



PENRITH HIGH SCHOOL

**2014
HSC TRIAL EXAMINATION**

Mathematics Extension 2

General Instructions:

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Answer all Questions in the booklets provided

Total marks–100

SECTION I Pages 2–5

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

SECTION II Pages 6–11

90 marks

- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section

Student Name: _____

Teacher Name: _____

This paper MUST NOT be removed from the examination room

Assessor: Mr Ferguson

Section I

10 marks

Attempt Questions 1–10

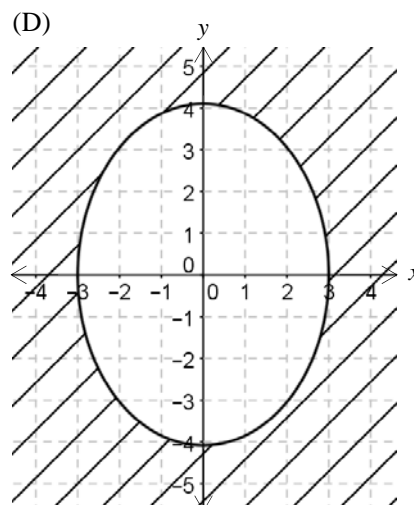
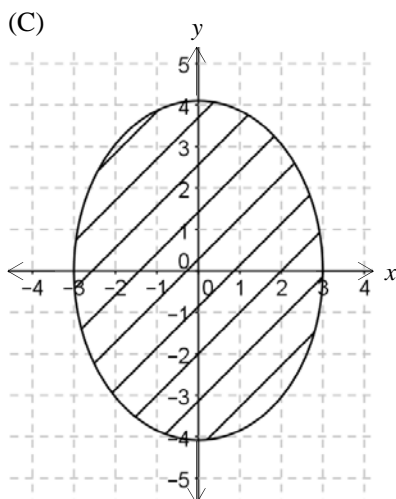
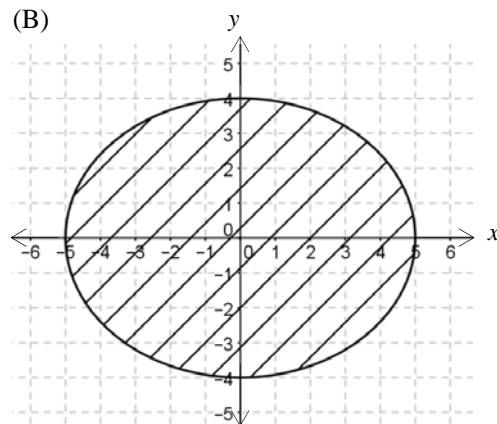
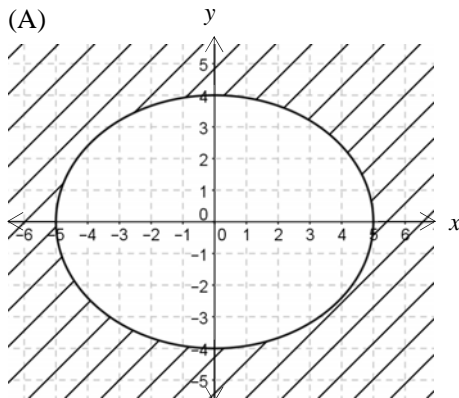
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

1 What is $-\sqrt{3} + i$ expressed in modulus-argument form?

- (A) $\sqrt{2}(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$
- (B) $2(\cos \frac{\pi}{6} + i \sin \frac{\pi}{6})$
- (C) $\sqrt{2}(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6})$
- (D) $2(\cos \frac{5\pi}{6} + i \sin \frac{5\pi}{6})$

2 The sketch of the locus of an equation $|z-3| + |z+3| \leq 10$ where $z = x + iy$ can best be represented by.



3 Which of the following expressions is equivalent to $\int_0^2 \sqrt{4-x^2} dx$.

- (A) π
- (B) 2π
- (C) 4π
- (D) 8π

4 Which expression is equal to $\int \frac{1}{\sqrt{4x^2-8x+5}} dx$?

- (A) $\frac{1}{2} \sin^{-1} 2(x-3) + C$
- (B) $\frac{1}{2} \cos^{-1} 2(x-3) + C$
- (C) $\frac{1}{2} \ln \left(x-1 + \sqrt{x^2 - 2x + \frac{5}{4}} \right) + C$
- (D) $\frac{1}{2} \ln \left(x-1 + \sqrt{x^2 - 2x - \frac{5}{4}} \right) + C$

5 If a, b, c, d , and e are real numbers and $a \neq 0$, then the polynomial equation $ax^7 + bx^5 + cx^3 + dx + e = 0$ has.

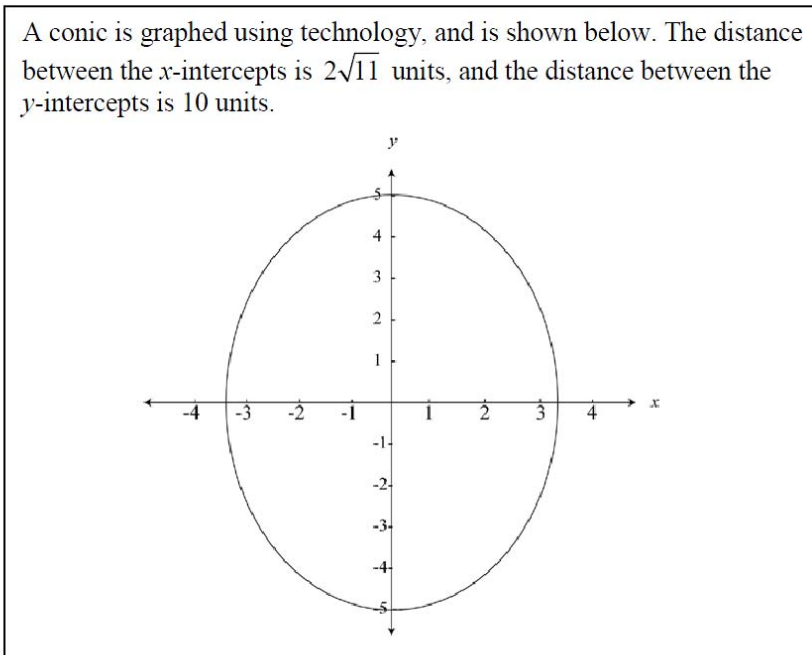
- (A) only one real root.
- (B) at least one real root.
- (C) an odd number of nonreal roots
- (D) no real roots

6 Suppose that a function $y = f(x)$ is given with $f(x) \geq 0$ for $0 \leq x \leq 4$. If the area bounded by the curves $y = f(x)$, $y = 0$, $x = 0$, and $x = 4$ is revolved about the line $y = -1$, then the volume of the solid of revolution is given by.

- (A) $\pi \int_0^4 [f(x-1)^2 - 1] dx$
- (B) $\pi \int_0^4 [(f(x)-1)^2 - 1] dx$
- (C) $\pi \int_0^4 [f(x+1)^2 - 1] dx$
- (D) $\pi \int_0^4 [(f(x)+1)^2 - 1] dx$

Use the following information to answer the next question.

A conic is graphed using technology, and is shown below. The distance between the x -intercepts is $2\sqrt{11}$ units, and the distance between the y -intercepts is 10 units.



7 The equation of the graph shown above is.

(A) $\frac{x^2}{25} + \frac{y^2}{44} = 1$

(B) $\frac{x^2}{44} + \frac{y^2}{25} = 1$

(C) $\frac{x^2}{25} + \frac{y^2}{11} = 1$

(D) $\frac{x^2}{11} + \frac{y^2}{25} = 1$

8 If $3x^2 + 2xy + y^2 = 2$, then the value of $\frac{dy}{dx}$ at $x = 1$ is

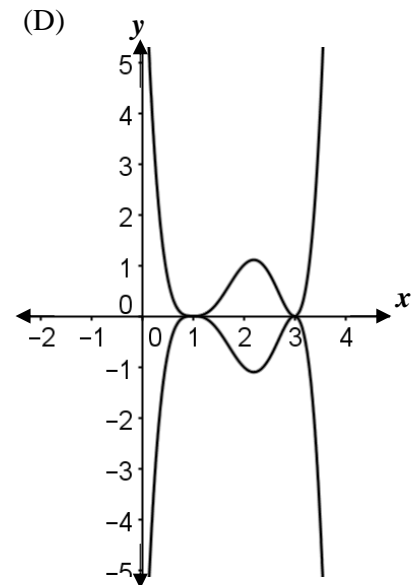
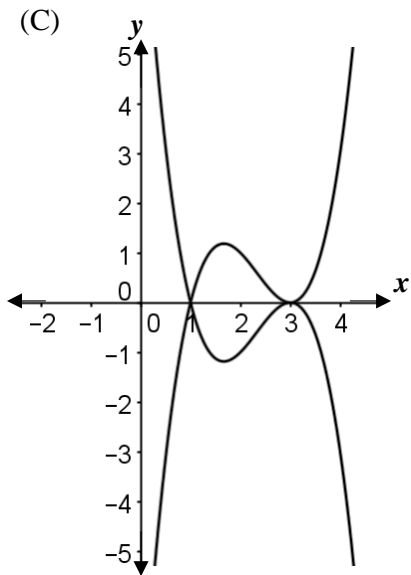
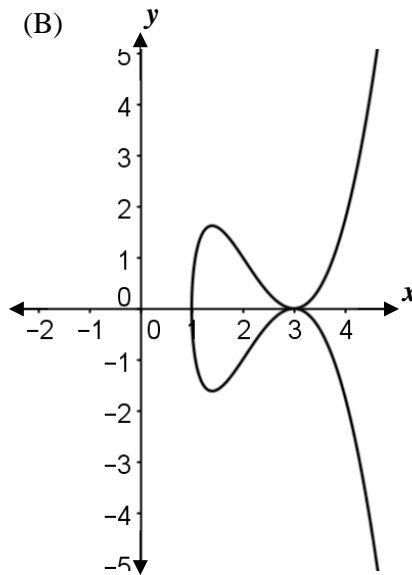
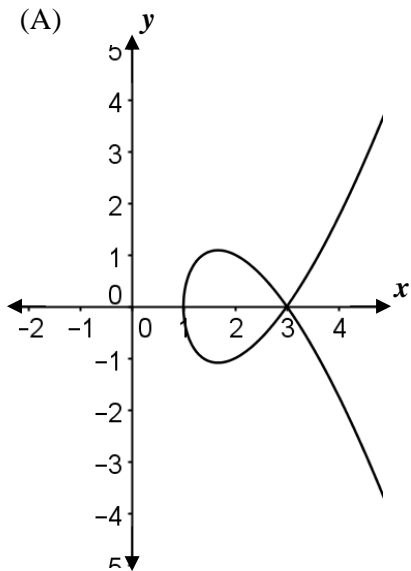
(A) -2

(B) 0

(C) 4

(D) not defined

9 Which diagram best represents the graph $y^2 = (x-1)(x-3)^2$?



10 A person is standing on the outer edge of a circular disc that is spinning. His relative position on the disc remains unchanged. Which description below best describes the situation?

- (A) The person is experiencing a force that is pushing him away from the centre of the disc.
- (B) The person is experiencing a force that is pushing him towards the centre of the disc.
- (C) The person is experiencing a force tangential to the edge of the disk in the direction of the motion of the disk.
- (D) The person is experiencing a force tangential to the edge of the disk in the opposite direction to the motion of the disk.

Section II

90 marks

Attempt Questions 11–16

Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a) If $z_1 = 3 + 4i$, $z_2 = 1 - i$, find

(i) $\overline{z_1 z_2}$ 1

(ii) $\left| \frac{z_1}{z_2} \right|$ 2

(iii) $\sqrt{z_1}$ 3

b) Sketch separately the following loci in an Argand plane and state the cartesian equations in each case.

(i) $\operatorname{Re}\left(\frac{z-2}{2}\right) = 0$ 2

(ii) $\arg(z+2) = -\frac{\pi}{6}$ 2

c) (i) Express $\frac{1+2x^2}{(2+x^2)(1+x^2)}$ in the form $\frac{A}{2+x^2} + \frac{B}{1+x^2}$ 2

(ii) Use the substitution $t = \tan x$ and your answer from part (i) to find $\int \frac{(1+\sin^2 x) dx}{1+\cos^2 x}$ 3

(Leave your answer in term of t)

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) If $\arg z_1 = \theta$ and $\arg z_2 = \phi$, show that $\arg \left(\frac{z_1}{z_2} \right) = \arg(z_1) - \arg(z_2)$ **3**

b) The equation $z^2 + (1+i)z + k = 0$ has root $1 - 2i$. Find the other root, and the value of k . **2**

c) Let α, β, γ be the roots (none of which is zero) of $x^3 + 3px + q = 0$

(i) Find expressions for $\alpha + \beta + \gamma$, $\alpha\beta + \beta\gamma + \alpha\gamma$ and $\alpha\beta\gamma$ **1**

(ii) Find an expression for $\frac{\alpha\beta}{\gamma} + \frac{\beta\gamma}{\alpha} + \frac{\gamma\alpha}{\beta}$ **2**

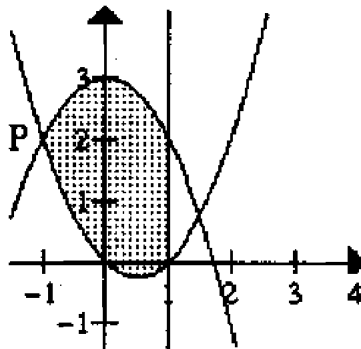
(iii) Find an expression for $\frac{\alpha\beta}{\gamma} \cdot \frac{\beta\gamma}{\alpha} + \frac{\alpha\beta}{\gamma} \cdot \frac{\gamma\alpha}{\beta} + \frac{\beta\gamma}{\alpha} \cdot \frac{\gamma\alpha}{\beta}$ **2**

(iv) Hence obtain a monic equation whose roots are $\frac{\alpha\beta}{\gamma}, \frac{\beta\gamma}{\alpha}, \frac{\gamma\alpha}{\beta}$ **2**

d) Show that $\int_0^1 \frac{dx}{9-x^2} = \frac{1}{6} \ln 2$ **3**

Question 13 (15 marks) Use a SEPARATE writing booklet.

- a) The shaded region bounded by $y = 3 - x^2$, $y = x^2 - x$ and $x = 1$ is rotated about the line $x = 1$. The point P is the intersection of $y = 3 - x^2$ and $y = x^2 - x$ in the second quadrant.



- i) Find the x coordinate of P . 1
- ii) Use the method of cylindrical shells to express the volume of the resulting solid of revolution as an integral [DO NOT SOLVE THE INTEGRAL] 3

b) (i) If $u_n = \frac{1}{n!} \int_0^1 x^n e^{-x} dx$, where $n \geq 0$, show that $\frac{1}{n!} = e(u_{n-1} - u_n)$ 3

(ii) Hence find the value of u_4 2

- c) If a, b, c are positive real numbers;
- i) Show that $a^2 + b^2 \geq 2ab$ 1
- ii) Hence prove $(a + b + c) \left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c} \right) \geq 9$ 2

- d) If z_1, z_2 are two complex numbers such that $|z_1 + z_2| = |z_1 - z_2|$, show that 3

$$\arg z_1 - \arg z_2 = \frac{\pi}{2}$$

Question 14 (15 marks) Use a SEPARATE writing booklet.

- a) $f(x)$ is defined by the equation $f(x) = x^2 \left(x - \frac{3}{2} \right)$, on the domain $-2 \leq x \leq 2$.

Note: each sketch should take about a third of a page.

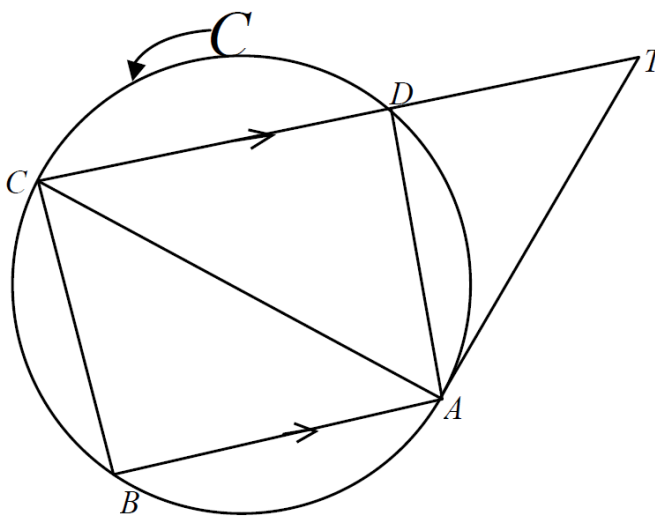
- i) Draw a neat sketch of $f(x)$, labelling all intersections with coordinate axes and turning points 2

- ii) Sketch $y = \frac{1}{f(x)}$ 2

- iii) Sketch $y = \sqrt{f(x)}$ 2

- iv) Sketch $y = \ln(f(|x|))$ 2

- b) The points A, B, C and D lie on the circle C . From the exterior point T , a tangent is drawn to point A on C . The line CT passes through D and TC is parallel to AB .



- i) Copy or trace the diagram onto your page.
- ii) Prove that $\triangle ADT$ is similar to $\triangle ABC$. 3

The line BA is produced through A to point M , which lies on a second circle. The points A, D, T also lie on this second circle and the line DM crosses AT at O .

- iii) Show that $\triangle OMA$ is isosceles. 2
- iv) Show that $TM = BC$. 2

Question 15 (15 marks) Use a SEPARATE writing booklet.

a) Given that $\int_0^a f(x) dx = \int_0^a f(a-x) dx$, find $\int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx$ 2

b) The hyperbola H has an equation $xy = 9$. $P\left(3p, \frac{3}{p}\right)$, where $p > 0$, and $Q\left(3q, \frac{3}{q}\right)$, where $q > 0$, are two distinct arbitrary points on H .

(i) Prove that the equation of the tangent at P is $x + p^2y = 6p$ 2

(ii) The tangents at P and Q intersect at T . Find the coordinates of T . 3

(iii) The chord PQ produced passes through the point $(0, 6)$. Given that the equation of this chord is $x + pqy = 3(p + q)$ find;

(a) Find the equation of the locus of T 3

(b) Give a geometrical description of this locus 1

c) A light inextensible string of length $3L$ is threaded through a smooth vertical ring which is free to turn. The string carries a particle at each end. One particle A of mass m is at rest at a distance L below the ring. The other particle B of mass M is rotating in a horizontal circle whose centre is A .

(i) Find m in terms of M . 2

(ii) Find the angular velocity of B in terms of g and L 2

Question 16 (15 marks) Use a SEPARATE writing booklet.

- a) Use mathematical induction to prove that for all n where n can be any positive integer that $(a-b)$ is a factor of $a^n - b^n$ **3**
- b) A car travels around a banked circular track of radius 90 metres at 54 km/h.
- (i) Draw a diagram showing all the forces acting on the car **1**
- (ii) Show that the car will have tendency to slip sideways if the angle at which the banked track is banked is $\tan^{-1}\left(\frac{1}{4}\right)$. **3**
- (iii) A second car of mass 1.2 tonnes travels around the same bend at 72 km/h. Find the sideways frictional force exerted by the road on the wheels of the car in Newtons. You may assume gravity = 10 m/s^2 . (Answer correct to 1 decimal place) **3**
- c) (i) Using $\tan(2\theta + \theta) = \tan 3\theta$, show that $\tan 3\theta = \frac{3 \tan \theta - \tan^3 \theta}{1 - 3 \tan^2 \theta}$ **2**
- (ii) Find the value of x for which $3 \tan^{-1} x = \frac{\pi}{2} - \tan^{-1} 3x$, **3**
- where $\tan^{-1} x$ and $\tan^{-1} 3x$ both lie between 0 and $\frac{\pi}{2}$

End of Exam

STANDARD INTEGRALS

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, a \neq 0$$

$$\int \sec^2 ax dx = \frac{1}{a} \tan ax, a \neq 0$$

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, a \neq 0$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a}, a \neq 0, -a < x < a$$

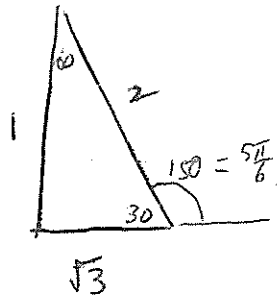
$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, x > 0$

Multiple Choice

1. $-\sqrt{3} + i$



$$2 \operatorname{cis} \frac{5\pi}{6}$$

= D.

2. sum of focal lengths is a constant $\Rightarrow 2a$
 $\therefore a = 5$.

B

3.
$$= \frac{4\pi}{4} = \pi$$

A.

4.
$$\int \frac{1}{\sqrt{4(x^2 - 2x + \frac{5}{2})}} dx$$

$$= \frac{1}{2} \int \frac{1}{\sqrt{x^2 - 2x + \frac{5}{2}}} dx$$

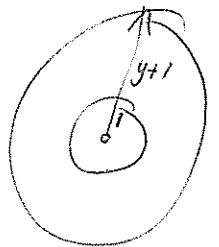
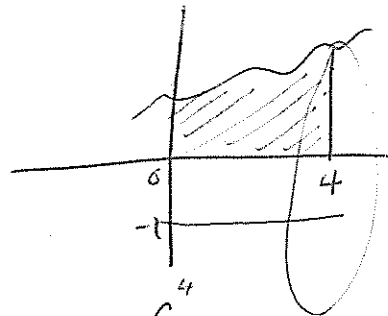
$$= \frac{1}{2} \int \frac{1}{\sqrt{(x-1)^2 + (\frac{\sqrt{3}}{2})^2}} dx$$

$$= \frac{1}{2} \ln \left[(x-1) + \sqrt{x^2 - 2x + \frac{5}{2}} \right] + c$$

= C

5. B
 unreal occur in conjugate pairs since real coefficients
 so since the degree of polynomial is odd there must be at least 1 real root.

6.



$$\pi \int_0^4 [(y+1)^2 - 1] dy$$

= D.

7. D

8. when $x = 1$.

D

$$3 + 2y + y^2 = 2$$

$$y^2 + 2y + 1 = 0$$

$$(y+1)^2 = 0$$

$$y = -1$$

$$\frac{d}{dx} \Rightarrow 6x + y \cdot 2 + 2x \frac{dy}{dx} + 2y \frac{dy}{dx} = 0$$

$$2x \frac{dy}{dx} + 2y \frac{dy}{dx} = -2y - 6x$$

$$\frac{dy}{dx} (2x + 2y) = -2y - 6x$$

$$\frac{dy}{dx} = \frac{-2y - 6x}{2x + 2y} \quad \text{when } x=1, y=-1$$

9. A

10. B.

\therefore not defined

Question 11

a(i) $\overline{z_1 z_2}$

$$\begin{aligned} z_1 z_2 &= (3+4i)(1-i) \\ &= 3-3i+4i-4i^2 \\ &= 7+i \end{aligned}$$

$$\therefore \overline{z_1 z_2} = 7-i$$

(ii) $\left| \frac{z_1}{z_2} \right|$

$$\begin{aligned} \frac{z_1}{z_2} &= \frac{3+4i}{1-i} \times \frac{1+i}{1+i} \\ &= \frac{3+3i+4i+4i^2}{1+1} \\ &= \frac{-1+7i}{2} \\ &= -\frac{1}{2} + \frac{7}{2}i \end{aligned}$$

$$\left| \frac{z_1}{z_2} \right| = \left| -\frac{1}{2} + \frac{7}{2}i \right|$$

$$= \sqrt{\left(-\frac{1}{2}\right)^2 + \left(\frac{7}{2}\right)^2}$$

$$= \sqrt{\frac{1}{4} + \frac{49}{4}}$$

$$= \sqrt{\frac{50}{4}}$$

$$= \frac{5\sqrt{2}}{2}$$

$$\text{OR } \frac{|z_1|}{|z_2|} = \frac{\sqrt{9+16}}{\sqrt{1+1}}$$

$$= \frac{5}{\sqrt{2}}$$

$$= \frac{5\sqrt{2}}{2}$$

(iii) $(x+iy)^2 = 3+4i$

$$x^2 + 2xyi - y^2 = 3+4i$$

$$x^2 - y^2 = 3 \quad 2xy = 4$$

$$\therefore xy = 2$$

$$x^2 = 4$$

$$x = \pm 2$$

$$x^2 - \left(\frac{2}{x}\right)^2 = 3$$

$$y = \frac{2}{x}$$

$$x^4 - 3x^2 - 4 = 0$$

$$(x^2+1)(x^2-4) = 0$$

$$\therefore \begin{aligned} x &= 2 \quad y = 1 \\ x &= -2 \quad y = -1 \end{aligned}$$

$$\therefore \sqrt{z_1} = \pm(2+i)$$

b

(i)

$$\operatorname{Re}\left(\frac{z-2}{2}\right) = 0$$

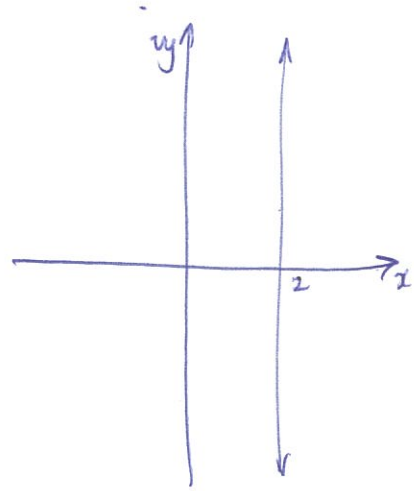
$$\frac{z-2}{2} = \frac{x+iy-2}{2}$$

$$= \frac{x-2}{2} + \frac{iy}{2}$$

$$\operatorname{Re}\left(\frac{z-2}{2}\right) = 0 \Rightarrow \frac{x-2}{2} = 0$$

$$\therefore x-2=0$$

$$\text{or } \underline{x=2}$$



$$(ii) \operatorname{arg}(z+2) = -\frac{\pi}{6}$$

$$m = \tan\frac{\pi}{6} = -\frac{1}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}} = -\frac{\sqrt{3}}{3}$$

For y intercept $\tan\frac{\pi}{6} = \frac{y}{2}$.

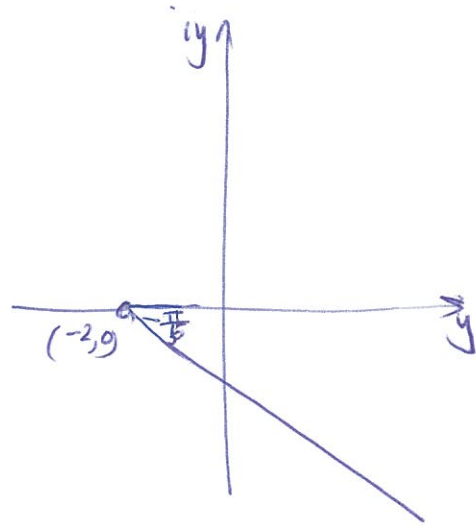
$$y = 2 \tan\frac{\pi}{6}$$

$$= \frac{2}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$$

$$= \frac{2\sqrt{3}}{3}$$

$$\therefore \text{equation is } y = -\frac{\sqrt{3}}{3}x - \frac{2\sqrt{3}}{3}$$

for $x > -2$,



$$C(i) \quad 1+2x^2 \equiv A(1+x^2) + B(2+x^2)$$

$$\equiv A+Ax^2+2B+Bx^2$$

$$= A+2B+(A+B)x^2$$

$$A+2B=1 \quad \text{--- (1)}$$

$$A+B=2 \quad \text{--- (2)}$$

$$\textcircled{1} - \textcircled{2} \quad B = -1$$

sub into (1)

$$A - 2 = 1$$

$$A = 3$$

$$\therefore \frac{3}{2+x^2} + \frac{-1}{1+x^2}$$

$$(ii) \quad \int \frac{(1+\sin^2 x)}{1+\cos^2 x} dx = \int \frac{1 + \frac{1-\cos 2x}{2}}{1 + \frac{1+\cos 2x}{2}} dx$$

$$= \int \frac{3 - \cos 2x}{3 + \cos 2x} dx$$

$$= \int \frac{3 - \frac{1-t^2}{1+t^2}}{3 + \frac{1-t^2}{1+t^2}} \cdot \frac{1}{1+t^2} dt$$

$$= \int \frac{3+3t^2 - 1+t^2}{(3+3t^2+1-t^2)(1+t^2)} dt$$

$$= \int \frac{1+2t^2}{(2+t^2)(1+t^2)} dt$$

Let $t = \tan x$

$$\frac{dt}{dx} = \sec^2 x$$

$$dt = \sec^2 x dx$$

$$dx = \frac{1}{1+t^2} dt$$

$$dx = \frac{1}{1+t^2} dt$$

From part (i)

$$= \int \frac{3}{2+t^2} dt - \int \frac{1}{1+t^2} dt$$

$$= \frac{3}{\sqrt{2}} \tan^{-1} \frac{t}{\sqrt{2}} - \tan^{-1} t + C$$

$$= \frac{3}{\sqrt{2}} \tan^{-1} \frac{\tan x}{\sqrt{2}} - x + C$$

Question 12

a) Let $z_1 = r_1(\cos \theta + i \sin \theta)$ and $z_2 = r_2(\cos \phi + i \sin \phi)$

$$\frac{z_1}{z_2} = \frac{r_1(\cos \theta + i \sin \theta)}{r_2(\cos \phi + i \sin \phi)}$$

$$\frac{z_1}{z_2} = \frac{r_1(\cos \theta + i \sin \theta)}{r_2(\cos \phi + i \sin \phi)} \times \frac{(\cos \phi - i \sin \phi)}{\cos \phi - i \sin \phi}$$

$$= \frac{r_1(\cos \theta \cos \phi - i \sin \phi \cos \theta + i \sin \theta \cos \phi - i^2 \sin \theta \sin \phi)}{\cos^2 \phi - i^2 \sin^2 \phi}$$

$$= \frac{r_1[(\cos \theta \cos \phi + \sin \theta \sin \phi) - i(\sin \phi \cos \theta - \sin \theta \cos \phi)]}{r_2 \cdot (\cos^2 \phi + \sin^2 \phi)}$$

$$= \frac{r_1[\cos(\theta - \phi) - i \sin(\theta - \phi)]}{r_2}$$

$$\frac{z_1}{z_2} = \frac{r_1}{r_2} [\cos(\theta - \phi) + i \sin(\theta - \phi)]$$

$$\arg\left(\frac{z_1}{z_2}\right) = \theta - \phi$$

$$= \arg z_1 - \arg z_2.$$

b) Let α, β be the roots

$$\alpha = 1 - 2i$$

$$\alpha + \beta = -\frac{b}{a}$$

$$1 - 2i + \beta = -1 - i$$

$$\beta = \underline{\underline{-2 + i}}$$

$$\alpha\beta = \frac{c}{a}$$

$$= k.$$

$$(-2 + i)(1 - 2i) = k$$

$$k = -2 + 4i + i - 2i^2$$

$$= \underline{\underline{5i}}$$

$$c) \quad x^3 + 3px + q = 0$$

$$(i) \quad \alpha + \beta + \gamma = 0$$

$$\alpha\beta + \beta\gamma + \alpha\gamma = 3p$$

$$\alpha\beta\gamma = -q$$

$$(ii) \quad \frac{\alpha\beta}{\gamma} + \frac{\beta\gamma}{\alpha} + \frac{\gamma\alpha}{\beta} = \frac{(\alpha\beta)^2 + (\beta\gamma)^2 + (\gamma\alpha)^2}{\alpha\beta\gamma}$$

$$= \frac{(\alpha\beta + \beta\gamma + \gamma\alpha)^2 - 2(\alpha\beta \cdot \beta\gamma + \alpha\beta \cdot \gamma\alpha + \beta\gamma \cdot \gamma\alpha)}{\alpha\beta\gamma}$$

$$= \frac{(3p)^2 - 2[\alpha\gamma\beta^2 + \alpha^2\beta\gamma + \gamma^2\beta\alpha]}{\alpha\beta\gamma}$$

$$= \frac{(3p)^2 - 2[\alpha\beta\gamma(\alpha + \beta + \gamma)]}{\alpha\beta\gamma}$$

$$= \frac{9p^2 - 0}{-q}$$

$$= \frac{9p^2}{q}$$

$$(iii) \quad \frac{\alpha\beta^3\gamma + \alpha^3\beta\gamma + \alpha\beta\gamma^3}{\alpha\beta\gamma}$$

$$= \frac{\alpha\beta\gamma(\alpha^2 + \beta^2 + \gamma^2)}{\alpha\beta\gamma}$$

$$= \alpha^2 + \beta^2 + \gamma^2$$

$$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$$

$$= 0 - 2(3p)$$

$$= -6p$$

(iv) (i) \Rightarrow sum of roots $-\frac{3p^2}{q}$

(ii) \Rightarrow sum of roots 2 at a time $-6p$

$$\therefore x^3 - \frac{9p^2}{q}x^2 - 6px + q = 0$$

product of roots

$$\frac{\alpha\beta}{\gamma} \times \frac{\beta\gamma}{\alpha} \times \frac{\gamma\alpha}{\beta} = \alpha\beta\gamma$$

$$= -q$$

$$(d) \int_0^1 \frac{dx}{9-x^2} = \frac{1}{6} \ln 2.$$

$$\int_0^1 \frac{1}{(3-x)(3+x)} dx.$$

$$\int_0^1 \frac{1}{(3-x)(3+x)} = \int \left[\frac{a}{(3-x)} + \frac{b}{(3+x)} \right] dx$$

$$1 \equiv a(3+x) + b(3-x)$$

$$\text{Let } x=3 \quad 1=6a$$

$$\text{Let } x=-3 \quad 1=6b$$

$$a = \frac{1}{6} \quad b = \frac{1}{6}$$

$$\int_0^1 \frac{1}{(3-x)(3+x)} = \frac{1}{6} \int_0^1 \frac{1}{3-x} + \frac{1}{6} \int_0^1 \frac{1}{3+x} dx.$$

$$= \left[-\frac{1}{6} \ln(3-x) \right]_0^1 + \left[\frac{1}{6} \ln(3+x) \right]_0^1$$

$$= \left[-\frac{1}{6} \ln 2 + \frac{1}{6} \ln 3 \right] + \left[\frac{1}{6} \ln 4 - \frac{1}{6} \ln 3 \right]$$

$$= -\frac{1}{6} \ln 2$$

$$= \frac{1}{6} [\ln 3 - \ln 2 + \ln 4 - \ln 3]$$

$$= \frac{1}{6} [\ln 4 - \ln 2]$$

$$= \frac{1}{6} \ln \frac{4}{2}$$

$$= \frac{1}{6} \ln 2.$$

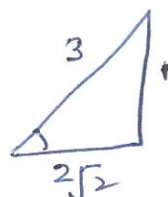
Alternative approach: Let $x = 3 \sin \theta$
 $dx = 3 \cos \theta d\theta$

$$\int_0^1 \frac{dx}{9-x^2} = \int_{\sin^{-1}(\frac{1}{3})}^{\sin^{-1}(\frac{1}{3})} \frac{1}{3} \sec \theta d\theta$$

$$= \frac{1}{3} \int \frac{\sec \theta (\tan \theta + \sec \theta)}{\tan \theta + \sec \theta} d\theta$$

$$= \frac{1}{3} \left[\ln(\tan \theta + \sec \theta) \right]_{\sin^{-1}(\frac{1}{3})}^{\sin^{-1}(\frac{1}{3})}$$

$$= \frac{1}{3} \left[\ln \left(\frac{1}{2\sqrt{2}} + \frac{3}{2\sqrt{3}} \right) \right] = \frac{1}{3} \left[\ln \frac{4}{2\sqrt{2}} \right] = \frac{1}{3} \ln \sqrt{2}$$



Question 13

a (i) P: $3 - x^2 = x^2 - x$

$$2x^2 - x - 3 = 0$$

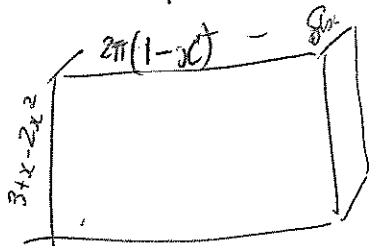
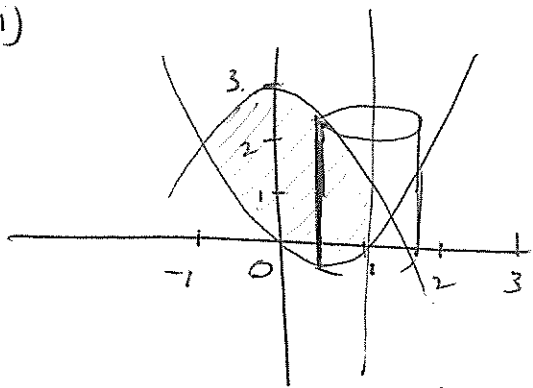
$$(x+1)(2x-3) = 0$$

$$\therefore x = -1, \frac{3}{2}$$

\therefore x coord of P is -1

(as P is in 2nd quadrant)

(ii)



$$h = (3 - x^2) - (x^2 - x)$$

$$= 3 + x - 2x^2$$

Volume $\int V = \int Ah$.

$$\int V = 2\pi(1-x)(3+x-2x^2) \delta x$$

$$V = \lim_{\delta x \rightarrow 0} \sum_{x=-1}^1 2\pi(3-2x-3x^2+2x^3) \delta x$$

$$= 2\pi \int_{-1}^1 3-2x-3x^2+2x^3 dx$$

$$b(i) \quad U_n = \frac{1}{n!} \int_0^1 x^n e^{-x} dx \quad \begin{array}{l} u = x^n \\ \frac{du}{dx} = n x^{n-1} \end{array} \quad \begin{array}{l} \frac{dv}{dx} = e^{-x} \\ v = -e^{-x} \end{array}$$

$$U_n = \frac{1}{n!} \left\{ \left[e^{-x} x^n \right]_0^1 - \int_0^1 -e^{-x} n x^{n-1} dx \right\}$$

$$= \frac{1}{n!} \left\{ -e^{-1} + n \int_0^1 e^{-x} x^{n-1} dx \right\}$$

$$= -\frac{1}{en!} + \frac{1}{(n-1)!} \int_0^1 e^{-x} x^{n-1} dx$$

$$U_n = -\frac{1}{en!} + U_{n-1}$$

$$\therefore -\frac{1}{en!} = U_{n-1} - U_n$$

$$\frac{1}{n!} = e(U_{n-1} - U_n)$$

$$(ii) \quad U_n = U_{n-1} - \frac{1}{en!}$$

$$U_4 = U_3 - \frac{1}{4!e}$$

$$= U_2 - \frac{1}{3!e} - \frac{1}{4!e}$$

⋮

$$= U_0 - \frac{1}{1!e} - \frac{1}{2!e} - \frac{1}{3!e} - \frac{1}{4!e}$$

$$= \frac{1}{0!} \int_0^1 x^0 e^{-x} dx - \frac{1}{e} - \frac{1}{2e} - \frac{1}{6e} - \frac{1}{24e}$$

$$= \int_0^1 e^{-x} dx - \frac{41}{24e}$$

$$= -e^{-1} - (-e^0) - \frac{41}{24e}$$

$$= -\frac{1}{e} + 1 - \frac{41}{24e}$$

$$= -\frac{65}{24e} + 1$$

$$c) (i) (a-b)^2 \geq 0$$

$$a^2 - 2ab + b^2 \geq 0$$

$$a^2 + b^2 \geq 2ab$$

$$(ii) a\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + b\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) + c\left(\frac{1}{a} + \frac{1}{b} + \frac{1}{c}\right) \geq 9.$$

$$\text{L.H.S. } 1 + \frac{a}{b} + \frac{a}{c} + 1 + \frac{b}{a} + \frac{b}{c} + 1 + \frac{c}{a} + \frac{c}{b}$$

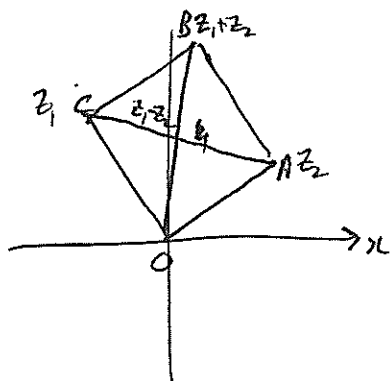
$$3 + \frac{a}{b} + \frac{b}{a} + \frac{a}{c} + \frac{c}{a} + \frac{b}{c} + \frac{c}{b}$$

$$3 + \frac{a^2+b^2}{ab} + \frac{a^2+c^2}{ac} + \frac{b^2+c^2}{cb}$$

$$\geq 3 + \frac{2ab}{ab} + \frac{2ac}{ac} + \frac{2bc}{cb}$$

$$\geq 9.$$

d)



$$\text{since } |z_1 + z_2| = |z_1 - z_2|$$

$$|AC| = |OB|$$

OACB is a parallelogram with equal diagonals

\therefore it is a rectangle

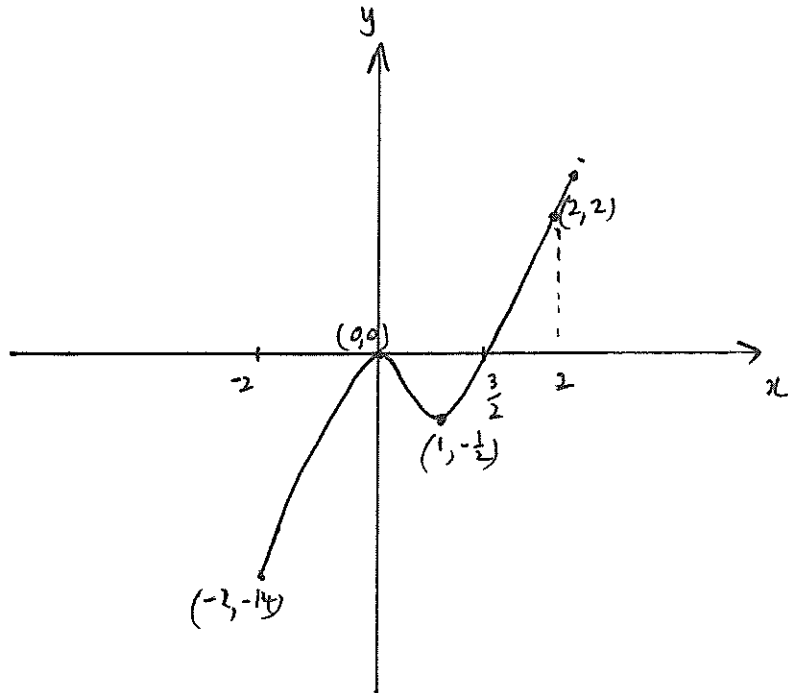
$\therefore \angle AOC$ is a right angle

$$\angle xOC - \angle xOA = 90^\circ$$

$$\text{or } \arg z_1 - \arg z_2 = \frac{\pi}{2}.$$

Question 14

(i)



$$y = x^3 - \frac{3x^2}{2}$$

$$y' = 3x^2 - 3x$$

$$\text{Let } y' = 0$$

$$0 = 3x^2 - 3x$$

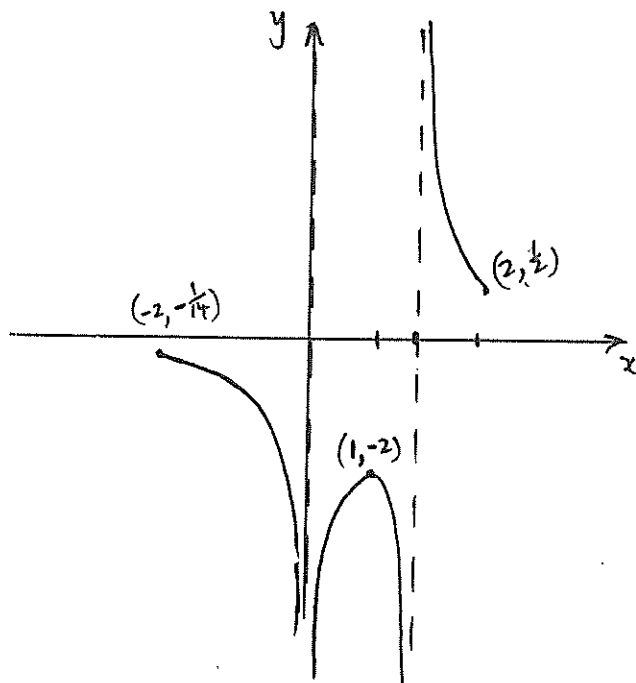
$$0 = 3x(x-1)$$

$$x = 0 \text{ or } x = 1$$

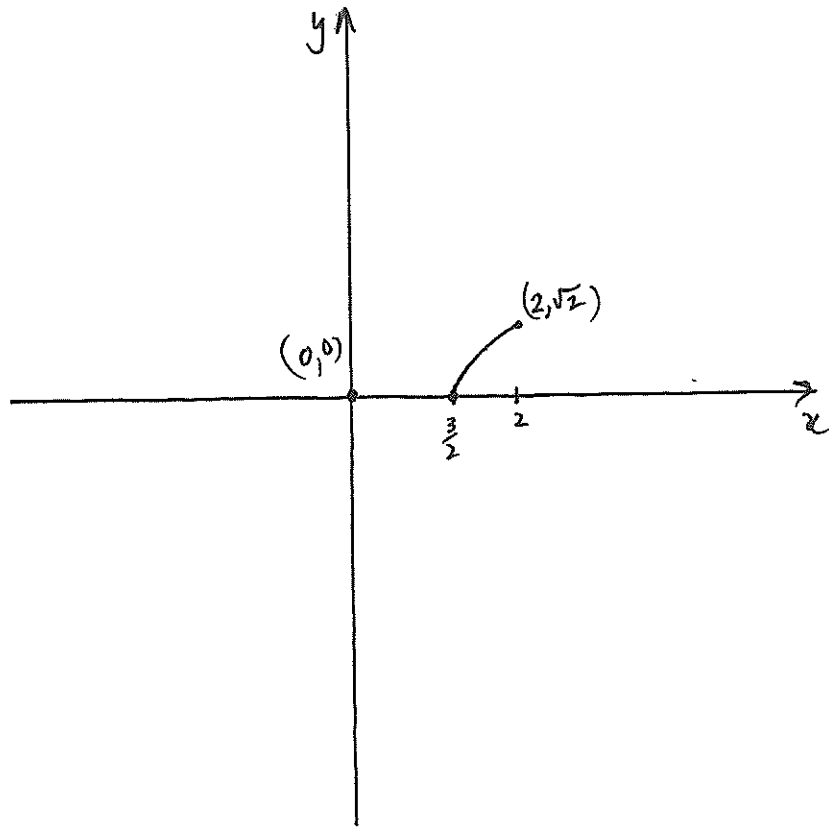
$$y = 0 \quad y = 1\left(-\frac{1}{2}\right) = -\frac{1}{2}$$

$$(0,0) \quad (1, -\frac{1}{2})$$

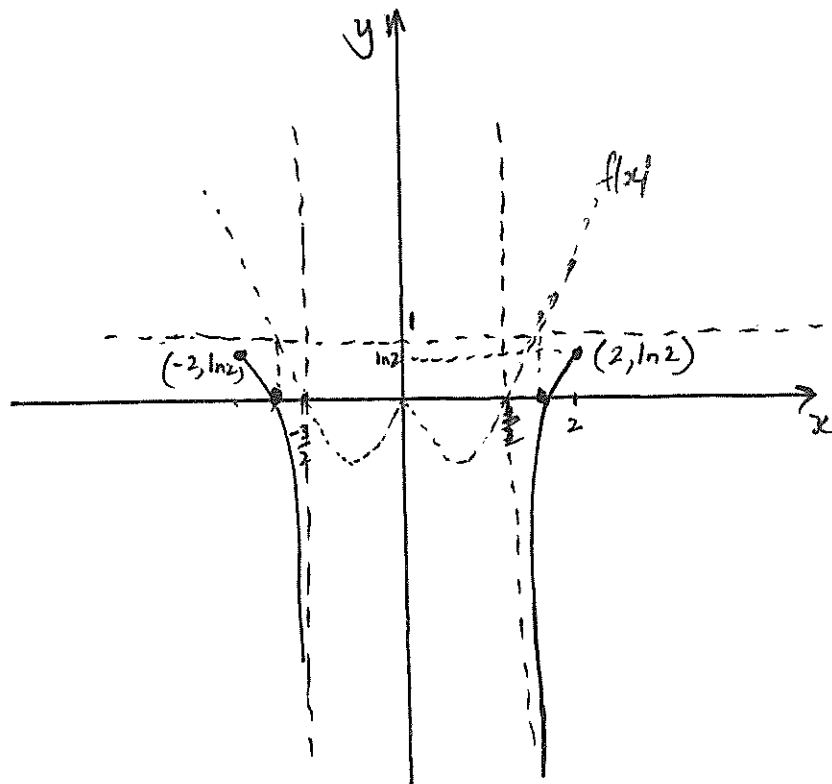
(ii)



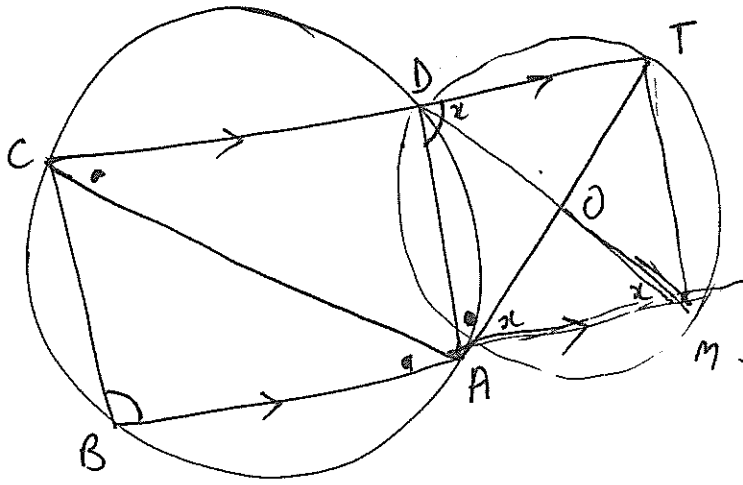
(ii)



(iv)



b(i)



(i) In $\triangle ADT$ and $\triangle ABC$

$$\angle TAD = \angle DCA \quad (\text{angle in the alternate segment are equal})$$

$$\angle DCA = \angle CAB \quad (\text{alternate angles on parallel lines } DC \text{ and } AB \text{ are equal})$$

$$\therefore \angle TAD = \angle CAB \quad (\text{angle})$$

$$\angle TDA = \angle CBA \quad (\text{exterior angle of a cyclic quadrilateral is equal to the interior opposite angle})$$

$\therefore \triangle ADT \cong \triangle ABC$ (Two angles in one triangle are equal to two angles in the other.)

(ii)

$$\text{Let } \angle TDM = x$$

$$\angle TAM = x \quad (\text{angle in the same segment are equal})$$

$$\angle OMA = x \quad (\text{alternate angles on parallel lines } TD \text{ and } AM \text{ are equal})$$

$$\therefore \angle OAM = \angle OMA$$

$\therefore \triangle OMA$ is isosceles.

(iv)

$$\angle TMA = \angle TDA \quad (\text{exterior opposite angles of a cyclic quadrilateral are equal})$$

$$\angle TDA = 180^\circ - \angle CBA$$

$$\therefore \angle CBA = 180^\circ - \angle TMA$$

$\therefore \angle CBA = 180^\circ - \angle TMA$ $\therefore CB \parallel TM$ since co-interior angles are supplementary

$\therefore TMBC$ is a parallelogram $\therefore TM = CB$ (opposite sides of parallelogram)

Question 15

$$\begin{aligned}
 (a) \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx &= \int_0^{\pi} \frac{(\pi - x) \sin(\pi - x)}{1 + \cos^2(\pi - x)} dx \\
 &= \int_0^{\pi} \frac{(\pi - x) \sin x}{1 + \cos^2 x} dx \quad \text{since } \sin(\pi - x) = \sin x. \\
 &\quad \text{and } \cos^2(\pi - x) = (-\cos x)^2 = \cos^2 x \\
 &= \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} dx - \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx \\
 \therefore 2 \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx &= \int_0^{\pi} \frac{\pi \sin x}{1 + \cos^2 x} dx \\
 \therefore \int_0^{\pi} \frac{x \sin x}{1 + \cos^2 x} dx &= \frac{\pi}{2} \int_0^{\pi} \frac{\sin x}{1 + \cos^2 x} dx \\
 &= -\frac{\pi}{2} \left[\tan^{-1}(\cos x) \right]_0^{\pi} \quad \text{since } \frac{d}{dx}(\cos x) = -\sin x. \\
 &= -\frac{\pi}{2} \left[\tan^{-1}(-1) - \tan^{-1}(1) \right] \\
 &= \frac{\pi^2}{4}.
 \end{aligned}$$

$$(b) (i) \quad x = 3p \quad \frac{dx}{dp} = 3 \quad \frac{dy}{dp} = -\frac{3}{p^2}$$

$$\begin{aligned}
 \frac{dy}{dx} &= \frac{dy}{dp} \times \frac{dp}{dx} \\
 &= -\frac{3}{p^2} \times \frac{1}{3} \\
 &= -\frac{1}{p^2}
 \end{aligned}$$

$$\begin{aligned}
 y - \frac{3}{p} &= -\frac{1}{p^2} (x - 3p) \\
 p^2 y - 3p &= -x + 3p \\
 x + p^2 y &= 6p
 \end{aligned}$$

$$(i) \quad x + p^2 y = 6p \quad \text{--- (1)}$$

$$x + q^2 y = 6q \quad \text{--- (2)}$$

$$\textcircled{1} - \textcircled{2} \quad (p^2 - q^2)y = 6(p - q)$$

$$\frac{(p-q)(p+q)y}{(p-q)} = 6$$

$$(p+q)y = 6$$

$$y = \frac{6}{p+q}$$

sub into (1)

$$x + \frac{6p^2}{p+q} = 6p$$

$$x = 6p - \frac{6p^2}{p+q}$$

$$(p+q)x = 6p(p+q) - 6p^2$$

$$(p+q)x = 6p^2 + 6pq - 6p^2$$

$$x = \frac{6pq}{p+q}$$

$$\therefore T\left(\frac{6pq}{p+q}, \frac{6}{p+q}\right)$$

$$(iii) \quad x + pqy = 3(p+q)$$

sub (0, b)

$$6pq = 3(p+q)$$

$$\frac{pq}{p+q} = \frac{1}{2}$$

$$x = \frac{6pq}{p+q} \quad y = \frac{6}{p+q}$$

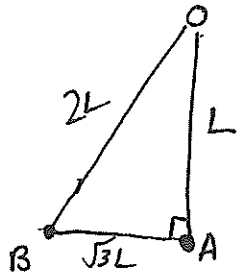
$$x = 3 \quad y = \frac{6}{p+q}$$

$$T\left(3, \frac{6}{p+q}\right)$$

$$\therefore x = 3$$

vertical line passing through $x=3$.

C,



$$T = mg$$



vertically $T \sin 30 = Mg$

$$\frac{1}{2}T = Mg$$

$$\text{or } \frac{1}{2}mg = Mg$$

$$m = 2M$$

horizontally

$$T \cos 30 = Mr\omega^2$$

$$\frac{\sqrt{3}}{2}mg = M(\sqrt{3}L)\omega^2$$

$$\frac{\sqrt{3}}{2}mg = \left(\frac{m}{2}\sqrt{3}L\right)\omega^2$$

$$\omega = \sqrt{\frac{g}{L}}$$

Question 1b

(a)
Test for $n=1$

$a-b$ is a factor of a^1-b^1

Assume true for $n=k$

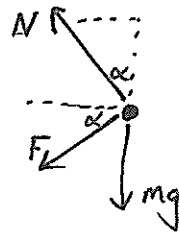
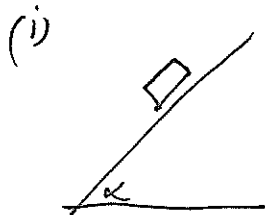
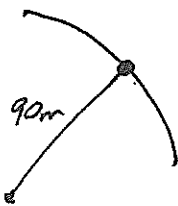
$$a^k - b^k = (a-b)F \quad \text{where } F \text{ is an integer.}$$

Test for $n=k+1$

$$\begin{aligned} a^{k+1} - b^{k+1} &= a^{k+1} - a^k b + a^k b - b^{k+1} \\ &= a^k(a-b) + b(a^k - b^k) \\ &= a^k(a-b) + b(a-b)F \quad (\text{from assumption}) \\ &= (a-b)(a^k + bF) \end{aligned}$$

since $[a^k + bF]$ is another polynomial in a and b ,
we have shown what we set out to prove.

b)



N - Normal reaction
 F - frictional force
 mg - weight

(i) Vertically $N \cos \alpha = F \sin \alpha + mg$
 $N \cos \alpha - F \sin \alpha = mg$ — (1)

Horizontally $N \sin \alpha + F \cos \alpha = \frac{mv^2}{r}$ — (2)

From (1) $N \cos \alpha \sin \alpha - F \sin^2 \alpha = mg \sin \alpha$ (3)

and (2) $N \sin \alpha \cos \alpha + F \cos^2 \alpha = \frac{mv^2}{r} \cos \alpha$ (4)

(4) - (3)

$$F(\cos^2 \alpha + \sin^2 \alpha) = \frac{mv^2}{r} \cos \alpha - mg \sin \alpha$$

$$F = \frac{mv^2}{r} \cos \alpha - mg \sin \alpha \quad *$$

no sideways slip when $F=0$

$$\therefore mg \sin \alpha = \frac{mv^2}{r} \cos \alpha$$

$$\tan \alpha = \frac{v^2}{rg}$$

$$v = \frac{54 \times 1000}{60 \times 60} \text{ ms}^{-1} \quad r = 90.$$

$$\tan \alpha = \frac{15 \times 15}{90 \times 10}$$

$$= \frac{225}{900}$$

$$= \frac{1}{4}$$

$$\alpha = \tan^{-1}\left(\frac{1}{4}\right)$$

(ii) now $F = m \cos \alpha \left(\frac{v^2}{r} - g \tan \alpha \right) *$

$$F = 1200 \cdot \frac{4}{\sqrt{17}} \left(\left(\frac{72 \times 1000}{60 \times 60} \right)^2 \cdot \frac{1}{90} - \frac{10}{4} \right) \quad \cos \alpha = \frac{4}{\sqrt{17}}$$

$$= \frac{4800}{\sqrt{17}} \left(\frac{400}{90} - \frac{10}{4} \right)$$

$$= \frac{4800}{\sqrt{17}} \left(\frac{800 - 450}{180} \right)$$

$$= \frac{4800}{\sqrt{17}} \times \frac{350}{180}$$

$$\doteq 2263.665 \dots$$

$$= 2263.7 \text{ N}$$

(c).

$$(i) \tan(2\alpha + \alpha) = \frac{\tan 2\alpha + \tan \alpha}{1 - \tan 2\alpha \tan \alpha}$$

$$= \left(\frac{2\tan \alpha}{1 - \tan^2 \alpha} + \tan \alpha \right) \div \left(1 - \frac{2\tan^2 \alpha}{1 - \tan^2 \alpha} \right)$$

$$= \left(\frac{2\tan \alpha + \tan \alpha - \tan^3 \alpha}{1 - \tan^2 \alpha} \right) \div \left(\frac{1 - \tan^2 \alpha - 2\tan^2 \alpha}{1 - \tan^2 \alpha} \right)$$

$$= \frac{3\tan \alpha - \tan^3 \alpha}{1 - 3\tan^2 \alpha}$$

(ii) Let $\alpha = \tan^{-1} x$

$$\therefore 3\alpha = 3\tan^{-1} x$$

$$= \frac{\pi}{2} - \tan^{-1} 3x$$

$$\tan^{-1} 3x = \frac{\pi}{2} - 3\alpha$$

$$\tan(\tan^{-1} 3x) = \tan\left(\frac{\pi}{2} - 3\alpha\right)$$

$$3x = \cot 3\alpha$$

$$3x = \frac{1 - 3\tan^2 \alpha}{3\tan \alpha - \tan^3 \alpha} \quad (\text{from (i)})$$

$$3x = \frac{1 - 3x^2}{3x - x^3} \quad (\alpha = \tan^{-1} x \Rightarrow x = \tan \alpha)$$

$$3x(3x - x^3) = 1 - 3x^2$$

$$9x^2 - 3x^4 = 1 - 3x^2$$

$$3x^4 - 12x^2 + 1 = 0$$

$$\therefore x^2 = \frac{6 \pm \sqrt{33}}{3}$$

$$\therefore x = \sqrt{\frac{6 \pm \sqrt{33}}{3}}$$