6/ $$(3+\sqrt{5})^4 - 6(3+\sqrt{5})^3 + 8(3+\sqrt{5})^2 - 24(3+\sqrt{5}) + 16$$

$$= 81 + 108\sqrt{5} + \cancel{270} + 60\sqrt{5} + 25$$

$$-162 - 162\sqrt{5} - \cancel{270} - 150\sqrt{5} \cancel{+72} + 48\sqrt{5} + 40$$

$$\cancel{-72} - 24\sqrt{5} + 16$$

$$= 0$$

6/ $x^4 - 6x^3 + 8x^2 - 24x + 16$

roots $2i, -2i, \alpha, \beta$

$\alpha + \beta = 6$ $\quad$ $4\alpha\beta = 16$

$\alpha\beta = 4$

$x^2 - 6x + 4 = 0$

$x = \frac{6 \pm \sqrt{20}}{2}$

$x = 3 \pm \sqrt{5}$

9b) $P(x) = x^3 + kx^2 + 6$

a) show $P(2i) = (6-4k) - 8i$

b) $P(x) = (x^2+4)Q(x) - 4x - 6$

$P(2i) = -8i - 6$

$\therefore\ 6 - 4k - 8i = -8i - 6$

$4k = 12$

$\underline{\underline{k = 3}}$

11) $P(x) = x^3 + x^2 + 6x - 3$

$P(-2i) = -7 - 4i$

$P(ki) = -k^3 i - k^2 + 6ki - 3 = -k^2 - 3 + i(-k^3 + 6k)$

$P(-ki) = k^3 i - k^2 - 6ki - 3 = -k^2 - 3 + i(k^3 - 6k)$

$\therefore P(2i) = -7 + 4i$

$$x^3 + x^2 + 6x - 3 = (x^2+4)Q(x) + ax + b.$$

$$P(-2i) = -7-4i$$

$$-2ia + b = -7-4i$$

$$b = -7$$

$$a = 2$$

$$\therefore \underline{R(x) = 2x - 7}$$

12b) $P(z) = z^8 - \frac{5}{2}z^4 + 1$

a) $iw, \frac{1}{w}, w$ are roots

$w, \overline{w}, iw, i\overline{w}, \frac{1}{w}, \overline{\left(\frac{1}{w}\right)}, \frac{i}{w}, \overline{\frac{i}{w}}$

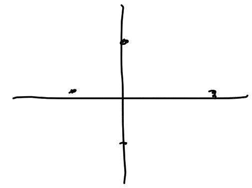
$$P(z) = \left(z^4 - \frac{5}{4}\right)^2 - \frac{9}{16}$$

$$= \left(z^4 - \frac{5}{4} + \frac{3}{4}\right)\left(z^4 - \frac{5}{4} - \frac{3}{4}\right)$$

$$= \left(z^4 - \frac{1}{2}\right)\left(z^4 - 2\right)$$

$$z^4 - 2 = 0$$

$$z^4 = 2$$



$$z^4 = 1$$

$$z = \sqrt[4]{2},\ -\sqrt[4]{2},\ \sqrt[4]{2}\,i,\ -\sqrt[4]{2}\,i,$$

$$\frac{1}{\sqrt[4]{2}},\ -\frac{1}{\sqrt[4]{2}},\ -\frac{i}{\sqrt[4]{2}},\ \frac{i}{\sqrt[4]{2}}$$

10b) $P(x) = x^3 - x^2 + mx + n$

a) $P(-i) = (1+n) + i(1-m)$

b) $P(x) = (x^2+1)Q(x) + 6x - 3$

$P(-i) = -6i - 3$

$\therefore (1+n) + i(1-m) = -6i - 3$

$n + 1 = -3 \qquad 1 - m = -6$

$\underline{n = -4} \qquad \underline{m = 7}$

13/ $P(x) = x^4 + Ax^2 + B$

$x^4 \geq 0$, $Ax^2 \geq 0$, $B \neq 0$

$\therefore P(x) \neq 0$

$\therefore$ no real zeroes

b) if ic, id are zeroes $\therefore -ic$ and $-id$

are the other zeroes

as coefficients are real

$\therefore$ solutions are conjugate pairs

c) Prove: $c^4 + d^4 = A^2 - 2B$

$P(x) = x^4 + Ax^2 + B$

$P(ic) = 0 \qquad c^4 - Ac^2 + B = 0$

$P(id) = 0 \qquad d^4 - Ad^2 + B = 0$

$$c^4 + d^4 - A(c^2 + d^2) + 2B = 0$$

$$\sum \alpha\beta = A$$

$$A = (ic)(-ic) + (ic)(id) + (ic)(-id) + (-ic)(id) + (-ic)(-id) + (id)(-id)$$

$$A = c^2 + d^2$$

$$\therefore \quad c^4 + d^4 - A^2 + 2B = 0$$

$$\underline{c^4 + d^4 = A^2 - 2B}$$

14. $P(x) = x^3 + cx + d$

zeroes: $k < 0, \ \alpha, \ \bar{\alpha}$

$P'(x) = 3x^2 + c$

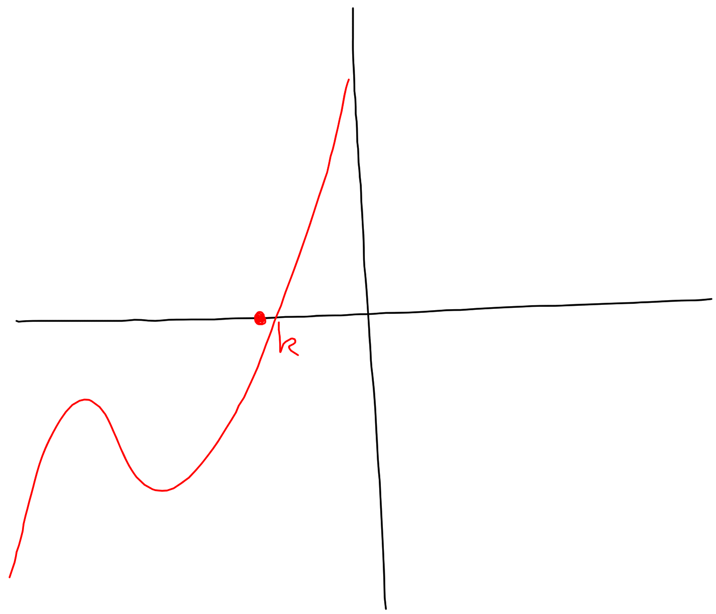
let one of the turning points be β

$\therefore P'(\beta) = 0$

$3\beta^2 + c = 0$

$\therefore c < 0$

c)



d) $\sum \alpha = 0$

$k + a + ib + a - ib = 0$

$k + 2a = 0$

$2a = -k$

$\therefore 2a > 0$

$\underline{a > 0}$

e) $d = 8a^3 + 2ac$

$x^3 + cx + d$

roots $k,\ a \pm ib$

$$(a+ib)^3 + c(a+ib) + d = 0$$

$$a^3 + 3a^2bi - 3ab^2 - b^3 i + ac + bci + d = 0$$

$$a^3 - 3a^2bi - 3ab^2 + b^3 i + ac - bci + d = 0$$

$$2a^3 - 6ab^2 + 2ac + 2d = 0$$

$$\mathrm{Im}(P(x)) = 0$$

$$3a^2b - b^3 + bc = 0$$

$$3a^2 - b^2 + c = 0$$

$$b^2 = 3a^2 + c$$

$$a^3 - 3a(3a^2 + c) + ac + d = 0$$

$$a^3 - 9a^3 - 3ac + ac + d = 0$$

$$-8a^3 - 2ac + d = 0$$

$$\underline{d = 8a^3 + 2ac}$$

15) $f(x) = x^3 - 3x + k$, $k > 2$

a) $f'(x) = 3x^2 - 3$

$= 3(x+1)(x-1)$

$f''(x) = 6x$

$\underline{\underline{x=-1}}$ $f(-1) = 2+k > 4$

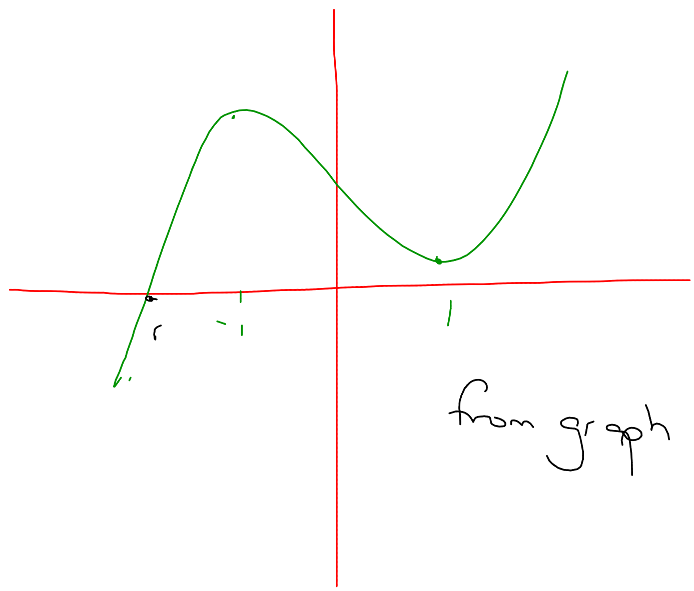
$f''(-1) = -6 < 0$

$\therefore$ max tp.

$\underline{\underline{x=1}}$ $f(1) = -2+k > 0$

$f''(1) = 6 > 0$

$\therefore$ min tp.



from graph $r < -1$

c) roots; r, $a \pm ib$

$\sum \alpha\beta = -3$ $\sum \alpha = 0$

$r + a + ib + a - ib = 0$

$r = -2a$

$r(a+ib) + r(a-ib) + a^2+b^2 = -3$

$2ar + a^2 + b^2 = -3$

$-4a^2 + a^2 + b^2 = -3$

$-3a^2 + b^2 = -3$

$\underline{b^2 = 3(a^2-1)}$

d) $k = 2702$

$$P(x) = x^3 - 3x + 2702$$

$$b^2 = 3(a^2 - 1)$$

roots $a+ib, a-ib, -2a$

$$-2a(a^2+b^2) = 2702$$

$$-2a(a^2+3a^2-3) = -2702$$

$$-8a^3 + 6a = -2702$$

$$4a^3 - 3a - 1351 = 0$$

1351

7 193