Name: ____________________

Teacher: ____________________

# Knox Grammar School
# 2015
**Trial Higher School Certificate Examination**

# Mathematics Extension 1

**General Instructions**

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen only
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11-14, show relevant mathematical reasoning and/or calculations

**Teachers:**
**Mr Vuletich**
**Mr Bradford**
**Mrs Dempsey**
**Ms Yun**

**Total marks – 70**

**Section I: Pages 2 – 5**

**10 marks**
- Attempt Questions 1-10
- Allow about 15 minutes for this section

**Section II: Pages 6 – 10**

**60 marks**
- Attempt Questions 11-14
- Allow about 1 hour 45 minutes for this section

**Write your Name, your Board of Studies Student Number and your Teacher's Name on the front cover of each writing booklet**

**This paper MUST NOT be removed from the examination room**

**Number of Students in Course: 63**

## Section I

**10 marks**
**Attempt Questions 1-10**
**Allow about 15 minutes for this section**

Use the multiple-choice answer sheet for Questions 1 – 10.

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**1** The point $P$ divides the interval joining $A(4,1)$ and $B(-1,11)$ externally in the ratio 2:3. Which of these are the coordinates of $P$?

(A) $(2,5)$

(B) $(14,-19)$

(C) $(1, 7)$

(D) $(-11,31)$

**2** What is $\lim_{x\to 0}\frac{5\sin 3x}{x}$?

(A) 15

(B) $\frac{5}{3}$

(C) $\frac{3}{5}$

(D) $\frac{1}{15}$

**3** Which of the following is the solution to the inequation $\frac{x-3}{x}\leq 0$?

(A) $x\leq 3$

(B) $x<0$ or $x\geq 3$

(C) $0<x\leq 3$

(D) $0\leq x\leq 3$

**4** Differentiate $\dfrac{1}{\sqrt{4-x^2}}$.

(A) $\sin^{-1}\dfrac{x}{2}$

(B) $\dfrac{1}{2}\sin^{-1}\dfrac{x}{2}$

(C) $\dfrac{-1}{2\sqrt{(4-x^2)^3}}$

(D) $\dfrac{x}{\sqrt{(4-x^2)^3}}$

**5** The expression $\sin x-\sqrt{3}\cos x$ can be written in the form $2\sin(x+\alpha)$.
Find the value of $\alpha$ .

(A) $\alpha=\dfrac{\pi}{6}$

(B) $\alpha=-\dfrac{\pi}{6}$

(C) $\alpha=\dfrac{\pi}{3}$

(D) $\alpha=-\dfrac{\pi}{3}$

**6** The equation of motion of a particle moving in Simple Harmonic Motion is given by $\ddot{x}=1-3x$. Which of the following statements is true?

(A) The period of motion is $\frac{2\pi}{\sqrt{3}}$ and the centre is $x=\dfrac{1}{3}$

(B) The period of motion is $\frac{2\pi}{3}$ and the centre is $x=3$

(C) The period of motion is $\frac{-2\pi}{3}$ and the centre is $x=3$

(D) The period of motion is $\frac{2\pi}{3}$ and the centre is $x=\dfrac{1}{3}$.

**7** What is the exact value of $\tan^{-1}\left(\tan\frac{5\pi}{6}\right)$ ?

(A) $-\frac{1}{\sqrt{3}}$

(B) $\sqrt{3}$

(C) $\frac{5\pi}{6}$

(D) $-\frac{\pi}{6}$

**8** What is the coefficient of $x$ in $\left(x^2-\frac{2}{x}\right)^5$ ?

(A) –160

(B) –80

(C) –32

(D) 3

**9** Mark, Greg and four friends arrange themselves at random in a circle.
What is the probability that Mark and Greg are not together?

(A) $\frac{1}{120}$

(B) $\frac{2}{5}$

(C) $\frac{3}{5}$

(D) $\frac{119}{120}$

**10** By using symmetry arguments, what is the value of $\int_{-a}^{a} \cos^{-1} x dx$ where $-1 \leq a \leq 1$ ?

(A) $0$

(B) $\frac{a\pi}{2}$

(C) $a\pi$

(D) $2a\pi$

## Section II

**60 marks**
**Attempt Questions 11-14**
**Allow about 1 hour and 45 minutes for this section**

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

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**Question 11** (15 marks) Use a SEPARATE writing booklet.

(a) Find the size of the acute angle between the lines $2x+y=5$ and $3x-y=1$. **2**

(b) Using the substitution $u=2x^2+1$, or otherwise, find $\int x(2x^2+1)^{\frac{5}{4}}\,dx$. **3**

(c) Let $P(x)=(x+1)(x-3)Q(x)+a(x+1)+b$, where $Q(x)$ is a polynomial and $a$ and $b$ are real numbers.
When $P(x)$ is divided by $(x+1)$ the remainder is $-11$.
When $P(x)$ is divided by $(x-3)$ the remainder is $1$.

(i) What is the value of $b$? **1**

(ii) What is the remainder when $P(x)$ is divided by $(x+1)(x-3)$? **2**

(d) Let $f(x)=\sin^{-1}(x+4)$.

(i) State the domain of the function $f(x)$. **1**

(ii) Find the gradient of the graph of $y=f(x)$ at the point where $x=-4$. **2**

(iii) Sketch the graph of $y=f(x)$. **2**

(e) In a factory that makes metal containers it is found that 2% have defects. Find the probability (leaving answers in index form) that a random sample of twenty such metal containers should contain:

(i) no defective containers. **1**

(ii) not more than one defective container. **1**

**Question 12** (15 marks) Use a SEPARATE writing booklet.

(a) Find the exact value of the volume of the solid of revolution formed when the region bounded by the curve $y = \sin 2x$, the $x$-axis and the line $x = \frac{\pi}{8}$ is rotated about the $x$-axis. **3**

(b) (i) Find the vertical and horizontal asymptotes of the hyperbola $y = \frac{x-1}{x-3}$ and hence sketch the graph of $y = \frac{x-1}{x-3}$, carefully showing any intercepts with the coordinate axes. **3**

(ii) Hence, or otherwise, find the values of $x$ for which $\frac{x-1}{x-3} \le 2$ **2**

(c) A team of 4 players consists of 2 men and 2 women.
A total of 7 players are available of which 4 are women and 3 are men.

(i) How many different teams of 4 players can be selected? **1**

(ii) If 2 of the players are husband and wife and wish to play in the same team, how many different teams can now be selected if the husband and wife are on the team? **1**

(d) Consider the function $f(x) = e^x - e^{-x}$.

(i) Show that $f(x)$ is increasing for all values of $x$. **1**

(ii) Show that the inverse function is given by: **3**

$$f^{-1}(x) = \log_e\left(\frac{x + \sqrt{x^2 + 4}}{2}\right)$$

(iii) Hence, or otherwise, solve $e^x - e^{-x} = 3$. **1**
Give your answer correct to two decimal places.

**Question 13** (15 marks) Use a SEPARATE writing booklet.

(a) Use mathematical induction to prove that $3^{3n} + 2^{n+2}$ is divisible by 5, for all positive integers. **3**

(b) (i) Show that the equation $e^{-x} = \sin 2x$ has a root lying between 1 and 2. **1**

(ii) By taking 1.5 as a first approximation, use one application of Newton's method to obtain a better approximation to this root. Give your answer correct to three decimal places. **2**

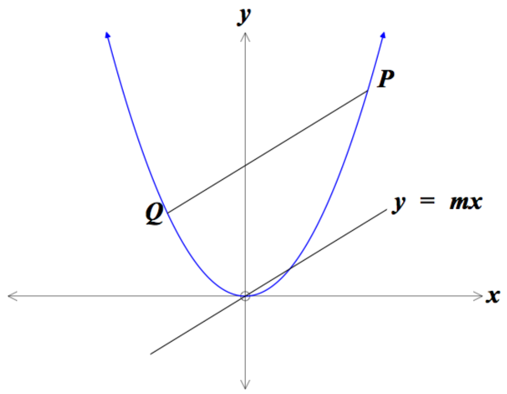
(c) A particle is moving in simple harmonic motion about the origin $O$. Its displacement $x$ cm from $O$ at time $t$ seconds satisfies the equation:

$$\ddot{x} = -\pi x$$

(i) If the velocity of the particle is $v$ cm per second, derive an expression for $v^2$ as a function of $x$, given that the amplitude of the motion is 4 cm. **2**

(ii) Where is the particle when its speed is 5 cm per second, giving your answer correct to 4 significant figures? **1**

(d) Two parametric points $P(2p, p^2)$ and $Q(2q, q^2)$ lie on the parabola $x^2 = 4y$, where $P$ lies in the 1st quadrant, $Q$ in the 2nd quadrant and the line through $PQ$ is parallel to the line $y = mx$, where $m > 0$ as shown in the diagram below.




(i) Show that $p + q = 2m$. **1**

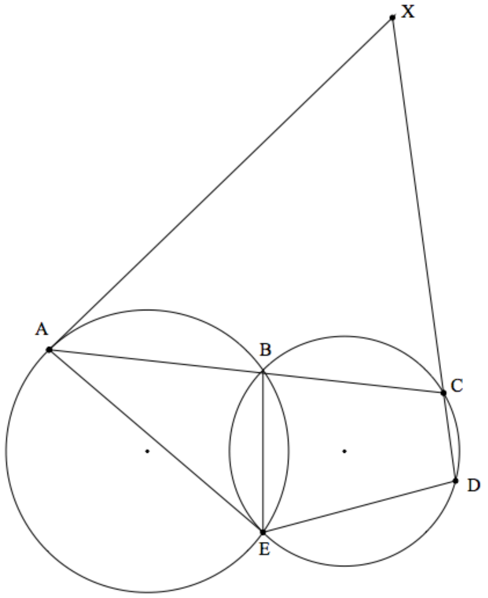
The equation of the normal to the parabola at the point $P$ is $x + py = p^3 + 2p$. (Do NOT prove this)

(ii) Find the coordinates of $N$, the point of intersection of the normals to the curve from $P$ and $Q$. **2**

(iii) Determine the Cartesian equation of the locus of $N$ as the line $PQ$ moves parallel to the line $y = mx$. Describe the locus of $N$ if $m = 0$. **3**

**Question 14** (15 marks) Use a SEPARATE writing booklet.

(a)




Two circles intersect at *B*, *E* and *AX* is a tangent to the circle *ABE*. *AB* produced meets the second circle at *C*, and *XC* meets the circle *BCDE* again at *D* as shown.

Let $\angle XAC = \alpha°$ and $\angle AXC = \beta°$.

Copy or trace the diagram into your writing booklet.

(i) Give the reason why $\angle AEB = \alpha°$ **1**

(ii) Prove that *AXDE* is a cyclic quadrilateral. **2**

(b) Tom and Jerry play a game by each tossing a fair coin. The game consists of tossing the two coins together, until for the first time either two heads appear when Tom wins, or two tails appear when Jerry wins.

(i) What is the probability that Tom wins at or before the third toss? **1**

(ii) Hence, or otherwise, show that the probability that Tom wins at or before the $n^{\text{th}}$ toss is $\frac{1}{2} - \frac{1}{2^{n+1}}$ . **2**

**Question 14 continues on page 10**

Question 14 (continued)

(c) (i) Use the binomial theorem to write out the expansion of $(1+x)^n$ . **1**

(ii) Hence prove that $\sum_{r=1}^{n} r\, {}^nC_r\, 2^{r-1} = n.3^{n-1}$ . **2**

(d) Two particles are projected from the same point on level ground with the same speed $V$ m/s and with angles of projection $\alpha$ and $\frac{\pi}{2}-\alpha$ to the horizontal.

The equations of motion for a particle projected at an angle of $\alpha$ are

$$x = Vt\cos\alpha \text{ and } y = Vt\sin\alpha - \frac{1}{2}gt^2$$

(Do NOT prove these equations)

(i) Prove that the maximum height $h_1$ of a particle with angle of projection $\alpha$ **2**

is $h_1 = \dfrac{V^2\sin^2\alpha}{2g}$.

The greatest heights they reach are $h_1$ and $h_2$ .

(ii) Show that, for any $\alpha$ , **1**

$h_1 + h_2 = \frac{1}{2}R$, where $R$ is the maximum range.

(iii) Explain why both particles have the same range. Furthermore, given $\tan\alpha = \frac{3}{4}$ , $V$ = 200 m/s and $g$ = 10 m/s$^2$, what time must elapse between the instants of projection if the particles collide as they strike the ground? **3**

**End of paper**

# STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \textbf{ if } n < 0$$

$$\int \frac{1}{x} \, dx = \ln x \quad , \quad x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln\left(x + \sqrt{x^2 - a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln\left(x + \sqrt{x^2 + a^2}\right)$$

*Note* $\ln x = \log_e x, \quad x > 0$

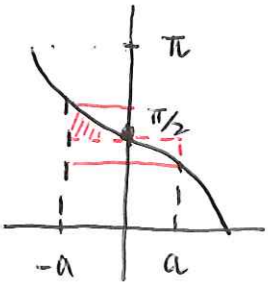
# 2015 Year 12 Mathematics Ext 1 HSC Task 5 Solutions

| Suggested Solution (s) | Comments |
|---|---|
| **SECTION I** | |
| 1) $(4,1)$ $(-1,11)$ <br> $-2$ $3$ <br> $\left(\frac{12+2}{-1}, \frac{-22+3}{1}\right)$ <br> $= (14, -19)$ | (B) |
| 2) $5\times 3 \lim_{x\to 0} \frac{\sin 3x}{3x}$ <br> $= 15\times 1$ <br> $= 15$ | (A) |
| 3) $\frac{x-3}{x} \le 0$ <br> $x(x-3) \le 0$ <br> $0 < x \le 3$ $x \ne 0$ | (C) |
| 4) $\frac{d}{dx}(4-x^2)^{-\frac{1}{2}}$ <br> $= -\frac{1}{2}(4-x^2)^{-3/2}(-2x)$ <br> $= \frac{x}{\sqrt{(4-x^2)^3}}$ | (D) |
| 5) $\sin x - \sqrt{3}\cos x$ <br> $= 2\sin x\cos\alpha + 2\cos\alpha\sin x$ <br> $2\cos\alpha = 1$ $2\sin\alpha = -\sqrt{3}$ <br> $\alpha = -\frac{\pi}{3}$ | (D) |
| 6) $\ddot{x} = 1-3x$ <br> $= -3(x-\frac{1}{3})$ <br> period $= \frac{2\pi}{\sqrt{3}}$ centre $x=\frac{1}{3}$ | (A) |
| 7) $\tan^{-1}[\tan(\frac{5\pi}{6})]$ <br> $= \tan^{-1}(\tan(-\frac{\pi}{6}))$ <br> $= -\frac{\pi}{6}$ | (D) |
| 8) $T_{k+1} = {}^5C_k (x^2)^{5-k}(-\frac{2}{x})^k$ <br> $= {}^5C_k x^{10-2x}(-2)^k x^{-k}$ <br> $= {}^5C_k(-2)^k x^{10-3k}$ <br> ie) $10-3k=9$ <br> $k=3$ <br> ∴ coefficient of $x$ <br> $= {}^5C_3(-2)^3 = -80$ | (B) |
| 9) M, G, A, B, C & D <br> Can form a ring of 6 in 5! ways <br> seat M - 1 way <br> G - 3 ways <br> seat rest 4! ways <br> P(not together) $= \frac{1\times 3\times 4!}{5!}$ <br> $= \frac{3}{5}$ | (C) |

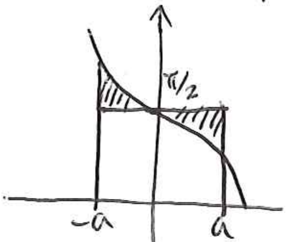
**2015 Year 12 Mathematics Ext 1 HSC Task 5 Solutions**

| Suggested Solution (s) | Comments | Suggested Solution (s) | Comments |
|---|---|---|---|

10) $\int_{-a}^{a} \cos^{-1} x \, dx$




because of the symmetry of $\cos^{-1} x$, the area is also equivivalent to




area is a rectangle

$A = 2a \times \frac{\pi}{2}$

$= a\pi$

Comment: (C)

SECTION II

**Question 11**

(a) $\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{(-2)-(3)}{1+(-2)(3)}\right|$ ✓

$= 1$

$\therefore \theta = \frac{\pi}{4}$ or $45°$ ✓

(b) $u = 2x^2 + 1$

$\frac{du}{dx} = 4x$ ie) $\frac{1}{4}du = x\,dx$

$\therefore \int x(2x^2+1)^{5/4}\,dx$

$= \int u^{5/4} \cdot \frac{1}{4} du$ ✓

$= \frac{1}{4} u^{9/4} \cdot \frac{4}{9} + c$ ✓

$= \frac{1}{9} u^{9/4} + c$

$= \frac{1}{9}(2x+1)^{9/4} + c$ ✓

(c)

$P(x) = (x+1)(x-3)\,Q(x) + a(x+1) + b$

i) $P(-1) = -11$

$(-1+1)(-1-3)Q(x) + a(-1+1) + b = -11$

$0 + 0 + b = -11$

$\therefore b = -11$ ✓

Comment: wrong formula −1

ii) $P(3) = 1$

ie) $(3+1)(3-3)Q(x) + a(3+1) + b = 1$

$a(3+1) - 11 = 1$

$4a = 12$

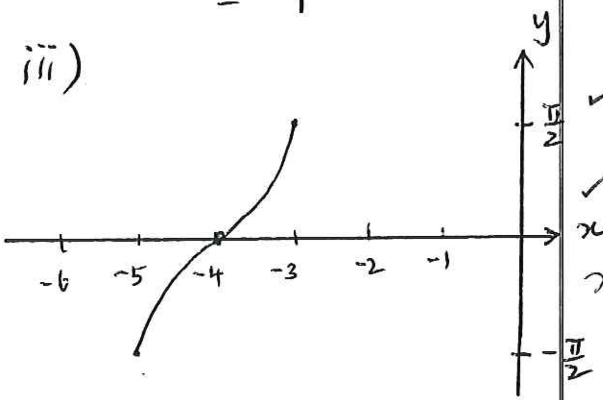
$a = 3$ ✓

When $P(x)$ is divided by $(x+1)(-3)$, the remainder is $a(x+1) + b$

$3(x+1) - 11$

$= 3x - 8$ ✓

# 2015 Year 12 Mathematics Ext 1 HSC Task 5 Solutions

| Suggested Solution (s) | Comments |
|---|---|
| 11(d) $f(x) = \sin^{-1}(x+4)$ | |
| i) domain $-1 \leq x+4 \leq 1$ | |
| $-5 \leq x \leq -3$ | ✓ |
| ii) $f'(x) = \dfrac{1}{\sqrt{1-(x+4)^2}}$ | ✓ |
| $f'(-4) = \dfrac{1}{\sqrt{1-(-4+4)^2}}$ | |
| $= 1$ | ✓ |
| iii)   | ✓ Range<br>✓ Shape with x-intercept |
| e) $P(D) = \frac{1}{50}$ $P(\tilde{D}) = \frac{49}{50}$ | |
| consider $\left(\frac{49}{50} + \frac{1}{50}\right)^{20}$ | |
| i) $P(X=0) = {}^{20}C_0 \left(\frac{49}{50}\right)^{20}$ | |
| $= \left(\frac{49}{50}\right)^{20}$ | ✓ |
| ii) $P(X \leq 1) = P(X=0) + P(X=1)$ | |
| $= \left(\frac{49}{50}\right)^{20} + {}^{20}C_1 \left(\frac{49}{50}\right)^{19}\left(\frac{1}{50}\right)$ | |
| $= \left(\frac{49}{50}\right)^{20} + 20\left(\frac{49}{50}\right)^{19}\left(\frac{1}{50}\right)$ | |
| $= \left(\frac{49}{50}\right)^{19}\left[\frac{49}{50} + \frac{20}{50}\right]$ | ✓ |

$= \left(\frac{49}{50}\right)^{19}\left(\frac{69}{50}\right)$ $\quad (0.98)^{20} + \binom{20}{1}(0.02)(0.98)^{19}$ or equivalent acceptable.

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QUESTION 12

(a) $V = \pi \int y^2 \, dx$

$= \pi \int_0^{\frac{\pi}{8}} \sin^2 2x \, dx \qquad \cos 2A = 1 - 2\sin^2 A$

$= \frac{\pi}{2} \int_0^{\frac{\pi}{8}} 1 - \cos 4x \, dx \qquad \therefore \sin^2 2x = \frac{1-\cos 4x}{2}$

$= \frac{\pi}{2} \left[ x - \frac{\sin 4x}{4} \right]_0^{\frac{\pi}{8}}$

$= \frac{\pi}{2}\left( \left(\frac{\pi}{8} - \frac{1}{4}\sin\frac{\pi}{2}\right) - (0 - \sin 0) \right)$

$= \frac{\pi}{2}\left(\frac{\pi}{8} - \frac{1}{4}\right)$

$= \frac{\pi}{16}(\pi - 2) \quad u^3$ (2)

$\frac{\pi^2}{16} - \frac{\pi}{8}$

(b) i) $y = \frac{x-1}{x-3}$

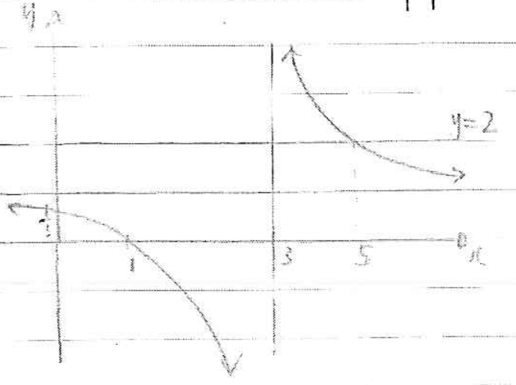
vertical asymptote : $x = 3$

$y = \frac{x-3+2}{x-3}$

$= 1 + \frac{2}{x-3}$ (3)

as $x \to \infty$, $\frac{2}{x+3} \to 0 \therefore y \to 1$

$\therefore$ horizontal asymptote : $y = 1$




$y$-int $= \frac{1}{3}$

$x$-int $= 1$

At point of intersection with $y = 2$

$2 = \frac{x-1}{x-3}$

$2x - 6 = x - 1$

$x = 5$ (2)

$\therefore \frac{x-1}{x-3} \leq 2$ for $x < 3$ and $x \geq 5$

(c) 4 players 2 men, 2 women.
7 available 4W, 3M.

(i) number of teams $= {}^4C_2 \times {}^3C_2$
$= 18$ (1)

(ii) number of teams $= {}^3C_1 \times {}^2C_1$
$= 6$. (1)

(d) $f(x) = e^x - e^{-x}$

i) $f'(x) = e^x + e^{-x}$
$> 0$ for all $x \in \mathbb{R}$. (1)
$\therefore f(x)$ increasing for all $x \in \mathbb{R}$.

(ii) Let $y = e^x - e^{-x}$
$= e^x - \frac{1}{e^x}$
$= \frac{e^{2x} - 1}{e^x}$

$f^{-1}$: $x = \frac{e^{2y} - 1}{e^y}$

$e^y x = e^{2y} - 1$

$e^{2y} - e^y x - 1 = 0$

Let $u = e^y$

$u^2 - ux - 1 = 0$

$u = \frac{x \pm \sqrt{x^2 - 4(1)(-1)}}{2}$

$= \frac{x \pm \sqrt{x^2+4}}{2}$ (3)

$\therefore e^y = \frac{x \pm \sqrt{x^2+4}}{2}$

$e^y > 0 \therefore e^y = \frac{x + \sqrt{x^2+4}}{2}$

$\therefore y = \ln\left(\frac{x + \sqrt{x^2+4}}{2}\right)$

$\therefore f^{-1}(x) = \ln\left(\frac{x + \sqrt{x^2+4}}{2}\right)$ as req'd

(iii) $e^x - e^{-x} = 3$

$f^{-1}(3) = \ln\left(\frac{3 + \sqrt{3^2+4}}{2}\right)$ (1)

$= \ln\left(\frac{3 + \sqrt{13}}{2}\right)$

$= 1.1947$

$= 1.19$ (2dp)

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QUESTION 13.

(a) RTP $3^{3n}+2^{n+2}$ is divisible by 5

Step 1: Prove true for $n=1$

when $n=1$, $3^{3n}+2^{n+2}$
$=3^3+2^3$
$=35$
which is divisible by 5
$\therefore$ true for $n=1$ (3)

Step 2:
Assuming true for $n=k$
i.e. $3^{3k}+2^{k+2}=5m$ for some integer, m

Prove true for $n=k+1$

$3^{3(k+1)}+2^{k+1+2}$
$=3^{3k+3}+2^{k+3}$
$=3^{3k}.3^3+2^k.2^3$
$=3^3(5m-2^{k+2})+2^k.2^3$
$=135m-27.2^{k+2}+2\times2^{k+2}$
$=135m-25\times2^{k+2}$
$=5(27m-5.2^{k+2})$
which is divisible by 5

Since true for $n=1$ and true for $n=k+1$, assuming true for $n=k$, then true for $n=1+1=2$, $n=2+1=3$ etc $\therefore$ true for all positive integers

(b) (i) Let $f(x)=e^{-x}-\sin 2x$
$f(1)=e^{-1}-\sin 2$ $\quad f(2)=e^{-2}-\sin 4$
$=-0.541...$ $\quad =0.892...$
$<0$ $\quad >0$ (1)
$\therefore$ root lies between $x=1$ and $x=2$

(ii) $a_2=a_1-\frac{f(a_1)}{f'(a_1)}$ $\quad f'(x)=-e^{-x}-2\cos 2x$
$=1.5-\frac{e^{-1.5}-\sin 3}{-e^{-1.5}-2\cos 3}$ (2)
$=1.4533...$
$=1.453$ (to 3dp)

(c) $\ddot{x}=-\pi x$.

$\ddot{x}=-n^2x$, $n=\sqrt{\pi}$
or: $v^2=n^2(a^2-x^2)$
$=\pi(16-x^2)$

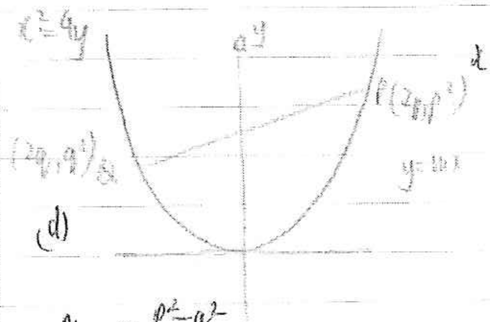
(i) $\frac{1}{2}v^2=\int-\pi x\,dx$
$=\frac{-\pi x^2}{2}+c$.

amplitude $=4$ $\therefore$ when $v=0$, $x=4$
$\therefore 0=-\frac{\pi 4^2}{2}+c$
$\therefore c=8\pi$
$\therefore v^2=-\pi x^2+16\pi$ (2)

or use
$v^2=n^2(a^2-x^2)$
$=\pi^2(16-x^2)$
when $x=0$, $v=4\pi$

(ii) when $v=5$, $25=-\pi x^2+16\pi$
$\pi x^2=16\pi-25$
$x^2=16-\frac{25}{\pi}$ (1)
$=\frac{16\pi-25}{\pi}$
$x=\pm\sqrt{\frac{16\pi-25}{\pi}}$
$=\pm2.8358...$
$=\pm2.836$ (to 4sf)

(d)




$m_{PQ}=\frac{p^2-q^2}{2p-2q}$
$m=\frac{p+q}{2}$

$\therefore p+q=2m$ as req'd (1)

(ii) $x+py=p^3+2p$ — eqn of normal at P ①
At: $x+qy=q^3+2q$ — ②

$(p-q)y=p^3-q^3+2p-2q$
$(p-q)y=(p-q)(p^2+pq+q^2)+2(p-q)$
$y=p^2+pq+q^2+2$

$x+p(p^2+pq+q^2+2)=p^3+2p$
$x=-p^2q-pq^2$
$=-pq(p+q)$ (2)

$\therefore N(-pq(p+q),\ p^2+pq+q^2+2)$

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(iii) for locus of N

$x = -pq(p+q)$
$= -pq(2m)$ from i)

Now, $y = p^2+q^2+pq+2$
$= (p+q)^2 - pq + 2$
$= (2m)^2 + \frac{x}{2m} + 2$

$y = 4m^2 + \frac{x}{2m} + 2$ (3)

$y = \frac{x}{2m} + 4m^2 + 2$

If $m = 0$, $\therefore m_{PQ} = 0$ i.e. PQ is horizontal
and $p+q = 0$
$\therefore q = -p$.
$\therefore$ locus of N must be $x = 0$. *

* since $p+q = 2m$
$N = (-2mpq, m^2+2-pq)$
$\therefore$ when $m=0$, $N(0, 2-pq)$
since $p>0$, $q<0$ (or vice versa), $pq<0$.
$\therefore 2-pq > 2$
$\therefore$ locus is $x=0$ for $y>2$.

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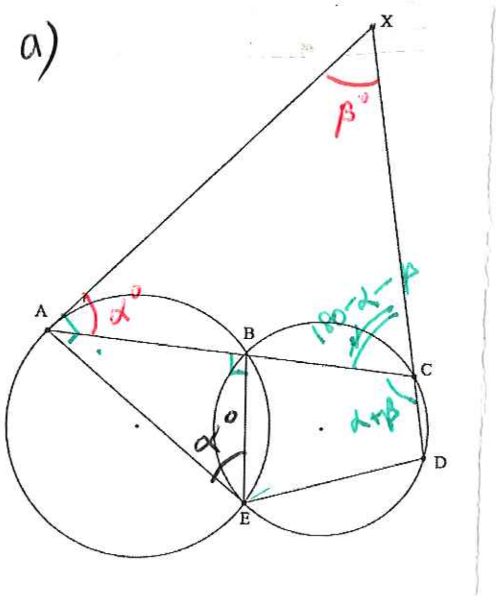
## 2015 Year 12 Mathematics Ext 1 HSC Task 5 Solutions

| Suggested Solution (s) | Comments | Suggested Solution (s) | Comments |
|---|---|---|---|
| **Question 14**<br>a)<br><br>i) $\angle AEB = \alpha^\circ$<br>(angle between tangent & chord of contact is equal to the angle in the alternate segment)<br>ii) $\angle XCA = 180 - (\alpha + \beta)$<br>(angle sum of a triangle)<br>$\angle DEB = 180 - (\alpha + \beta)$<br>(exterior angle of a cyclic quadrilateral equals interior opposite angle)<br>$\angle AED = \angle AEB + \angle DEB$<br>$= \alpha + 180 - \alpha - \beta$<br>$= 180 - \beta$<br>$\therefore \angle AXC + \angle AED = 180^\circ$<br>$\therefore$ AXDE is a cyclic quadrilateral as opposite angles are supplementary ✓ | ✓<br>✓ | b) i) $\frac{1}{4} + \frac{1}{4}\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{2}\right)^2$<br>$= \frac{7}{16}$ ✓<br>ii) $\frac{1}{4} + \frac{1}{4}\left(\frac{1}{2}\right) + \frac{1}{4}\left(\frac{1}{2}\right)^2 + \dots + \frac{1}{4}\left(\frac{1}{2}\right)^{n-1}$<br>It's a G.P. with $a = \frac{1}{4}$ $r = \frac{1}{2}$<br>$S_n = \frac{a(r^n - 1)}{r - 1}$ ✓<br>$= \frac{\frac{1}{4}\left[\left(\frac{1}{2}\right)^n - 1\right]}{\frac{1}{2} - 1} = \frac{\frac{1}{2^2}\left(\frac{1}{2^n} - 1\right)}{-\frac{1}{2}}$<br>$= \frac{1}{2}\left(1 - \frac{1}{2^n}\right)$ ✓<br>$= \frac{1}{2} - \frac{1}{2^{n+1}}$ as required.<br>OR<br>Tom & Jerry winning by $n^{th}$ game<br>$P(X) = \frac{1}{2}(1 - P(\text{no one has won}))$<br>$= \frac{1}{2}\left(1 - \frac{1}{2^n}\right)$<br>$= \frac{1}{2} - \frac{1}{2^{n+1}}$ | |

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## 2015 Year 12 Mathematics Ext 1 HSC Task 5 Solutions

| Suggested Solution (s) | Comments | Suggested Solution (s) | Comments |
|---|---|---|---|

Question 14 (c)

i) $(1+x)^n = \sum_{r=0}^{n} {}^nC_r x^n$

$= {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$ ✓

ii) Differentiating both sides with respect to $x$

$n(1+x)^n$

$= {}^nC_1 + 2\,{}^nC_2 x + 3\,{}^nC_3 x + \dots + n.\,{}^nC_n x^{n-1}$ ✓

Now put $x = 2$

$n(3)^{n-1} = {}^nC_1 + 2\,{}^nC_2 \cdot 2 + 3\,{}^nC_3 2^2 + \dots + n\,{}^nC_n 2^{n-1}$ ✓

$n.3^{n-1} = \sum_{r=1}^{n} r\,{}^n\,2^{r-1}$

Q 14 (d) (i)

$\dot{x} = V\cos\alpha \quad \dot{y} = V\sin\alpha - gt$

At maximum height $\dot{y} = 0$

ie) $V\sin\alpha - gt = 0$ ✓

$t = \frac{V\sin\alpha}{g}$ —①

sub ① into $y(t)$

$y = V\left(\frac{V\sin\alpha}{g}\right)\sin\alpha - \frac{1}{2}g\left(\frac{V\sin\alpha}{g}\right)^2$ ✓

$= \frac{V^2\sin^2\alpha}{g} - \frac{V^2\sin^2\alpha}{2g}$

$= \frac{V^2\sin^2\theta}{2g}$

Q 14 (d) (ii)

greatest height for projection $\left(\frac{\pi}{2} - \alpha\right)$

$= \frac{V^2\sin^2\left(\frac{\pi}{2} - \alpha\right)}{2g}$

$= \frac{V^2\cos^2\alpha}{2g}$

$h_1 + h_2 = \frac{V^2\sin^2\alpha}{2g} + \frac{V^2\cos^2\alpha}{2g}$

$= \frac{V^2(\sin^2\alpha + \cos^2\alpha)}{2g}$ ✓

$= \frac{V^2}{2g}$

$= \frac{1}{2}\left(\frac{V^2}{g}\right)$

$= \frac{1}{2}R$ where R is max. range $= \frac{V^2}{g}$.

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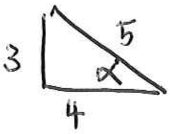
**2015 Year 12 Mathematics Ext 1 HSC Task 5 Solutions**

| Suggested Solution (s) | Comments | Suggested Solution (s) | Comments |
|---|---|---|---|
| Q14 (d) (iii)<br><br><br>$\cos\alpha = \frac{4}{5}$<br>$\sin\alpha = \frac{3}{5}$<br><br>Time of flight for particle projected @ $\alpha$<br>$= \frac{2v\sin\alpha}{g}$<br>$= \frac{2\times 200 \times \frac{3}{5}}{10}$<br>$= 24$ sec<br><br>@ $\left(\frac{\pi}{2}-\alpha\right)$<br>$\Rightarrow \frac{2v\sin\left(\frac{\pi}{2}-\alpha\right)}{g}$<br>$= \frac{2v\cos\alpha}{g}$<br>$= \frac{2\times 200\times\frac{4}{5}}{10}$<br>$= 32$ seconds | ✓<br><br>✓ | Range for particle with<br>• angle of projection $\alpha$<br>$= \frac{2v^2\cos\alpha\sin\alpha}{g}$<br>• angle of $\left(\frac{\pi}{2}-\alpha\right)$<br>$= \frac{2v^2\sin\left(\frac{\pi}{2}-\alpha\right)\cos\left(\frac{\pi}{2}-\alpha\right)}{g}$<br>$= \frac{2v^2\cos\alpha\sin\alpha}{g}$<br>= range for particle with angle of projection $\alpha$<br><br>$\therefore$ 8 seconds must elapse between the instants of projection if the particle collide so they strike the ground. | ✓ |