Sydney Girls High School
2014

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I

10 Marks

- Attempt Questions 1 – 10
- Answer on the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II

90 Marks

- Attempt Questions 11 – 16
- Answer on the blank paper provided
- Begin a new page for each question
- Allow about 2 hours and 45 minutes for this section

Name: ………………………………………………………	**THIS IS A TRIAL PAPER ONLY**
Teacher: …………………………………………………	It does not necessarily reflect the format or the content of the 2014 HSC Examination Paper in this subject.

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1 – 10

1. The substitution $t = \tan\dfrac{\theta}{2}$ is used to find $\int \sec\theta \, d\theta$. Which of the following gives the correct expression for the required integral?

   (A) $\displaystyle\int \frac{dt}{2\left(1-t^2\right)}$

   (B) $\displaystyle\int \frac{2t\,dt}{1-t^2}$

   (C) $\displaystyle\int \frac{2dt}{1-t^2}$

   (D) $\displaystyle\int \frac{4t\,dt}{\left(1+t^2\right)^2}$

2. The distance between the foci for the ellipse $\dfrac{x^2}{100}+\dfrac{y^2}{36}=1$ is :

   (A) 8

   (B) 16

   (C) 20

   (D) 25

3. Which of the following is an expression for $\displaystyle\int \frac{dx}{\sqrt{7-6x-x^2}}$?

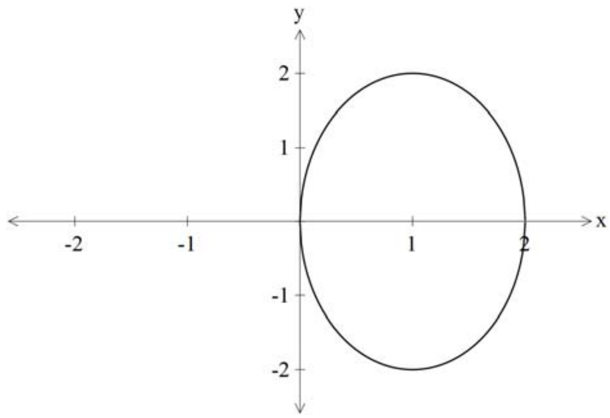
   (A) $\sin^{-1}\left(\dfrac{x+3}{2}\right)+C$

   (B) $\sin^{-1}\left(\dfrac{x+3}{4}\right)+C$

   (C) $\sin^{-1}\left(\dfrac{x-3}{4}\right)+C$

   (D) $\sin^{-1}\left(\dfrac{x-3}{2}\right)+C$

4. The region enclosed by the ellipse $(x-1)^2+\frac{y^2}{4}=1$ is rotated about the $y-$axis to form a solid.



Which integral represents the volume of the solid when applying the method of slicing?

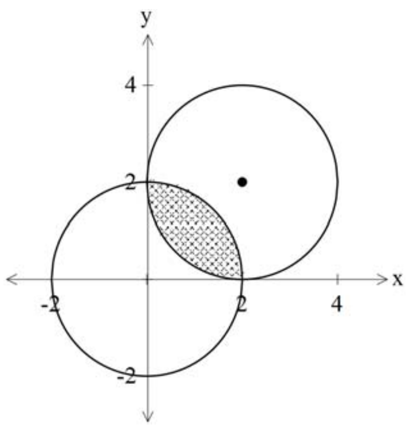
(A) $V=\int_{-2}^{2}\pi\sqrt{4-y^2}\,dy$

(B) $V=\int_{-2}^{2}2\pi\sqrt{1-y^2}\,dy$

(C) $V=\int_{-2}^{2}\pi\sqrt{1-y^2}\,dy$

(D) $V=\int_{-2}^{2}2\pi\sqrt{4-y^2}\,dy$

5. The shaded area in the Argand diagram below can be represented by the inequalities :



(A) $|z|\le 4$ and $|z-2+2i|\le 4$

(B) $|z|\le 2$ and $|z-2-2i|\le 2$

(C) $|z|\le 4$ and $|z-2-2i|\le 4$

(D) $|z|\le 2$ and $|z-2+2i|\le 2$

6. What is the solution to the inequation $\frac{x(5-x)}{x-4} \geq -3$?

(A) $2 \leq x < 4$ or $x \geq 6$

(B) $4 < x \leq 6$ or $x \leq 2$

(C) $4 > x \leq 5$ or $x \leq 1$

(D) $1 \leq x < 4$ or $x \geq 5$

7. The gradient of the tangent at $(2,1)$ on the curve $2x^2 + 3xy + y^2 = 15$ is :

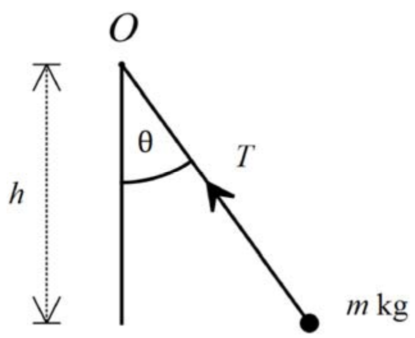
(A) -1

(B) $-\frac{8}{5}$

(C) $-\frac{11}{2}$

(D) $-\frac{11}{8}$

8. A particle of mass m moves in a horizontal circle with angular speed ω at a distance h below the point O. Which of the following reflects the relationship between ω and h?



(A) $h = \omega^2 g$

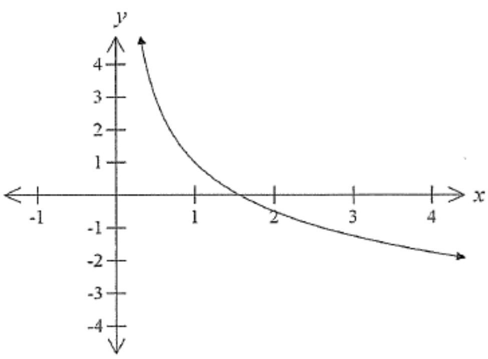
(B) $h = \omega g$

(C) $g = \omega^2 h$

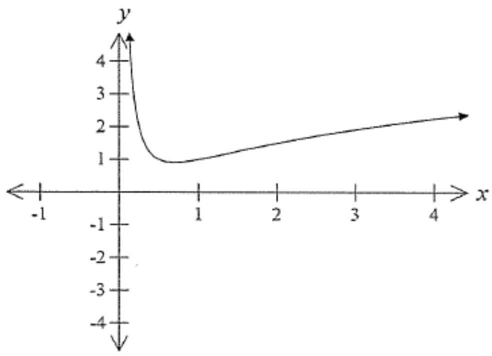
(D) $g = \omega h$

9. Which of the following is the graph of $y = \log_2 x + \frac{1}{x}$?

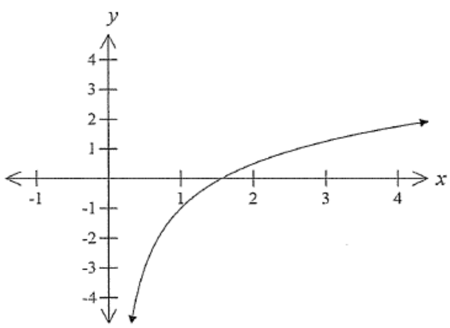
(A)



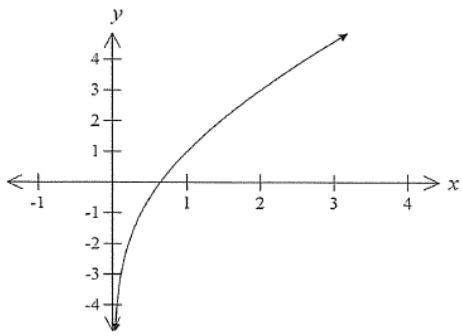
(B)



(C)



(D)



10. The polynomial $P(x) = x^4 + cx^2 + dx + 28$ has a double root at $x = 2$. What are the values of c and d ?

(A) $c = -5$ and $d = -12$

(B) $c = -5$ and $d = 12$

(C) $c = -11$ and $d = 12$

(D) $c = -11$ and $d = -12$

BLANK PAGE

Section II

90 marks

Attempt Questions 11 – 16
Allow about 2 hours and 45 minutes for this section

Answer on the blank paper provided. Begin a new page for each question.
Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks)

(a) (i) Find numbers A, B and C such that **2**

$$\frac{13}{(x+2)(x^2+9)} = \frac{A}{x+2} + \frac{Bx+C}{x^2+9}.$$

(ii) Hence find $\int \frac{13dx}{(x+2)(x^2+9)}$. **2**

(b) (i) Express $z = \frac{1+2i}{1-3i}$ in modulus-argument form. **2**

(ii) Hence, find the value of z^7, expressing your answer in the form $a+ib$, where a and b are real. **2**

(c) (i) Expand and simplify $(4+3i)^2$. **1**

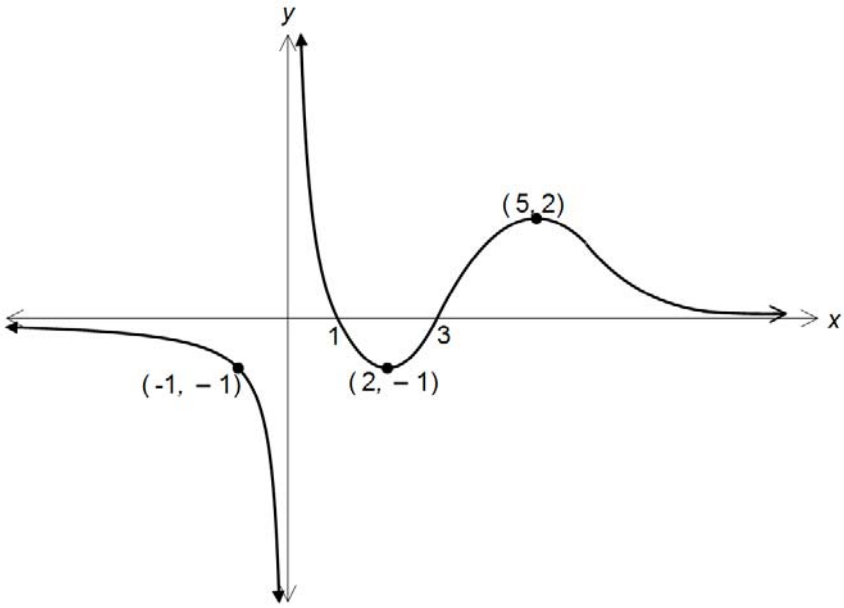
(ii) Hence, solve the equation $2z^2 - iz - 1 - 3i = 0$, expressing the solutions in the form $x+iy$. **2**

(d) $(1+3i)$ is a root of the polynomial $P(x) = x^4 + 2x^3 + 11x^2 + 22x + 90$. **4**

Find all the other roots.

Question 12 (15 marks)

(a) The diagram shows the graph of $y = f(x)$.



Sketch the following curves on separate half-page diagrams :

(i) $y = f(-x)$ **1**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = f(x) - |f(x)|$ **2**

(iv) $y = e^{f(x)}$ **2**

(b) Find $\displaystyle\int \frac{2x+1}{\sqrt{x^2-2x+10}}\,dx$. **3**

(c) $P\left(2p, -\dfrac{2}{p}\right)$ and $Q\left(2q, -\dfrac{2}{q}\right)$ are points on the rectangular hyperbola $xy = -4$.

(i) Show that the equation of the chord PQ is $x - pqy = 2p + 2q$. **2**

(ii) Find the locus of the mid-point of the chord given that the chord passes through the point $(0, -6)$. **3**

Question 13 (15 marks)

(a) (i) If $I_n = \int_0^1 \left(1-x^2\right)^{\frac{n}{2}} \, dx$, where n is a positive integer, show that **4**

$$I_n = \frac{n}{n+1} I_{n-2} .$$

(ii) Hence, evaluate I_5 . **2**

(b) (i) By considering the expansion of $\left(\cos\theta + i\sin\theta\right)^5$, show that **3**

$$\cos 5\theta = 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta .$$

(ii) Hence, solve the equation $16x^4 - 20x^2 + 5 = 0$. Express the solutions in the form $x = \cos\alpha$. **3**

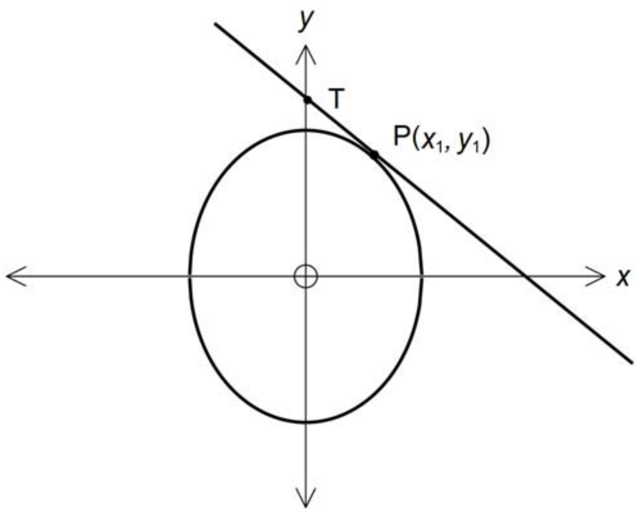
(c) A string of length 40 cm can just sustain a weight of mass 12 kg without breaking. One end of the string is fixed to a point P on a smooth horizontal table and the other end has a mass of 5 kg attached to it. The 5 kg mass revolves uniformly on the table about the point P . Determine the maximum number of complete revolutions the mass can make in a minute without breaking the string. (Use $g = 9.8$ m/s^2.) **3**

Question 14 (15 marks)

(a) The region bounded by $y = x^4 + 1$, the $x-$axis, the $y-$axis and the line $x = 1$ is rotated about the line $x = 3$. Use the method of cylindrical shells to find the volume of the solid generated. **4**

(b) z is a point in the first quadrant of the Argand diagram which lies on the circle $|z-3| = 3$. Given $\arg(z) = \theta$, find $\arg(z^2 - 9z + 18)$ in terms of θ. **2**

(c) The point $P(x_1, y_1)$ lies on the ellipse $\dfrac{x^2}{10} + \dfrac{y^2}{16} = 1$ where $x_1 \neq 0$ and $y_1 \neq 0$.



(i) Show that the equation of the tangent at P is $\dfrac{xx_1}{10} + \dfrac{yy_1}{16} = 1$. **2**

(ii) The tangent at P cuts the $y-$axis at T. Determine the coordinates of the point T. **1**

(iii) Find the coordinates of the foci, S and S'. **2**

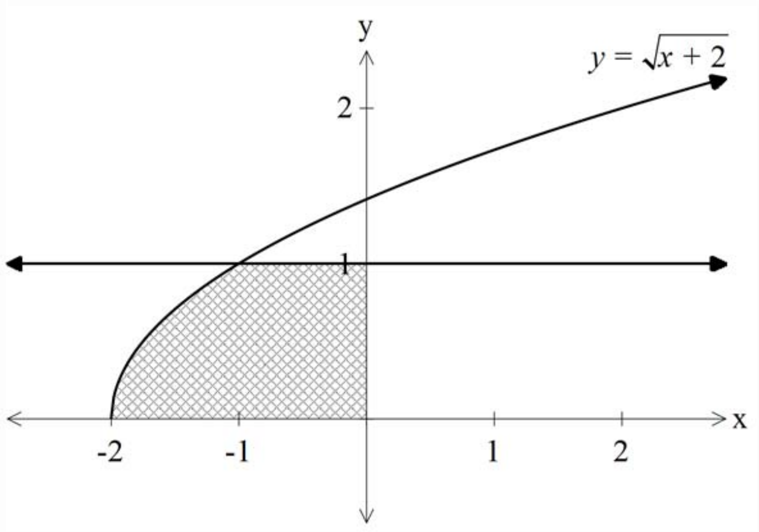
(iv) Find the equations of the directrices. **1**

(v) Show that $\dfrac{PS}{PS'} = \dfrac{TS}{TS'}$. **3**

Question 15 (15 marks)

(a) Evaluate $\int_1^9 e^{\sqrt{x}}\,dx$. **4**

(b) The base of a solid is the area (as shaded below) enclosed by the curve $y=\sqrt{x+2}$, the $x-$axis, the $y-$axis and the line $y=1$.



Each cross-section of the solid perpendicular to the $x-$ axis is a regular hexagon with one side in the base of the solid. Find the volume of the solid. **4**

(c) The polynomial $P(x)=2x^3-4x^2+7x+5$ has roots α, β and γ. Find the value of :

(i) $\alpha^3+\beta^3+\gamma^3$ **3**

(ii) $\alpha^2\beta+\alpha^2\gamma+\beta^2\alpha+\beta^2\gamma+\gamma^2\alpha+\gamma^2\beta$ **2**

(d) a and b are real non-zero numbers and $\omega=\dfrac{az+b}{bz+a}$. Given $|z|=1$, show that $|\omega|=1$. **2**

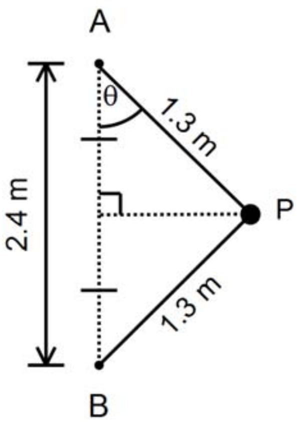
Question 16 (15 marks)

(a) The equation $px^3 + qx + r = 0$ has roots α, β and γ. Find the polynomial equation with roots $\dfrac{1}{-\alpha\beta-\alpha\gamma}$, $\dfrac{1}{-\alpha\beta-\beta\gamma}$ and $\dfrac{1}{-\alpha\gamma-\beta\gamma}$. **4**

Express your answer in simplest form.

(b) A particle of mass m kg, is attached at P by two strings, each of length 1.3 m, to two fixed points, A and B, which are 2.4 m apart and lie on a vertical line, as shown in the diagram below.

The particle moves with constant speed v ms^{-1} in a horizontal circle about the midpoint of AB so that both pieces of string experience tension. Let T_1 and T_2 represent the tension in AP and BP respectively. The acceleration due to gravity is g ms^{-2}.



(i) Draw a diagram showing all the forces acting on P. **1**

(ii) Resolve the forces on P in both the horizontal and vertical directions. **2**

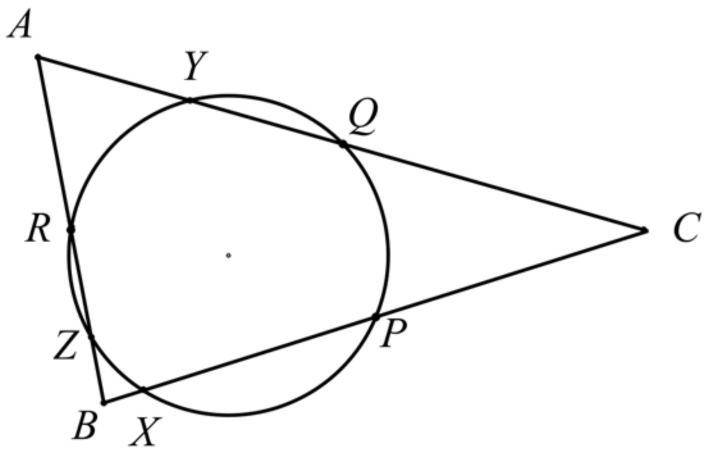
(iii) Find the tension in each part of the string in terms of m, v and g. **2**

(iv) Show that $v > \dfrac{\sqrt{30g}}{12}$. **2**

Question 16 continues on the next page

Question 16 (continued)

(c) P, Q and R are the midpoints of BC, AC and AB respectively. The circle drawn through P, Q and R intersects BC, AC and AB a second time at X, Y and Z respectively as shown below.



(i) Explain why $RPCQ$ is a parallelogram. **1**

(ii) Show ΔXQC is isosceles. **2**

(iii) Show $AX \perp BC$. **1**

End of paper

BLANK PAGE

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Sydney Girls High School

Mathematics Faculty

Multiple Choice Answer Sheet
2014 Trial HSC Mathematics Extension 2

Completely fill the response oval representing the most correct answer.

1.	A ○	B ○	C ●	D ○
2.	A ○	B ●	C ○	D ○
3.	A ○	B ●	C ○	D ○
4.	A ○	B ○	C ○	D ●
5.	A ○	B ●	C ○	D ○
6.	A ○	B ●	C ○	D ○
7.	A ○	B ○	C ○	D ●
8.	A ○	B ○	C ●	D ○
9.	A ○	B ●	C ○	D ○
10.	A ●	B ○	C ○	D ○

Question 11

Comments

(a) (i) $\dfrac{13}{(x+2)(x^2+9)} = \dfrac{A}{x+2} + \dfrac{Bx+C}{x^2+9}$

$13 = A(x^2+9) + (Bx+C)(x+2)$

let $x = -2$ $\quad 13 = 13A \quad A = 1$

coeff. of x^2 $\quad 0 = A + B \quad B = -A = -1$

constant $\quad 13 = 9A + 2C \quad C = \dfrac{13-9}{2} = 2$

(ii) $\displaystyle\int \frac{13\,dx}{(x+2)(x^2+9)} = \int\left(\frac{1}{x+2} + \frac{-x+2}{x^2+9}\right)dx$

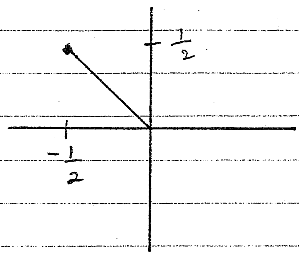
$= \ln(x+2) - \frac{1}{2}\ln(x^2+9) + \frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right) + C$

(a) (ii) Common mistake was a minus sign in front of $\frac{2}{3}\tan^{-1}\left(\frac{x}{3}\right)$

(b) (i) $z = \dfrac{1+2i}{1-3i} \times \dfrac{1+3i}{1+3i}$

$= \dfrac{1+2i+3i-6}{1+9}$

$= -\dfrac{1}{2} + \dfrac{i}{2}$



$|z| = \sqrt{\frac{1}{2}^2 + \frac{1}{2}^2} = \dfrac{1}{\sqrt{2}}$ $\qquad \arg(z) = \pi - \dfrac{\pi}{4} = \dfrac{3\pi}{4}$

$\therefore\ z = \frac{1}{\sqrt{2}}\operatorname{cis}\frac{3\pi}{4}$

(ii) $z^7 = \left(\frac{1}{\sqrt{2}}\operatorname{cis}\frac{3\pi}{4}\right)^7$

$= \frac{1}{8\sqrt{2}}\operatorname{cis}\frac{21\pi}{4}$

$= \frac{1}{8\sqrt{2}}\operatorname{cis}\left(\frac{-3\pi}{4}\right)$

$= \frac{1}{8\sqrt{2}}\left(-\frac{1}{\sqrt{2}} - \frac{1}{\sqrt{2}}i\right)$

$= -\frac{1}{16} - \frac{1}{16}i$

(b) (i) A number of students calculate arg(z) incorrectly as $-\frac{\pi}{4}$. A diagram would have eliminated this mistake.

(c) (i) $(4+3i)^2 = 16 + 24i + 9i^2 = 7 + 24i$

(ii) $z = \dfrac{i \pm \sqrt{i^2 - 4\times 2(-1-3i)}}{2\times 2}$

$= \dfrac{i \pm \sqrt{-1+8+24i}}{4}$

$= \dfrac{i \pm \sqrt{7+24i}}{4}$

$= \dfrac{i \pm (4+3i)}{4}$ using (i)

$= \dfrac{i+4+3i}{4}$ or $\dfrac{i-4-3i}{4}$

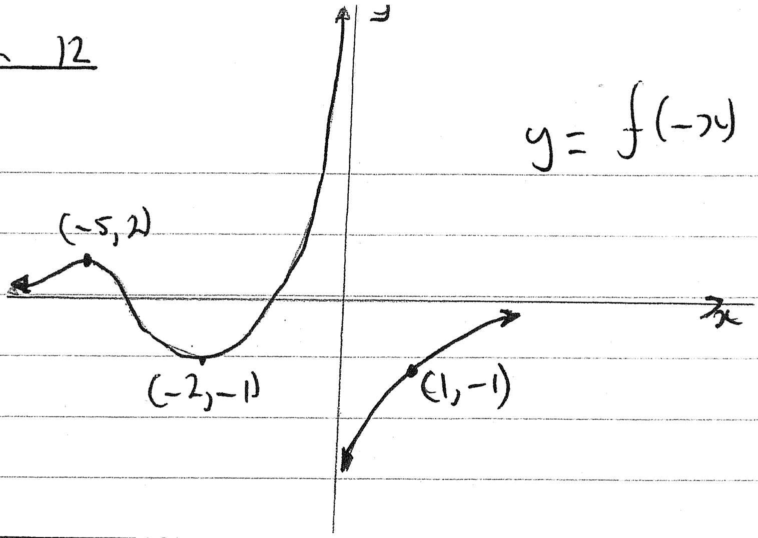
i.e. $z = 1 + i$ or $-1 - \frac{1}{2}i$

(c) (ii) Some students did not think to use the quadratic formula and some did not make the link to the result in (i).

Question 11 (continued)	Comments
(d) Since coeff. of $P(x)$ are real and $1+3i$ is a root, then $1-3i$ is also a root. Let roots be $\alpha, \beta, 1 \pm 3i$ Sum of roots: $\alpha+\beta+1+3i+1-3i=-2$ $\therefore \alpha+\beta=-4$ Product of roots: $\alpha\beta(1+3i)(1-3i)=90$ $\alpha\beta(1+9)=90$ $\therefore \alpha\beta=9$ $\therefore$ Eqn. with roots α and β is $z^2+4z+9=0$ $(z+2)^2=-5$ $z=-2\pm\sqrt{-5}$ $\therefore$ The roots of $P(x)$ are $1+3i, 1-3i, -2+\sqrt{5}i, -2-\sqrt{5}i$.	(d) Some solutions were too long-winded.

Question 12

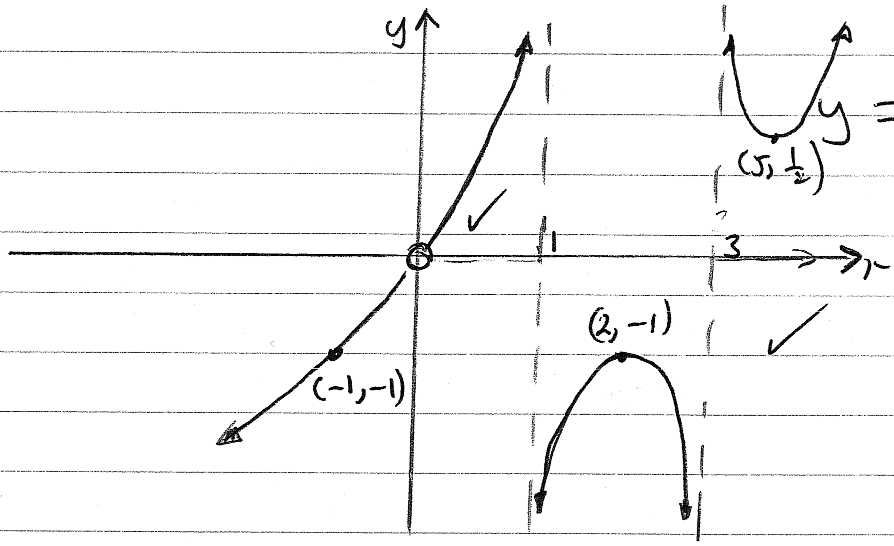
a) i)



① correct answer.

①

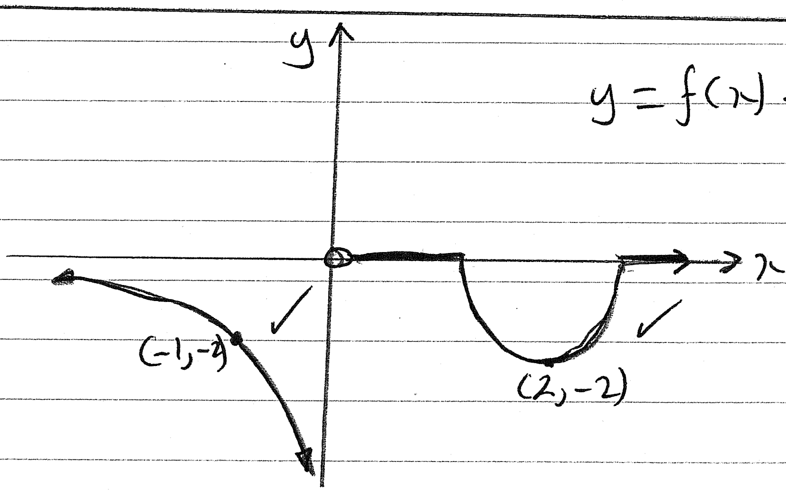
ii)



②

②

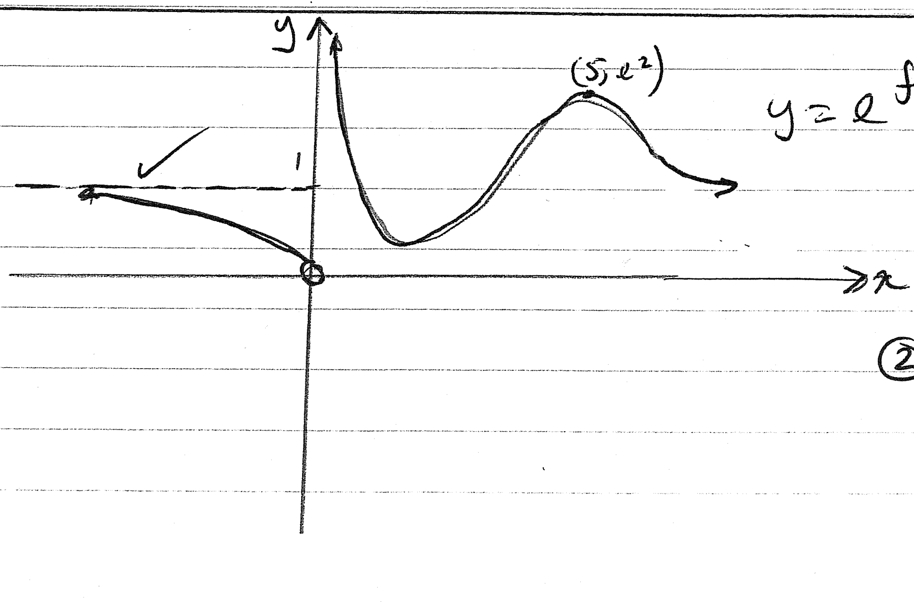
iii)



②

②

iv)



②

②

b) $x^2 - 2x + 10 = (x-1)^2 + 3^2$ (3)

$I = \int \frac{2x-2}{\sqrt{(x-1)^2+3^2}} + \int \frac{3\,dx}{\sqrt{(x-1)^2+3^2}}$ ✓

$= I_1 + I_2$

let $u = x^2 - 2x + 10$

$du = (2x-2)\,dx$

$I_1 = \int \frac{du}{u^{\frac{1}{2}}}$

$= 2u^{\frac{1}{2}}$

$= 2\sqrt{x^2-2x+1}$ ✓

$I_2 = 3\log_e[(x-1) + \sqrt{x^2-2x+10}] + C$ ✓

$\therefore \quad I = 2\sqrt{2x^2-2x+10} + 3\log_e[(x-1) + \sqrt{x^2-2x+10}] + C$

c) i) $m = \left(\frac{-2}{p} + \frac{2}{q}\right) \div (2p - 2q)$ ✓

$= \frac{1}{pq}$

(2)

<u>Eqn PQ</u>

$y + \frac{2}{p} = \frac{1}{pq}(x - 2p)$

$pqy + 2q = x - 2p$

$x - pqy = 2p + 2q$ ✓

ii) sub $x = 0, y = -6$ in above

$0 + 6pq = 2(p+q)$

$3pq = p + q$ ✓ (1)

<u>Co-ords of midpoint</u>

$x = \frac{2p+2q}{2}$, $y = \left(\frac{-2}{p} - \frac{2}{q}\right) \div 2$

$x = p + q$ $\quad y = -\frac{p+q}{pq}$ ✓ (3)

sub (1) in above

$y = \frac{-3pq}{pq}$

$y = -3$ ✓

13 (a) (i) Let $u = (1-x^2)^{\frac{n}{2}}$ $\quad v' = 1$

$u' = \frac{n}{2}(1-x^2)^{\frac{n}{2}-1} \times -2x \quad v = x$

$I_n = \left[+x(1-x^2)^{\frac{n}{2}}\right]_0^1 + n\int x^2(1-x^2)^{\frac{n-2}{2}}\,dx - n\int(1-x^2)^{\frac{n-2}{2}}\,dx + nI_{n-2}$

$= 0 + n\int(x^2-1)(1-x^2)^{\frac{n-2}{2}}\,dx + nI_{n-2}$

$= -nI_n + nI_{n-2}$

$(1+n)I_n = nI_{n-2}$

$I_n = \frac{n}{n+1}I_{n-2}$ Q.E.D.

(ii) $I_5 = \frac{5}{6}I_3$

$= \frac{5}{6}\times\frac{3}{4}I_1$

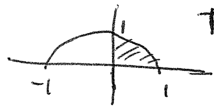
$= \frac{2}{3}\times\frac{\pi}{4}$

$= \frac{5\pi}{32}$

$I = \int_0^1 (1-x^2)^{\frac{1}{2}}dx$

$= \frac{\pi\times 1^2}{4}$

$= \frac{\pi}{4}$



This is easier than using $u = \sin x$.

(b) (i) $(\cos\theta + i\sin\theta)^5 = \cos^5\theta + 5i\cos^4\theta\sin\theta - 10\cos^3\theta\sin^2\theta - 10i\cos^2\theta\sin^3\theta + 5\cos\theta\sin^4\theta + i\sin^5\theta$

$= \cos 5\theta + i\sin 5\theta$

$\cos 5\theta = \cos^5\theta - 10\cos^3\theta(1-\cos^2\theta) + 5\cos\theta(1-\cos^2\theta)^2$

$= \cos^5\theta - 10\cos^3\theta + 10\cos^5\theta + 5\cos\theta - 10\cos^3\theta + 5\cos^5\theta$

$= 16\cos^5\theta - 20\cos^3\theta + 5\cos\theta$

(ii) Let $\cos 5\theta = 0$

$\cos\theta(16\cos^4\theta - 20\cos^2\theta + 5) = 0$

$5\theta = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \frac{7\pi}{2}, \frac{9\pi}{2}$

$\theta = \frac{\pi}{10}, \frac{3\pi}{10}, \frac{5\pi}{10}, \frac{7\pi}{10}, \frac{9\pi}{10}$

$\therefore x = \cos\frac{\pi}{10}, \cos\frac{3\pi}{10}, \cos\frac{7\pi}{10}, \cos\frac{9\pi}{10}$

$x = 0$ is not a solution $\therefore \cos\frac{\pi}{2}$ can't be either.

(c) $T \le 12\times 9.8$

≤ 117.6 N

$mrw^2 \le 117.6$

$5\times 0.4\times w^2 \le 117.6$

$w^2 \le 58.8$

$w \le \sqrt{58.8}$

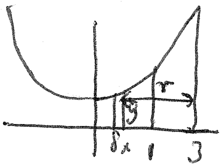
max rev $= \frac{\sqrt{58.8}}{2\pi}\times 60$

$= 73.2251207$

$\doteqdot 73$

The answer requires the no. of rev in 1 min not 1 sec.

14(a)



Students had more success using 1st principles rather than formulae.

$V_{shell} = \pi\left\{(3-x)^2 - (3-x-\delta x)^2\right\}$

$= \pi\left\{(6-2x)\delta x\right\} y$

$V_{solid} = 2\pi \int_0^1 (3-x) y \, dx$

$= 2\pi \int_0^1 (3-x)(x^4+1)\, dx$

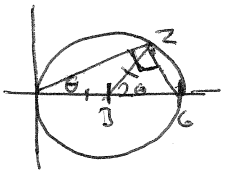
$= 2\pi \int_0^1 (3x^4 + 3 - x^5 - x)\, dx$

$= 2\pi \left[\frac{3x^5}{5} + 3x - \frac{x^6}{6} - \frac{x^2}{2}\right]_0^1$

$= 2\pi \times \left(\frac{3}{5} + 3 - \frac{1}{6} - \frac{1}{2}\right)$

$= \frac{88\pi}{15}$ ////

(b)



The geometric approach was much easier than the algebraic.

$\arg(z^2 - 9z + 18)$

$= \arg\left\{(z-6)(z-3)\right\}$

$= \arg(z-6) + \arg(z-3)$

$= \theta + \frac{\pi}{2} + 2\theta$

$= 3\theta + \frac{\pi}{2}$ //

(c) (i) $\frac{2x}{10} + \frac{2y}{16} \times \frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{x}{5} \times 8/y$

$= -\frac{8x}{5y}$

$m_t = \frac{8x_1}{5y_1}$

$y - y_1 = \frac{8x_1}{5y_1}(x - x_1)$

$5y_1 y - 5y_1^2 = 8x_1 x - 8x_1^2$

$8x_1 x + 5y_1 y = 8x_1^2 + 5y_1^2$

$\frac{x_1 x}{10} + \frac{y_1 y}{16} = \frac{x_1^2}{10} + \frac{y_1^2}{16} = 1$ // Q.E.D.

(ii) $\frac{y y_1}{16} = 1$

$x = 0, \; y = \frac{16}{y_1}$ ✓

(iii) $10 = 16(1 - e^2)$

$\frac{5}{8} = 1 - e^2$

$e^2 = \frac{3}{8}$

$e = \frac{\sqrt{3}}{2\sqrt{2}}$

$= \frac{\sqrt{6}}{4}$

foci are $\left(0, \pm \frac{\sqrt{6}}{4} \times 4\right)$

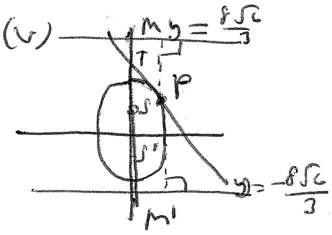
$(0, \pm\sqrt{6})$ //

(iv) $y = \pm \frac{4}{\frac{\sqrt{6}}{4}}$

$= \pm \frac{16}{\sqrt{6}}$

$= \pm \frac{8\sqrt{6}}{3}$ ✓

(v)



$\frac{PS}{PS'} = \frac{ePM}{ePM'}$

This is easier than trying to find PS or PS'.

$= \frac{\sqrt{\frac{8\sqrt{6}}{3} - y_1}}{\sqrt{y_1 - -\frac{8\sqrt{6}}{3}}} \times \frac{\sqrt{6}}{\sqrt{6}}$

$= \frac{16 - \sqrt{6}\, y_1}{16 + \sqrt{6}\, y_1}$

$\frac{TS}{TS'} = \frac{\frac{16}{y_1} - \sqrt{6}}{\frac{16}{y_1} + \sqrt{6}}$

$= \frac{16 - \sqrt{6}\, y_1}{16 + \sqrt{6}\, y_1}$ ///

Q.E.D.

Question 15

a) $I = \int_1^9 e^{\sqrt{x}}\,dx$ $\quad$ when $x = 9$, $u = 3$

let $u = \sqrt{x}$ $\quad$ $x = 1$, $u = 1$ ✓

$x = u^2$

$dx = 2u\,du$

$I = 2\int u e^u \, du$ ✓

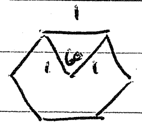
$= 2\left[ue^u - \int e^u\right] du$ [IBPts] ✓

$= 2\left[ue^u - e^u\right]_1^3$

$= 2\left[(3e^3 - e^3) - (e^1 - e^1)\right]$ $\quad$ (4)

$= 4e^3$ ✓

b) Volume for $x = 0$ to $x = -1$ is a hexagonal prism

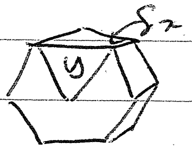


$V_1 = 6 \times \frac{1}{2} \times 1 \times 1 \times \sin 60° \times 1$

$= 3 \times \frac{\sqrt{3}}{2}$

$= \frac{3\sqrt{3}}{2}$ units^3 ✓

Volume for $x = -1$ to $x = -2$



$V_2 = \frac{3\sqrt{3}}{2}\int_{-2}^{-1} (x+2)\,dx$ ✓

$\text{Vslice} = 6 \times \frac{1}{2} y^2 \delta x \sin 60°$

$= \frac{3\sqrt{3}}{2} y^2 \delta x$

$= \frac{3\sqrt{3}}{2}\left[\frac{x^2}{2} + 2x\right]_{-2}^{-1}$ ✓

$= \frac{3\sqrt{3}}{2}\left[\left(\frac{1}{2} - 2\right) - (2 - 4)\right]$

$= \frac{3\sqrt{3}}{4}$ ✓ $\quad$ (4)

$\therefore$ Total Volume $= \frac{3\sqrt{3}}{2} + \frac{3\sqrt{3}}{4}$

$= \frac{9\sqrt{3}}{4}$ units^3

Q15

c) i) $2x^3 = 4x^2 - 7x - 5$

$x^3 = 2x^2 - \frac{7x}{2} - \frac{5}{2}$

$\alpha^3 = 2\alpha^2 - \frac{7\alpha}{2} - \frac{5}{2}$

$\beta^3 = 2\beta^2 - \frac{7\beta}{2} - \frac{5}{2}$ ✓

$\gamma^3 = 2\gamma^3 - \frac{7\gamma}{2} - \frac{5}{2}$

$(\alpha^2+\beta^2+\gamma^2)$

$= (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= (2)^2 - (-2)(\frac{7}{2})$ ✓

$= -3$

$\alpha^3+\beta^3+\gamma^3 = 2(\alpha^2+\beta^2+\gamma^2) - \frac{7}{2}(\alpha+\beta+\gamma) - 3(\frac{5}{2})$

$= 2(-3) - \frac{7}{2}(2) - \frac{5}{2}(3)$ (3)

$= -\frac{41}{2}$ ✓

ii) $\alpha^2\beta + \alpha^2\gamma + \beta^2\alpha, + \beta^2\gamma + \gamma^2\alpha + \gamma^2\beta$

$\alpha^2(\beta+\gamma) + \beta^2(\alpha+\gamma) + \gamma^2(\alpha+\beta)$

$\alpha+\beta+\gamma = 2$

$\alpha^2(2-\alpha) + \beta^2(2-\beta) + \gamma^2(2-\gamma)$

$= 2\alpha^2 - \alpha^3 + 2\beta^2 - \beta^3 + 2\gamma^2 - \gamma^3$

$= 2(\alpha^2+\beta^2+\gamma^2) - (\alpha^3+\beta^3+\gamma^3)$

$= 2(-3) - (-\frac{41}{2})$

$= \frac{29}{2}$

or using $(\alpha+\beta+\gamma)(\alpha\beta+\alpha\gamma+\beta\gamma) - 3(\alpha\beta\gamma)$

$= 2(\frac{7}{2}) - 3(\frac{-5}{2})$

$= 7 + \frac{15}{2}$

$= \frac{29}{2}$

(2)

Correct soln

d) $w = \frac{az+b}{bz+a}$ (1)

$|z| = 1 \Rightarrow z = \cos\theta + i\sin\theta$ (2)

$w = \frac{a\cos\theta + ai\sin\theta + b}{b\cos\theta + bi\sin\theta + a}$

$|w| = \frac{|(a\cos\theta+b) + ai\sin\theta|}{|(b\cos\theta+a) + bi\sin\theta|}$

$= \frac{\sqrt{(a\cos\theta+b)^2 + (ai\sin\theta)^2}}{\sqrt{(b\cos\theta+a)^2 + (bi\sin\theta)^2}}$

$= \frac{\sqrt{a^2\cos^2\theta + ab\cos\theta + b^2 + a^2\sin^2\theta}}{\sqrt{b^2\cos^2\theta + 2ab\cos\theta + a^2 + b^2\sin^2\theta}}$

$= \frac{\sqrt{a^2+b^2+2ab\cos\theta}}{\sqrt{a^2+b^2+2ab\cos\theta}}$

$|w| = 1$

or $w = \frac{az+b}{bz+a}$

$wbz + wa = az + b$

$wbz - az = b - wa$

$z(wb - a) = b - wa$

$|z| = \frac{|b-wa|}{|wb-a|}$ and $|z| = 1$

$\therefore b - wa = wb - a$

$b + a = w(b+a)$

$w = \frac{b+a}{b+a}$

$= 1$

Correct Sln (2)

QUESTION 16

Comments

(a)

$$\left.\begin{aligned} \alpha+\beta+\gamma &= 0 \\ \alpha\beta+\alpha\gamma+\beta\gamma &= \frac{q}{p} \\ \alpha\beta\gamma &= -\frac{r}{p} \end{aligned}\right\} \quad x = \alpha, \beta, \gamma$$

New eqn. has roots $y = \dfrac{1}{-\alpha\beta-\alpha\gamma}, \dfrac{1}{-\alpha\beta-\beta\gamma}, \dfrac{1}{-\alpha\gamma-\beta\gamma}$

$$= \frac{1}{-\alpha(\beta+\gamma)}, \frac{1}{-\beta(\alpha+\gamma)}, \frac{1}{-\gamma(\alpha+\beta)}$$

$$= \frac{1}{\alpha^2}, \frac{1}{\beta^2}, \frac{1}{\gamma^2}$$

$$\therefore\ y = \frac{1}{x^2} \quad \Rightarrow \quad x = \frac{1}{\sqrt{y}}$$

$$p\left(\frac{1}{\sqrt{y}}\right)^3 + q\left(\frac{1}{\sqrt{y}}\right) + r = 0$$

$$\frac{1}{\sqrt{y}}\left(\frac{p}{y} + q\right) = -r$$

$$\frac{1}{y}\left(\frac{p^2}{y^2} + \frac{2pq}{y} + q^2\right) = r^2$$

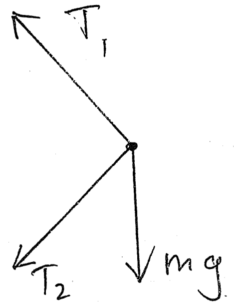
$$p^2 + 2pqy + q^2y^2 = r^2y^3$$

$$\therefore\ r^2y^3 - q^2y^2 - 2pqy - p^2 = 0$$

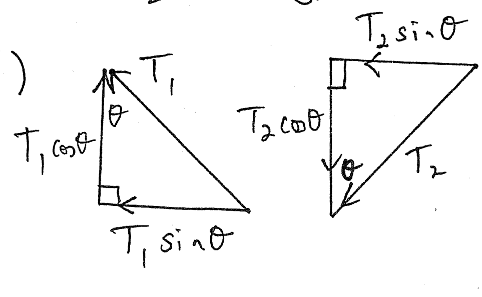
The solution is one of two common approaches to this question. The other approach used the relationship $-\alpha\beta-\alpha\gamma = \beta\gamma - \frac{q}{p}$ as well as $\beta\gamma = \frac{-r}{p\alpha}$.

Many students made small errors and many did not make a start - writing the sum, product etc would have guaranteed the 1st mark.

(b) (i)



(ii)



Horizontally:

$$T_1\sin\theta + T_2\sin\theta = \frac{mv^2}{r}$$

Vertically:

$$T_1\cos\theta - T_2\cos\theta - mg = 0$$

(b)(i) Marked generously - but the three vectors shown in the solution was expected (No more, no less!)

(ii) Take care with the position of angle θ.

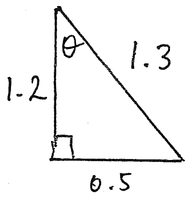
Question 16 (continued)

Comments

(b) continued

(iii)

$$\sin\theta(T_1+T_2)=\frac{mv^2}{r}$$
$$\cos\theta(T_1-T_2)=mg$$



$\sin\theta=\frac{5}{13}$

$\cos\theta=\frac{12}{13}$

$r=0.5$

$$T_1+T_2=\frac{13}{5}\left(\frac{mv^2}{0.5}\right)=\frac{26mv^2}{5}$$

$$T_1-T_2=\frac{13mg}{12}$$

$$2T_1=\frac{26mv^2}{5}+\frac{13mg}{12} \qquad \text{i.e. } T_1=\frac{13mv^2}{5}+\frac{13mg}{24}$$

$$2T_2=\frac{26mv^2}{5}-\frac{13mg}{12} \qquad T_2=\frac{13mv^2}{5}-\frac{13mg}{24}$$

(iv) $T_2>0$ (since both strings experience tension)

$$\frac{13mv^2}{5}-\frac{13mg}{24}>0 \quad \therefore \frac{v^2}{5}-\frac{g}{24}>0 \qquad v^2>\frac{5g}{24}$$

$$v>\sqrt{\frac{5g}{24}} \qquad v>\sqrt{\frac{30g}{144}} \qquad \text{i.e. } v>\frac{\sqrt{30g}}{12}$$

Comments: (b)(iii)

You must eliminate θ from your answers.
You do not need to find θ – you should use the triangle to determine sin θ and cos θ.

(ii) You had to <u>show</u> the given relationship.
Some solutions were left incomplete.

(c) (i) $\frac{AQ}{QC}=\frac{AR}{RB}=1$ (Q and R are midpts of AC and AB)

$\therefore$ RQ ∥ BC (ratio of intercepts is equal on parallel lines)

Similarly $\frac{BP}{PC}=\frac{BR}{RA}=1$ and PR ∥ AC.

$\therefore$ RPCQ is a parallelogram (2 pairs of opp. sides parallel)

(ii) ∠QCP = ∠QRP (opp. ∠s of parallelogram)

∠QRP = ∠QXP (∠s in same segment)

$\therefore$ ∠QCP = ∠QXP $\quad\therefore$ △XQC is isosceles (2 equal ∠s)

(iii) AQ = QC (given, Q is mid-point of AC)

QX = QC (sides opp. equal ∠s in isos. △)

$\therefore$ Circle passes through A, X, C with centre Q. (AQ = QC = QX)

$\therefore$ ∠AXC = 90° (∠ in semi-circle) $\quad\therefore$ AX ⊥ BC

Comments: (c)(i)

Poorly done in general – many "solutions" presented insufficient data.

(ii) This was not a difficult question if you use the result from (i).

(iii) More detail needed from most attempts.