STUDENT NAME: ___________________________

TEACHER: _________________________________

Founded 1982

THE HILLS GRAMMAR SCHOOL

TASK 4 Trial Examination 2015

YEAR 12

MATHEMATICS EXTENSION 2

Time Allowed: Three hours (plus five minutes reading time)

Weighting: 40%

Outcomes: E1, E2, E3, E4, E5, E6, E7, E8, E9

Instructions:

- Approved calculators may be used
- Attempt all questions
- Start all questions on a new sheet of paper
- The marks for each question are indicated on the examination
- Show all necessary working

MCQ	Question 11	Question 12	Question 13	Question 14	Question 15	Question 16	TOTAL
/10	/15	/15	/15	/15	/15	/15	/100

Section 1 Multiple Choice (10 Marks)

1 The $\displaystyle\int \frac{x}{\sqrt{9-4x^2}}dx$ is:

(A) $-\dfrac{\sqrt{9-4x^2}}{4}+c$

(B) $\dfrac{\sqrt{9-4x^2}}{4}+c$

(C) $-\dfrac{3\sqrt{9-4x^2}}{2}+c$

(D) $\dfrac{3\sqrt{9-4x^2}}{2}+c$

2 The $\displaystyle\int \frac{1}{x^2-6x+13}dx$ is:

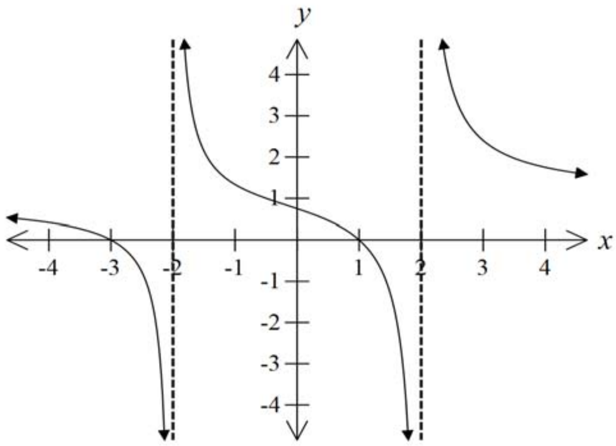
(A) $\tan^{-1}\dfrac{x-3}{2}+c$

(B) $\dfrac{1}{2}\tan^{-1}(x-3)+c$

(C) $\dfrac{1}{2}\tan^{-1}\dfrac{x-3}{2}+c$

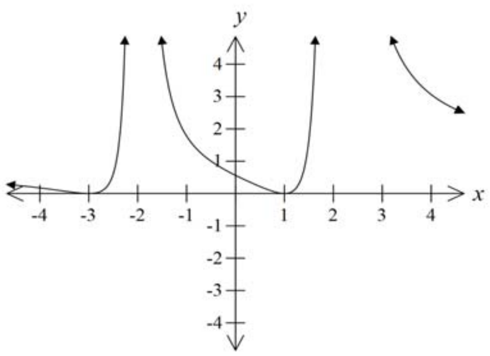
(D) $\dfrac{1}{4}\tan^{-1}\dfrac{x-3}{4}+c$

3 The diagram shows the graph of the function $y = f(x)$.

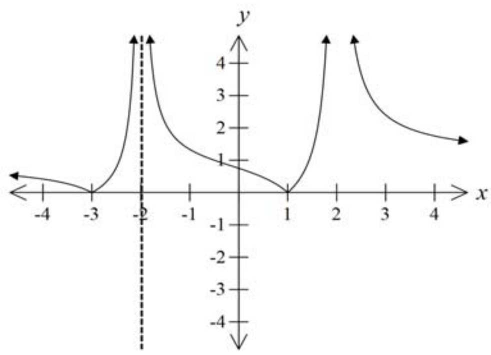


Which of the following is the graph of $y = |f(x)|$?

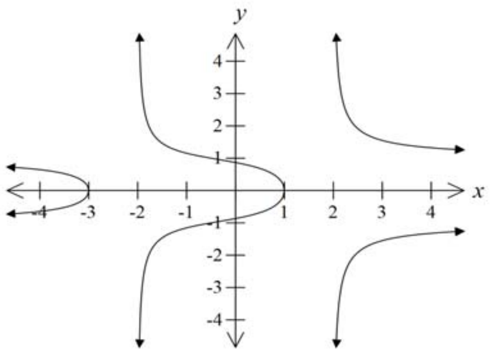
(A)



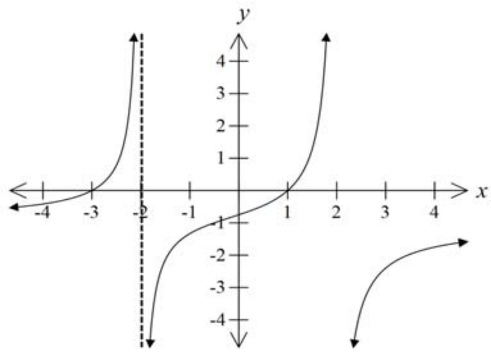
(B)



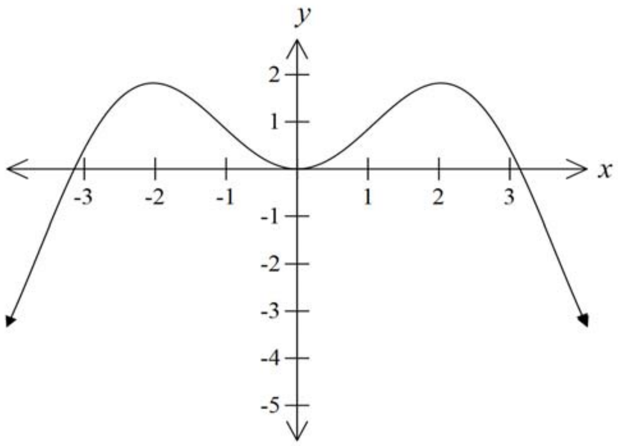
(C)



(D)

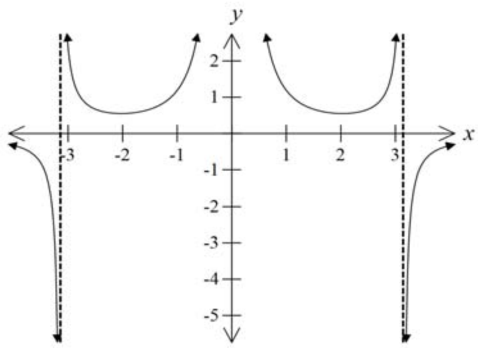


4 The diagram shows the graph of the function $y = f(x)$.

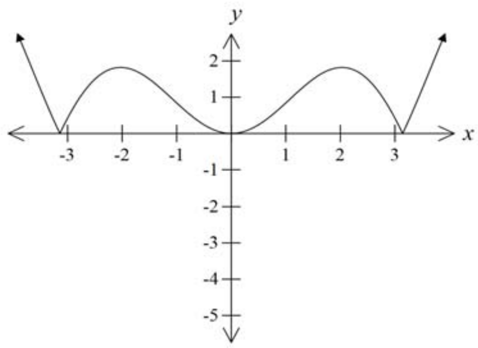


Which of the following is the graph of $y = \dfrac{1}{f(x)}$?

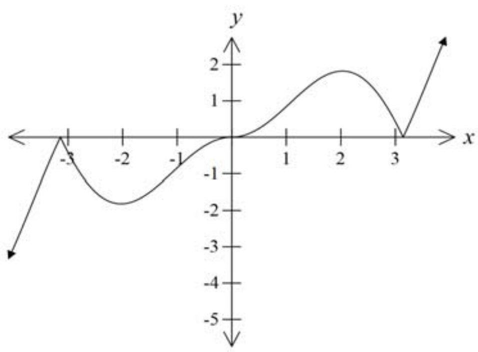
(A)



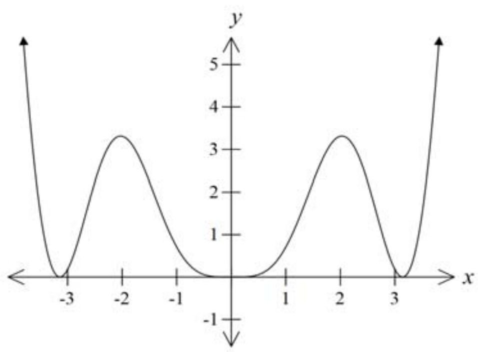
(B)



(C)



(D)



5 Let $z = 2 + i$ and $w = 1 - i$. What is the value of $3z + iw$?

(A) $5 - 4i$ (B) $5 + 4i$

(C) $7 + 4i$ (D) $7 - 4i$

6 It is given that $3 + i$ is a root of $P(z) = z^3 + az^2 + bz + 10$ where a and b are real numbers. Which expression factorises $P(z)$ over the real numbers?

(A) $(z-1)(z^2 + 6z - 10)$ (B) $(z-1)(z^2 - 6z - 10)$

(C) $(z+1)(z^2 + 6z + 10)$ (D) $(z+1)(z^2 - 6z + 10)$

7 For the ellipse with the equation $\dfrac{x^2}{4} + \dfrac{y^2}{3} = 1$. What is the eccentricity?

(A) $\dfrac{1}{4}$ (B) $\dfrac{1}{2}$

(C) $\dfrac{3}{4}$ (D) $\dfrac{9}{16}$

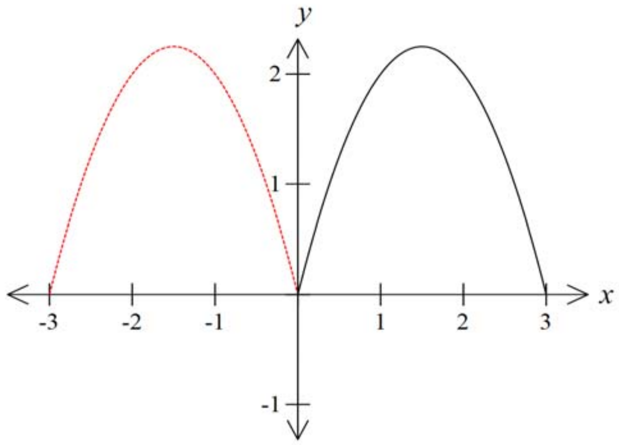
8 Consider the hyperbola with the equation $\dfrac{x^2}{4} - \dfrac{y^2}{3} = 1$.

What are the coordinates of the vertices of the hyperbola?

(A) $(0, \pm 2)$ (B) $(\pm 2, 0)$

(C) $(0, \pm 4)$ (D) $(\pm 4, 0)$

9 The area between the curve $y = 3x - x^2$, the x-axis, $x = 0$ and $x = 3$, is rotated about the y-axis to form a solid.



What is the volume of this solid?

(A) $\frac{9\pi}{4}$ cubic units

(B) $\frac{9\pi}{2}$ cubic units

(C) $\frac{27\pi}{4}$ cubic units

(D) $\frac{27\pi}{2}$ cubic units

10 A particle of mass m is moving in a straight line under the action of a force, $F = \frac{m}{x^3}(6 - 10x)$.

Which of the following is an expression for its velocity in any position, if the particle starts from rest at $x = 1$?

(A) $v = \pm\frac{1}{x}\sqrt{(-3 + 10x - 7x^2)}$

(B) $v = \pm x\sqrt{(-3 + 10x - 7x^2)}$

(C) $v = \pm\frac{1}{x}\sqrt{2(-3 + 10x - 7x^2)}$

(D) $v = \pm x\sqrt{2(-3 + 10x - 7x^2)}$

Section 2 **Marks**

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Let $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ and $\omega = \sqrt{3} + i$.

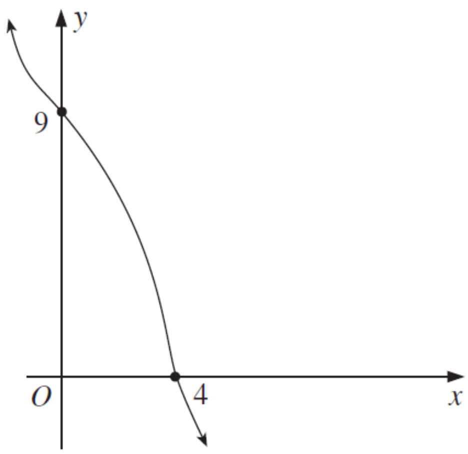
(i) Express ω in modulus-argument form. **1**

(ii) Hence, or otherwise, express $z^3\omega$ in modulus-argument form. **2**

(b) Sketch the region in the complex plane where the inequalities $|z + \overline{z}| \le 1$ *and* $|z - i| \le 1$ hold simultaneously. **2**

(c) Evaluate $\displaystyle\int_0^2 te^{-t}dt$. **3**

(d) The diagram shows the graph of the (decreasing) function $y = f(x)$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = |f(x)|$. **1**

(ii) $y = \dfrac{1}{f(x)}$. **2**

(iii) $y^2 = f(x)$. **2**

(iv) $y = f^{-1}(x)$. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Find the square roots of $3+4i$. **3**

(ii) Hence, or otherwise, solve the equation $z^2+iz-1-i=0$. **2**

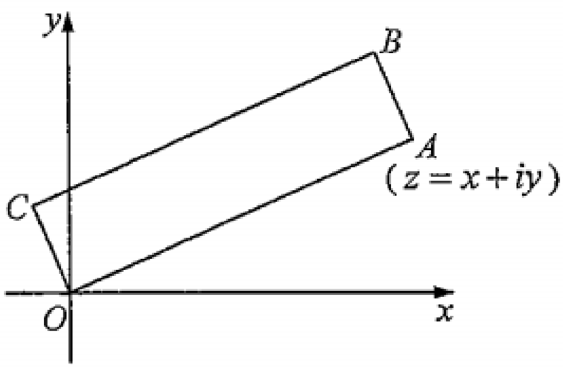
(b) Use the substitution $t=\tan\frac{\theta}{2}$ to show that $\displaystyle\int_{\frac{\pi}{2}}^{\frac{2\pi}{3}}\frac{d\theta}{\sin\theta}=\frac{1}{2}\ln 3$. **3**

(c) (i) Given that $\dfrac{16x-43}{(x-3)^2(x+2)}$ can be written as $\dfrac{a}{(x-3)^2}+\dfrac{b}{x-3}+\dfrac{c}{x+2}$

where a, b and c are real numbers, find a, b and c. **3**

(ii) Hence find $\displaystyle\int\frac{16x-43}{(x-3)^2(x+2)}\,dx$. **2**

(d) In the Argand diagram below, $OABC$ is a rectangle. O is the origin and the distance OA is four times the distance AB. The vertex A is represented by the complex number $z=x+iy$.



Find an expression for the complex number that represents the vertex B. Leave your answer in the form $a+ib$. **2**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $4x^3 - 27x + k = 0$ has a double root. Find the possible values of k. **2**

(b) Let α, β and γ be the roots of the equation $x^3 - 5x^2 + 5 = 0$.

(i) Find a polynomial equation with integer coefficients whose roots are $\alpha - 1, \beta - 1$ and $\gamma - 1$. **2**

(ii) Find a polynomial equation with integer coefficients whose roots are α^2, β^2 and γ^2. **2**

(iii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$. **2**

(c) (i) Let $a > 0$. Find the points where the line $y = ax$ and the curve $y = x(x - a)$ intersect. **1**

(ii) Let R be the region in the plane for which $x(x-a) \le y \le ax$. Sketch R. **1**

(iii) A solid is formed by rotating the region R about the line $x = -2a$. Use the method of cylindrical shells to find the volume of the solid. **5**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Find all the 5th roots of -1 in modulus-argument form. **2**

(ii) Sketch the 5th roots of -1 on an Argand diagram. **1**

(b) For each integer $n \geq 0$, let

$$I_n = \int_0^1 x^{2n+1} e^{x^2}\, dx .$$

(i) Show that for $n \geq 1$, $\quad I_n = \frac{e}{2} - nI_{n-1}$ **2**

(ii) Hence, or otherwise, calculate I_2. **2**

(c) If $5x^2 - y^2 + 4xy = 18$ defines a set of points:

(i) Using implicit differentiation show that is has no stationary points. **2**

(ii) Find the vertical tangents. **2**

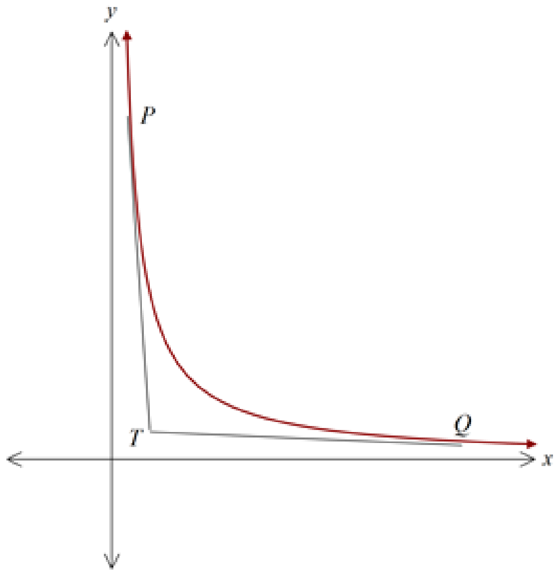
(iii) Find any intercepts. **1**

(iv) Find the oblique asymptotes. **2**

(v) Sketch the curve. **1**

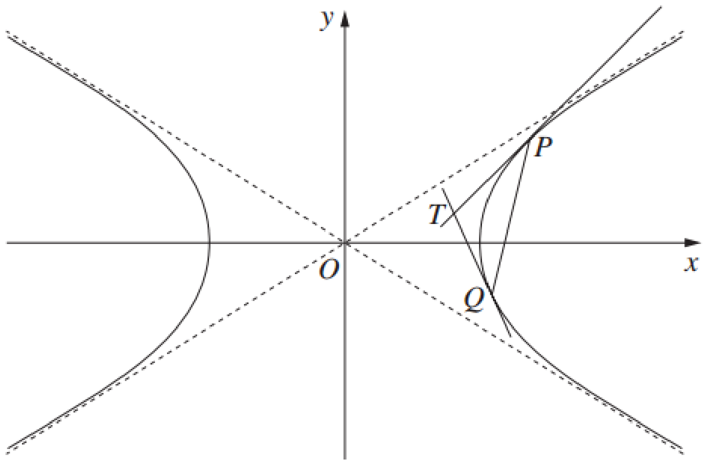
Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$, $p \neq q$, lie on the same branch of the hyperbola $xy = c^2$. The tangents at P and Q meet at the point T.



Find the equation of the tangent to the hyperbola at Q? **2**

(b) The points at $P(x_1, y_1)$ and $Q(x_2, y_2)$ lie on the same branch of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$.



The tangents at P and Q meet at $T(x_0, y_0)$.

(i) Show that the equation of the tangent at P is $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$. **2**

(ii) Hence show that the chord of contact, PQ, has equation $\frac{xx_0}{a^2} - \frac{yy_0}{b^2} = 1$. **2**

(iii) The chord PQ passes through the focus $S(ae, 0)$, where e is the eccentricity of the hyperbola. Prove that T lies on the directrix of the hyperbola. **1**

(c) In an alien universe, the gravitational attraction between two bodies is proportional to x^{-3}, where x is the distance between their centres.

A particle is projected upward from the surface of a planet with velocity u at time $t = 0$. Its distance x from the centre of the planet satisfies the equation $\ddot{x} = -\dfrac{k}{x^3}$.

(i) Show that $k = gR^3$, where g is the magnitude of the acceleration due to gravity at the surface of the planet and R is the radius of the planet. **1**

(ii) Show that v, the velocity of the particle, is given by $v^2 = \dfrac{gR^3}{x^2} - (gR - u^2)$. **3**

(iii) It can be shown that $x = \sqrt{R^2 + 2uRt - (gR - u^2)t^2}$. (Do NOT prove this.)

Show that if $u \geq \sqrt{gR}$ the particle will not return to the planet. **2**

(iv) If $u < gR$ the particle reaches a point whose distance from the centre of the planet is D, and then falls back.

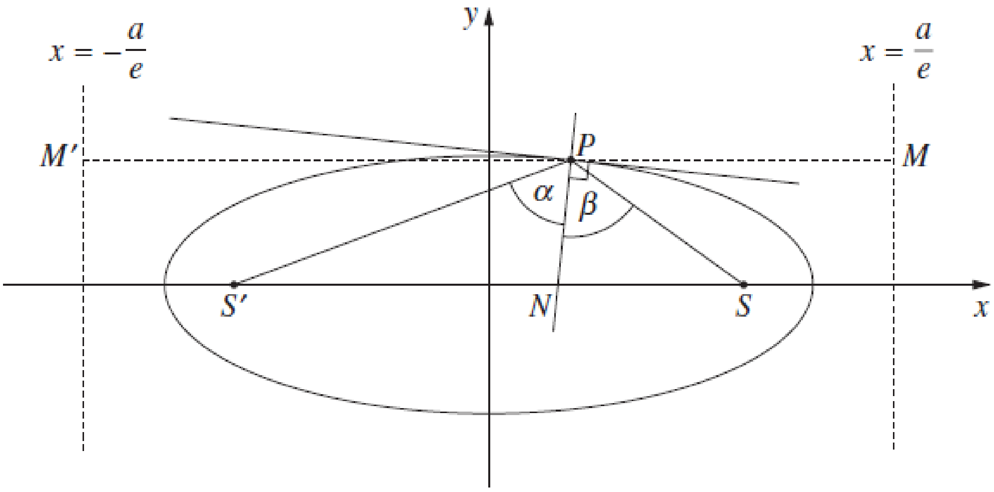
(1) Use the formula in part (ii) to find D in terms of u, R and g. **1**

(2) Use the formula in part (iii) to find the time taken for the particle to return to the surface of the planet in terms of u, R and g. **1**

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) A flywheel of radius 30cm makes 30 revolutions per second. Find the velocity and acceleration of a point on the rim. **2**

(b) The ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ has foci $S(ae,0)$ and $S'(-ae,0)$ where e is the eccentricity, with corresponding directrices $x=\frac{a}{e}$ and $x=-\frac{a}{e}$. The point $P(x_0,y_0)$ is on the ellipse. The points where the horizontal line through P meets the directrices are M and M',
as shown in the diagram below.



(i) Show that the equation of the normal to the ellipse at the point P is

$$y\text{-}y_0=\frac{a^2y_0}{b^2x_0}(x-x_0).$$ **2**

(ii) The normal at P meets the x-axis at N. Show that N has coordinates $(e^2x_0,0)$. **2**

(iii) Using the focus-directrix definition of an ellipse, or otherwise,
show that $\frac{PS}{PS'}=\frac{NS}{NS'}$ **2**

(iv) Let $\alpha=\angle S'PN$ and $\beta=\angle NPS$. By applying the sine rule to $\angle S'PN$ and to $\angle NPS$,
show that $\alpha=\beta$. **2**

(c) The gravitational force between two objects of masses m and M placed at a distance x apart is proportional to their masses and inversely proportional to the square of their distance, ie $F \propto \frac{Mm}{x^2}$. A satellite is launched so that it orbits the earth once a day. Take gravity at the earth's surface, $g = 9.8ms^{-2}$ and the radius of the earth, $R = 6400km$.

(i) Find the angular velocity of the satellite. **1**

(ii) Show that the centripetal force of the satellite $mr\omega^2$ is equal to $\frac{(6.4 \times 10^6)^2 \times 9.8m}{x^2}$. **2**

(iii) Hence find the height of the satellite. **1**

(iv) Find the linear velocity of the satellite. **1**

END OF ASSESSMENT

STANDARD INTEGRALS

$$\int x^n\,dx = \frac{1}{n+1}x^{n+1}, \quad n \neq -1; \quad x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x}\,dx = \ln x, \quad x > 0$$

$$\int e^{ax}\,dx = \frac{1}{a}e^{ax}, \quad a \neq 0$$

$$\int \cos ax\,dx = \frac{1}{a}\sin ax, \quad a \neq 0$$

$$\int \sin ax\,dx = -\frac{1}{a}\cos ax, \quad a \neq 0$$

$$\int \sec^2 ax\,dx = \frac{1}{a}\tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax\,dx = \frac{1}{a}\sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2+x^2}\,dx = \frac{1}{a}\tan^{-1}\frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2-x^2}}\,dx = \sin^{-1}\frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2-a^2}}\,dx = \ln\left(x+\sqrt{x^2-a^2}\right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2+a^2}}\,dx = \ln\left(x+\sqrt{x^2+a^2}\right)$$

NOTE : $\ln x = \log_e x, \quad x > 0$

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) SECTION I (MCQ)	**comments**

1. $\int x(9-4x^2)^{-\frac{1}{2}}dx = \frac{1}{-8\times\frac{1}{2}}(9-4x^2)^{\frac{1}{2}}$

$= -\frac{1}{4}\sqrt{9-4x^2}$

part (A)

2. $\int \frac{1}{x^2-6x+13}dx = \int \frac{1}{x^2-6x+9+4}dx$

$= \int \frac{1}{(x-3)^2+4}dx$

part (C) $= \frac{1}{2}\tan^{-1}\left(\frac{x-3}{2}\right)+c.$

3. part (B)

4. part (A)

5. $z=2+i \quad w=1-i$

$3z+w = 6+3i+i-i^2 = 7+4i$

part (C)

6. $P(z) = z^3+az^2+bz+10$

prod $=-10 \quad \alpha=3+i \quad \beta=3-i$

$\alpha\beta=10$

$\alpha\beta\gamma=-10 \Rightarrow \gamma=-1$

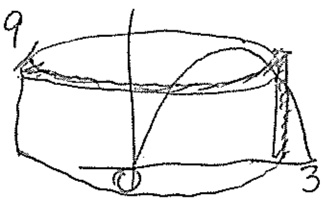
$\therefore P(z)=(z+1)(z^2-6z+10)$ part (D)

7. $b^2=a^2(1-e^2) \quad e<1$ ellipse

$e^2=\frac{a^2-b^2}{a^2}=\frac{4-3}{4}$ part (B) $e=\frac{1}{2}$

8. $\frac{x^2}{4}-\frac{y^2}{3}$ when $y=0 \quad x=\pm 2$

vertices $(\pm 2, 0)$ part (B)

9. 

Shells $\delta V = 2\pi r h\,\delta x$

$= 2\pi x(3x-x^2)\delta x$

$V = 2\pi\int_0^3 3x^2-x^3\,dx$

$= 2\pi\left[x^3-\frac{x^4}{4}\right]_0^3$

$= 2\pi\left(27-\frac{81}{4}\right) = \frac{27\pi}{2}$ part (D)

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) SECT 1	**comments**
10/ $F = \dfrac{m}{x^3}(6-10x)$ $\ddot{x} = \dfrac{6-10x}{x^3} = 6x^{-3} - 10x^{-2}$ $\frac{1}{2}v^2 = \dfrac{6x^{-2}}{-2} + \dfrac{10x^{-1}}{1} + c$ $v^2 = -6x^{-2} + 20x^{-1} + 2c$ when $x = 1$, $v = 0$ $0 = -6 + 20 + 2c \Rightarrow 2c = -14$ $v^2 = -6x^{-2} + 20x^{-1} - 14$ $= \dfrac{1}{x^2}(-6 + 20x - 14x^2)$ $v = \pm \dfrac{1}{x}\sqrt{2(-3 + 10x - 7x^2)}$ part (c)	

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) Question 11	comments

(a) (i) $\omega = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right) = 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= 2\operatorname{cis}\frac{\pi}{6}$ ① mark.

(ii) $z^3 = 2^3 \operatorname{cis} 3\times\frac{\pi}{3} = 8\operatorname{cis}\pi$ ① Mark

$z^3\omega = 8\operatorname{cis}\pi \cdot 2\operatorname{cis}\frac{\pi}{6}$

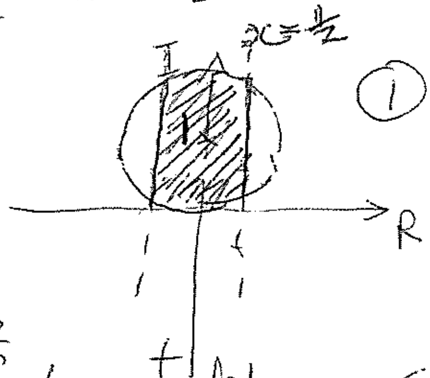
$= 16\operatorname{cis}\frac{7\pi}{6}$ ① Mark

$= 16\left(\cos\left(-\frac{5\pi}{6}\right) + i\sin\left(-\frac{5\pi}{6}\right)\right)$

(b) $|z+\bar{z}| \le 1$

$|2x| \le 1$

$-\frac{1}{2} \le x \le \frac{1}{2}$ ① Mark



① Mark.

(c) $\int_0^2 t e^{-t}\,dt$ ① Mark. $\begin{cases} u = t & v' = e^{-t} \\ u' = 1 & v = -e^{-t} \end{cases}$

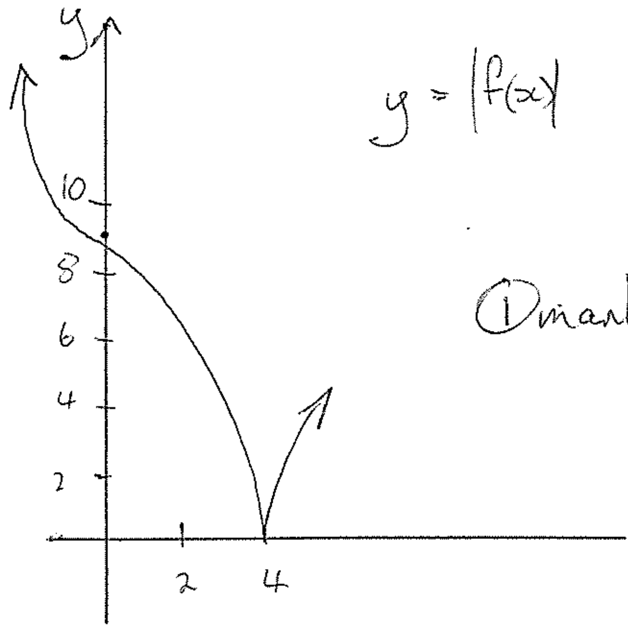
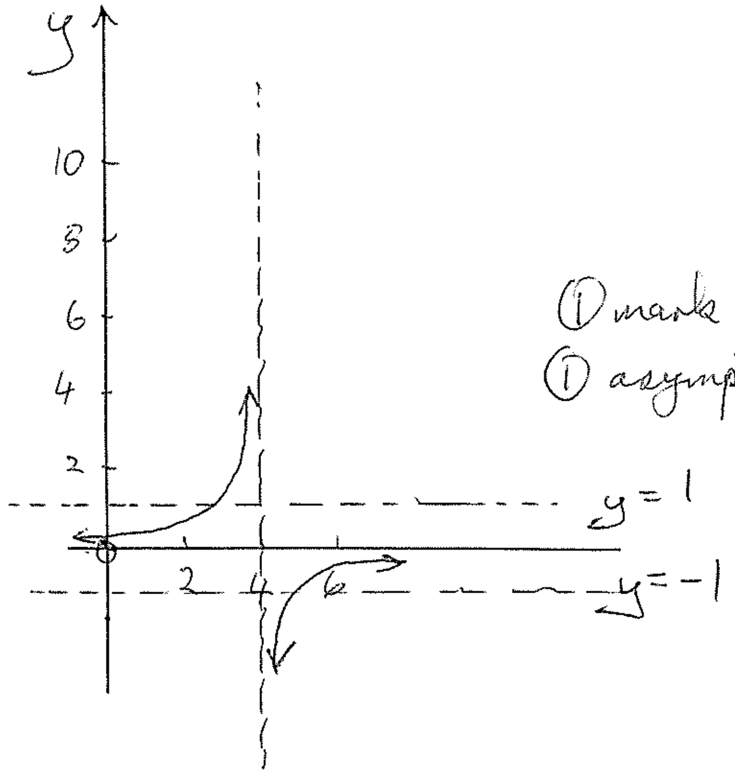
$= \left[-te^{-t}\right]_0^2 + \int_0^2 e^{-t}\,dt$ ① Mark

$= -2e^{-2} + \left[-e^{-t}\right]_0^2$

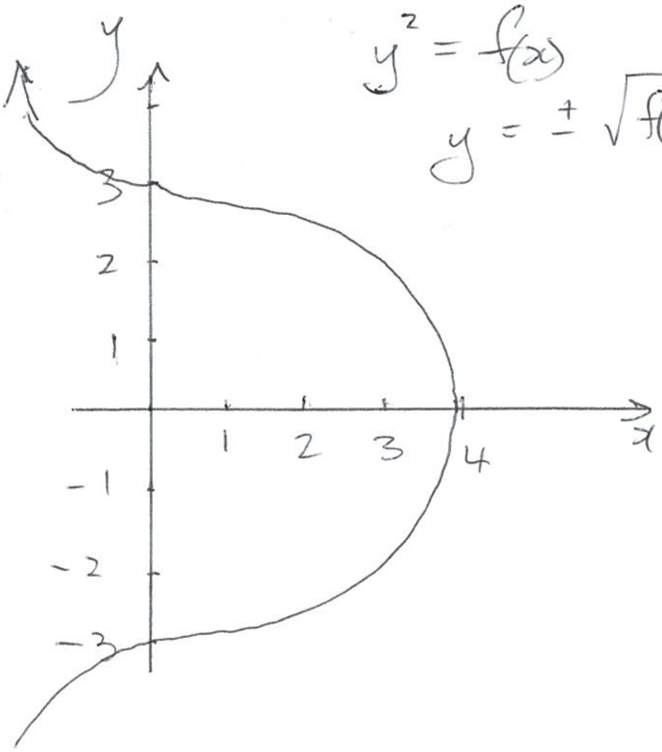
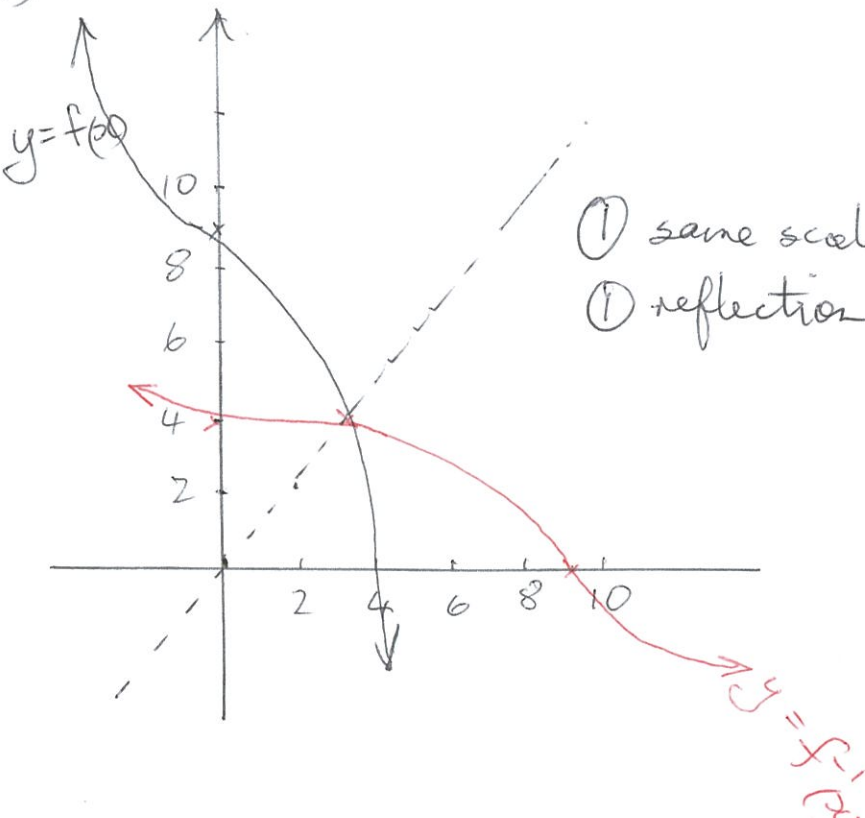
$= -2e^{-2} - e^{-2} + 1$

$= 1 - 3e^{-2}$ ① Mark.

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) Question 11	**comments**
(d) (i)  $y = \|f(x)\|$ ① mark	
ii)  ① mark shap ① asymptote	

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s)	**comments**
iii) $y^2 = f(x)$ $y = \pm\sqrt{f(x)}$  (iv)  (1) same scales (1) reflection	

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) Question 12	comments

(a)(i) $a + ib = \sqrt{3+4i}$

$a^2 - b^2 + 2abi = 3 + 4i$ ① mark

$a^2 - b^2 = 3 \qquad ab = 2$ ① mark

$a = \pm 2 \qquad b = \pm 1$ ① mark

$\therefore$ square roots are $\pm(2+i)$

(ii) $z^2 + iz - 1 - i = 0$

$z = \dfrac{-i \pm \sqrt{i^2 + 4(1+i)}}{2}$ ① mark

$= \dfrac{-i \pm \sqrt{-1+4+4i}}{2}$

$= \dfrac{-i \pm \sqrt{3+4i}}{2}$

$= \dfrac{-i \pm (2+i)}{2}$

$= \dfrac{2}{2}$ or $\dfrac{-2-2i}{2}$

$= 1, \ -1-i$ ① mark

(b) $t = \tan\dfrac{\theta}{2}$

$dt = \dfrac{1}{2}\sec^2\dfrac{\theta}{2}\, d\theta$

$dt = \dfrac{1}{2}\left(1 + \tan^2\dfrac{\theta}{2}\right)d\theta$

$d\theta = \dfrac{2}{1+t^2}\, dt$ deriv & limits ① mark

when $\theta = \dfrac{2\pi}{3}$ $\quad t = \sqrt{3}$

$= \dfrac{\pi}{2}$ $\quad t = 1$

$\displaystyle\int_1^{\sqrt{3}} \frac{\frac{2}{1+t^2}}{\frac{2t}{1+t^2}}\, d\theta = \int_1^{\sqrt{3}} \frac{1}{t}\, dt = \Big[\ln t\Big]_1^{\sqrt{3}}$

$= \ln\sqrt{3} = \dfrac{1}{2}\ln 3$

① mark ① mark

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) Question 12	comments
(c)(i) $\dfrac{16x-43}{(x-3)^2(x+2)} \equiv \dfrac{a}{(x-3)^2} + \dfrac{b}{(x-3)} + \dfrac{c}{x+2}$ $16x-43 \equiv a(x+2) + b(x-3)(x+2) + c(x-3)^2$ Let $x=3$ $48-43 = 5a \Rightarrow a = 1$ (1) mark Let $x=-2$ $-32-43 = 25c \Rightarrow c = -3$ (1) mark Let $x=0$ $-43 = 2a - 6b + 9c$ $-43 = 2 - 6b - 27$ $6b = 18 \Rightarrow b = 3$ (1) mark (ii) $\therefore \displaystyle\int \frac{16x-43}{(x-3)^2(x+2)}\,dx = \int \frac{1}{(x-3)^2} + \frac{3}{x-3} - \frac{3}{x+2}\,dx$ $= -(x-3)^{-1} + 3\ln(x-3) - 3\ln(x+2)$ $= \dfrac{-1}{x-3} + 3\ln\left(\dfrac{x-3}{x+2}\right) + c$ (1) mark (1) mark. (d) $\overrightarrow{AB} = \frac{1}{4} i \overrightarrow{AO}$ $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB}$ (1) mark $= x + iy + \frac{i}{4}(x+iy)$ $= (x - \frac{1}{4}y) + i(y + \frac{1}{4}x)$ (1) mark	

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) Question 13	**comments**

(a) $P(x) = 4x^3 - 27x + k$

$P'(x) = 12x^2 - 27$

for double root $12x^2 - 27 = 0$

$x^2 = \frac{27}{12} = \frac{9}{4}$

$x = \pm\frac{3}{2}$ (1) mark

when $x = \frac{3}{2}$ $P(x) = \cancel{4}\times\frac{27}{\cancel{8}2} - 27\times\frac{3}{2} + k$

$= -27\times\frac{2}{2} + k$

$k = 27$

$x = -\frac{3}{2}$ $P(x) = -\cancel{4}\times\frac{27}{2\cancel{8}} + \frac{27}{2}\times 3 + k$

$= \frac{2}{2}\times 27 + k$

$k = -27$ (1) mark

(b) $x^3 - 5x^2 + 5 = 0$

(i) roots α, β, γ

Equat with roots $\alpha - 1, \beta - 1, \gamma - 1$

$y = x - 1$ $x = 1 + y$ (1) mark

$(1+y)^3 - 5(1+y)^2 + 5 = 0$

$y^3 + 3y^2 + 3y + 1 - 5 - 10y - 5y^2 + 5 = 0$

$y^3 - 2y^2 - 7y + 1 = 0$ (1) mark

(ii) for roots $\alpha^2, \beta^2, \gamma^2$

$y = x^2 \Rightarrow x = \sqrt{y}$ (1) mark

$(y^{\frac{1}{2}})^3 - 5(y^{\frac{1}{2}})^2 + 5 = 0$

$y^{3/2} - 5y + 5 = 0$

$y^{3/2} = 5y - 5$

$y^3 = 25y^2 - 50y + 25$

$y^3 - 25y^2 + 50y - 25 = 0$ (1) mark

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) Quest 13	**comments**

(b)(iii) for roots $\alpha^3+\beta^3+\gamma^3$

$$\alpha^3 - 5\alpha^2 + 5 = 0$$
$$\beta^3 - 5\beta^2 + 5 = 0$$
$$\gamma^3 - 5\gamma^2 + 5 = 0$$

$$\alpha^3+\beta^3+\gamma^3 - 5(\alpha^2+\beta^2+\gamma^2) + 15 = 0$$

$$\alpha^3+\beta^3+\gamma^3 = 5(\alpha^2+\beta^2+\gamma^2) - 15$$

1 mark

$$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$$
$$= 5^2 - 2\times 0$$
$$= 25$$

$$\therefore \alpha^3+\beta^3+\gamma^3 = 5\times 25 - 15$$
$$= 110$$ 1 mark

(c) (i) $y = ax$ ①

$y = x(x-a) = x^2 - ax$ ②

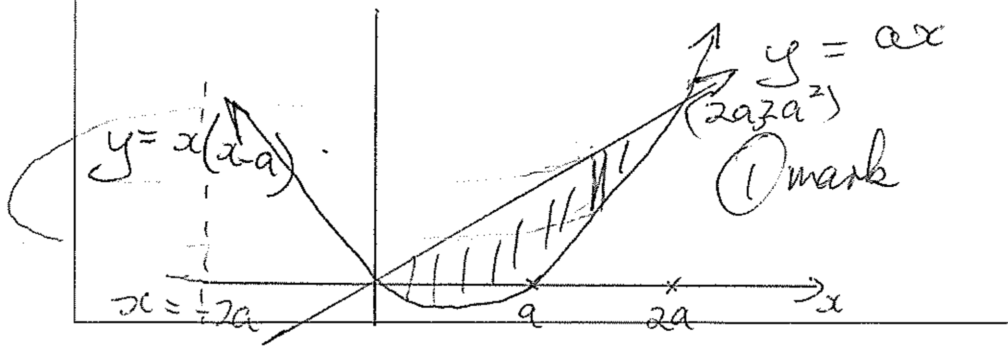
equate ① & ②

$$x^2 - 2ax = 0$$
$$x(x-2a) = 0$$
$$x = 0,\ 2a$$

when $x = 0$, $y = 0$

$x = 2a$, $y = 2a^2$

pts $(0,0)$ $(2a, 2a^2)$ 1 mark



Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) Question 13	comments
(c) $\delta V = \underbrace{2\pi r h}_{\text{curved surface area of cylinder}} . \delta x$ ① mark for r $\delta V = 2\pi(2a+x)(y_1-y_2)\delta x$ ① mark for y $= 2\pi(2a+x)(ax-x^2+ax)\delta x$ $V = 2\pi\int_0^{2a}(2a+x)(2ax-x^2)\,dx$ ① mark $= 2\pi\int_0^{2a} 4a^2x + \cancel{2ax^2} - \cancel{3ax^2} - x^3\,dx$ $= 2\pi\left[\cancel{4}2a^2\frac{x^2}{\cancel{2}} - \frac{x^4}{4}\right]_0^{2a}$ ① mark $= 2\pi\left[2a^2\times 4a^2 - \frac{16a^4}{4}\right]$ $= 2\pi(8a^4 - 4a^4)$ $= 8\pi a^4$ cu. units ① mark	

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) Quest 14	**comments**

(a)(i) $z^5 = -1 = \text{cis}(\pi + 2\pi k)$

$z = \text{cis}\left(\frac{\pi + 2\pi k}{5}\right)$ (1) mark

$k=0 \quad z = \text{cis}\frac{\pi}{5}$

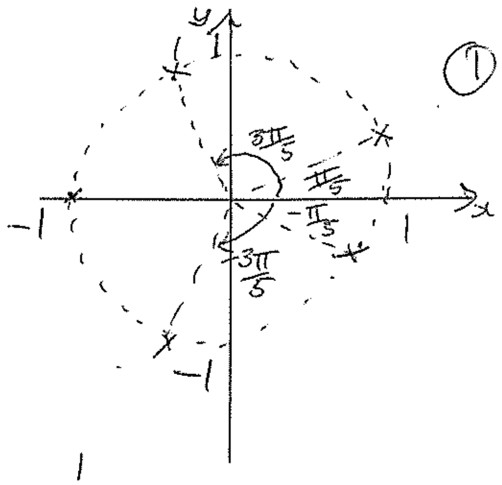
$k=1 \quad z = \text{cis}\frac{3\pi}{5}$

$k=-1 \quad z = \text{cis}-\frac{\pi}{5}$

$k=2 \quad z = \text{cis}\,\pi$

$k=-2 \quad z = \text{cis}-\frac{3\pi}{5}$

(1) mark



(1) mark

(b) $I_n = \int_0^1 x^{2n+1} e^{x^2} dx$ (1) mark

$u = x^{2n}, \quad v' = x e^{x^2}$

$u' = 2n x^{2n-1}, \quad v = \frac{1}{2} e^{x^2}$

$I_n = \left[\frac{1}{2} x^{2n} \cdot e^{x^2}\right]_0^1 - \int_0^1 n x^{2n-1} e^{x^2} dx$

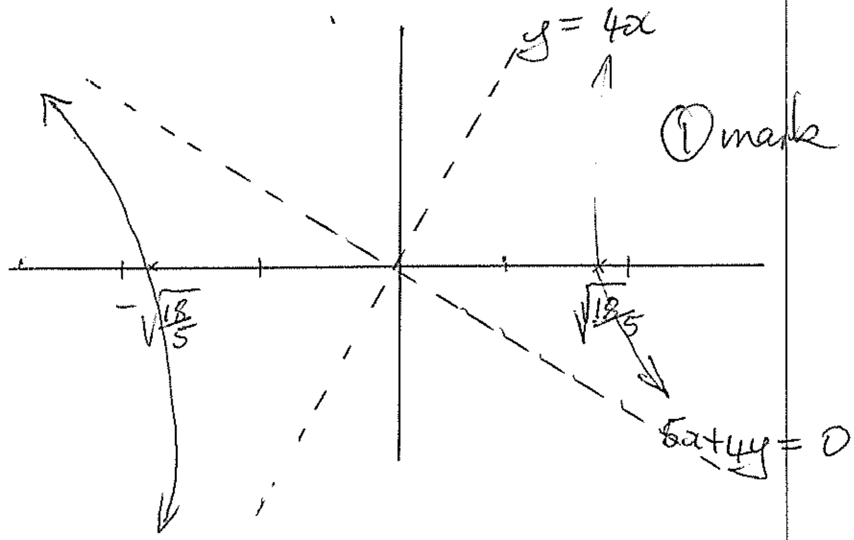
$= \frac{1}{2} e - n\int_0^1 x^{2n-1} e^{x^2} dx$ (1) mark

$I_n = \frac{e}{2} - n I_{n-1}$

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) Quest 14	comments
6 (ii) $I_2 = \frac{e}{2} - 2I_1$ $I_1 = \frac{e}{2} - I_0$ ① mark for pattern $I_0 = \int_0^1 x e^{x^2} dx$ $= \left[\frac{1}{2}e^{x^2}\right]_0^1 = \frac{1}{2}e - \frac{1}{2}$ $I_1 = \cancel{\frac{e}{2}} - \cancel{\frac{e}{2}} + \frac{1}{2}$ $I_2 = \frac{e}{2} - 2\times\frac{1}{2} = \frac{e}{2} - 1$ ① mark (c) $5x^2 - y^2 + 4xy = 18$ (i) $10x - 2y\frac{dy}{dx} + 4y + 4x\frac{dy}{dx} = 0$ $\frac{dy}{dx}(4x - 2y) = -10x - 4y$ $\frac{dy}{dx} = \frac{10x + 4y}{2y - 4x} = \frac{5x + 2y}{y - 2x}$ for stat pt $5x + 2y = 0$ ① mark $y = -\frac{5}{2}x$ $5x^2 - \frac{25}{4}x^2 - 4x \times \frac{5}{2}x = 18$ $5x^2 - \frac{25}{4}x^2 - 10x^2 = 18$ ① mark no real solutions (ii) for vert. tangents $y - 2x = 0$ $y = 2x$ ① mark $5x^2 - 4x^2 + 4x \times 2x = 18$ $13x^2 - 4x^2 = 18$ ① mark $x = \pm\sqrt{2}$ $y = \pm 2\sqrt{2}$	

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) Quest 14	comments
(c)(iii) when $x = 0$ $\quad -y^2 = 18$ $\therefore$ no y intercepts when $y = 0$ $\quad 5x^2 = 18$ $x = \pm\sqrt{\frac{18}{5}}$ (iv) for oblique asymptotes $5x - \frac{y^2}{x} + 4y = \frac{18}{y}$ ① mark as $x \to \infty$ $\quad 5x + 4y \Rightarrow 0$ $5x + 4y = 0$ is asymptote $\frac{5x^2}{y} - y + 4x = \frac{18}{y}$ as $y \to \infty$ $\quad -y + 4x \Rightarrow 0$ $4x - y = 0$ is asymptote ① mark  ① mark	

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) Question 15	comments
(a) $xy = c^2$ $y + x\frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{y}{x}$ (1) mark at $(cq, \frac{c}{q})$ equation of tang. $\frac{y - \frac{c}{q}}{x - cq} = -\frac{\frac{c}{q}}{cq}$ $\frac{y - \frac{c}{q}}{x - cq} = -\frac{1}{q^2}$ $q^2y - cq = -x + cq$ $x + q^2y = 2cq$ (1) mark (b) $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ $\frac{2x}{a^2} - \frac{2y}{b^2}\frac{dy}{dx} = 0$ $\frac{dy}{dx} = \frac{2x}{a^2} \div \frac{2y}{b^2}$ $= \frac{b^2x}{a^2y}$ (1) mark. at $P(x_1, y_1)$ equat of tangent $\frac{y - y_1}{x - x_1} = \frac{b^2x_1}{a^2y_1}$ $a^2yy_1 - a^2y_1^2 = b^2xx_1 - b^2x_1^2$ $b^2xx_1 - a^2yy_1 = b^2x_1^2 - a^2y_1^2$ $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} - \frac{y_1^2}{b^2}$ ie $\frac{xx_1}{a^2} - \frac{yy_1}{b^2} = 1$ (1) mark	

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) 15	**comments**
(b) similarly tangent thru Q is $\frac{x x_2}{a^2} - \frac{y y_2}{b^2} = 1$ If $T(x_0, y_0)$ satisfies both equats then $\frac{x_0 x_2}{a^2} - \frac{y_0 y_2}{b^2} = 1$ (1) mark and $\frac{x_0 x_1}{a^2} - \frac{y_0 y_1}{b^2} = 1$ then $\frac{x x_0}{a^2} - \frac{y y_0}{b^2} = 1$ satisfies P & Q (1) mark & hence it is chord of contact (iii) If PQ passes thru $S(ae, 0)$ $\frac{ae\, x_0}{a^2} - 0 = 1$ (1) mark. $x_0 = \frac{a}{e}$ which is on directrix (c) $\ddot{x} = -\frac{k}{x^3}$ (i) u ↑ ↓ g when $x = R$ $\ddot{x} = -g$ $-g = -\frac{k}{R^3} \Rightarrow k = gR^3$ (1) mark (ii) $\ddot{x} = -\frac{gR^3}{x^3}$ $\frac{d(\frac{1}{2}v^2)}{dx} = -gR^3 x^{-3}$ (1) mark $\frac{1}{2}v^2 = g\frac{R^3}{2} x^{-2} + c$ when $x = R$, $v = u$ $\frac{u^2}{2} = g\frac{R}{2} + c$ (1) mark $c = \frac{u^2 - gR}{2}$	

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) 15	comments
$v^2 = \frac{gR^3}{x^2} + u^2 - gR$ (1) mark. $v^2 = \frac{gR^3}{x^2} - (gR - u^2)$ (iii) $x = \sqrt{R^2 + 2uRt - (gR-u^2)t^2}$ If $u \geq \sqrt{gR}$ (1) mark $u^2 \geq gR$ then $x = \sqrt{R^2 + 2uRt + ct^2}$ where $c \geq 0$ (1) mark $\therefore x > R$ particle does not return (iv) i, $v^2 = \frac{gR^3}{x^2} - (gR - u^2)$ when $x = D$ $v = 0$ $0 = \frac{gR^3}{D^2} - (gR - u^2)$ $\frac{gR^3}{D^2} = gR - u^2$ $D^2 = \frac{gR^3}{gR - u^2}$ $D = \cancel{\pm} \sqrt{\frac{gR^3}{gR-u^2}}$ (1) mark 2, for time taken $x = R$ $R = \sqrt{R^2 + 2uRt - (gR-u^2)t^2}$ $\cancel{R^2} = \cancel{R^2} + 2uRt - (gR-u^2)t^2$ $(gR-u^2)t^2 - 2uRt = 0$ $t[(gR-u^2)t - 2uR] = 0$ $t = 0$ or $t = \frac{2uR}{gR-u^2}$ (1) mark	

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) Quest 16	comments
(a) $\omega = 30 \times 2\pi$ rad/sec $= 60\pi$ $V = r\omega = 1800\pi$ cm/sec. (1) mark $a = r\omega^2 = 30 \times (60\pi)^2$ $= 108{,}000\pi^2$ cm s^{-2} (b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ (i) $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$ $\frac{dy}{dx} = -\frac{2x}{a^2} \div \frac{2y}{b^2}$ $= -\frac{b^2}{a^2}\frac{x}{y}$ (1) mark equat of normal at P $\frac{y - y_0}{x - x_0} = + \frac{y_0}{x_0}\frac{a^2}{b^2}$ (1) mark $\therefore y - y_0 = \frac{a^2 y_0}{b^2 x_0}(x - x_0)$ (ii) for $y = 0$ $-\cancel{y_0} = \frac{a^2}{b^2}\frac{\cancel{y_0}}{x_0}(x - x_0)$ (1) mark $-b^2 x_0 = a^2 x - a^2 x_0$ $a^2 x = a^2 x_0 - b^2 x_0$ $x = x_0\left(\frac{a^2 - b^2}{a^2}\right)$ $x = e^2 x_0$ (1) mark	

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) Quest 16	comments
iii) $\frac{PS}{PM} = e = \frac{PS'}{PM'}$ 1 mark $\frac{PS}{PS'} = \frac{PM}{PM'} = \frac{\frac{a}{e} - x_0}{\frac{a}{e} + x_0}$ 1 mark. $\frac{NS}{NS'} = \frac{ae - e^2 x_0}{ae + e^2 x_0} = \frac{\frac{a}{e} - x_0}{\frac{a}{e} + x_0}$ $\therefore \frac{PS}{PS'} = \frac{NS}{NS'}$ iv) in $\Delta S'PN$ $\frac{\sin\alpha}{NS'} = \frac{\sin PNS'}{PS'}$ 1 in ΔSPN $\frac{\sin\beta}{NS} = \frac{\sin PNS}{PS}$ 2 1 mark but $\sin PNS = \sin PNS'$ $\therefore \sin\alpha \frac{PS'}{NS'} = \sin\beta \frac{PS}{NS}$ $\sin\alpha \frac{PS'}{PS} = \frac{NS'}{NS}\sin\beta$ 1 mark ie $\sin\alpha = \sin\beta$ from (iii) ie $\alpha = \beta$ (c)(i) $\omega = \frac{2\pi}{24 \times 3600} \doteqdot 7.3 \times 10^{-5}$ rad, s^{-1} 1 mark (ii) $F = k\frac{Mm}{x^2}$	

Suggested Solutions, Marking Scheme and Markers' comments

Suggested solution(s) quest 16	comments
c (ii) (contin) at earths surface $R = 6400\,km \quad F = mg$ $\cancel{m}g = \frac{kM\cancel{m}}{R^2}$ (1) mark $kM = gR^2$ $= 9.8 \times (6400 \times 1000)^2$ $= 9.8 \times (6.4 \times 10^6)^2$ $F = \frac{9.8 \times (6.4 \times 10^6)^2}{x^2} \cdot \frac{m}{1}$ (1) mark. for satellite in uniform motion $\cancel{m} x \omega^2 = \frac{9.8 \times (6.4 \times 10^6)^2 \cancel{m}}{x^2}$ (iii) Hence $x^3 = \frac{9.8 \times (6.4 \times 10^6)^2}{(7.3 \times 10^{-5})^2}$ $x \approx 4.22 \times 10^7\,m$ $= 42232\,km$ (1) mark $- R \approx 36{,}000\,km$ (iv) $v = r\omega$ $= 42200 \times 7.3 \times 10^{-5}\,km\,s^{-1}$ $\doteqdot 3\,km\,s^{-1}$	