0.609$ or 1.64π

∴ $t = 0.609\pi$ since $t<\pi$ } (1)

14i $\frac{dV}{dt} = \frac{1}{4}$ $V = \pi r^2(1-H)$ $= \frac{\pi}{4}(1-H)$ $= \frac{\pi}{4} - \frac{\pi H}{4}$ $\therefore \frac{dV}{dh} = -\frac{\pi}{4}$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dH}{dt}$

$\frac{1}{4} = -\frac{\pi}{4} \times \frac{dH}{dt}$

$\frac{dH}{dt} = -\frac{1}{\pi}$

$H = \int -\frac{1}{\pi}\, dt$

$H = -\frac{1}{\pi} t + c$

but $H=1$ when $t=0$ $\therefore c = 1$

$H = 1 - \frac{t}{\pi}$

Alternate methods

14i Yating $\frac{dV}{dt} = \frac{1}{4}$

$V = \int \frac{1}{4}\, dt$

$= \frac{-t}{4} + C$

But when $t=0$ $V = \frac{\pi}{4}$ $\therefore C = \frac{\pi}{4}$

$V = \frac{-t}{4} + \frac{\pi}{4}$

$V = \frac{\pi - t}{4}$

V is also $V = \pi r^2 H$

$= \frac{\pi}{4} H$

$\therefore \frac{\pi}{4} H = \frac{\pi - t}{4}$

$H = \frac{4}{\pi} \times \left(\frac{\pi - t}{4}\right)$

$= \frac{\pi - t}{\pi}$

$= 1 - \frac{t}{\pi}$

14ii) $\frac{dV}{dt} = 0.25$ $V = \pi h^2$ $\frac{dV}{dh} = 2\pi$

$\frac{dV}{dt} = \frac{dV}{dh} \times \frac{dh}{dt}$ $\frac{dh}{dt} = \frac{dV}{dt} \times \frac{dh}{dV}$ $\frac{dh}{dt} = \frac{\frac{1}{4}}{2\pi h}$

$\frac{1}{4} = 2\pi h \times \frac{dh}{dt}$

$\frac{1}{8\pi h} = \frac{dh}{dt}$

$\frac{dt}{dh} = 8\pi h$

$t = \int 8\pi h\, dh$

$t = 4\pi h^2 + c$

Since when $t=0$ $h=0$ $c=0$

$t = 4\pi h^2$

$\frac{t}{4\pi} = h^2$

$h = \sqrt{\frac{t}{4\pi}}$

$\int 2\pi h\, dh = \int \frac{1}{4}\, dt$

$\pi h^2 = \frac{t}{4} + c$

$\therefore t = 4\pi h^2$ $c=0$ since $h=0$ when $t=0$

$h^2 = \frac{t}{4\pi}$

$h = \sqrt{\frac{t}{4\pi}}$ $h > 0$