Rates of Change

In some cases two, or more, rates must be found to get the equation in terms of the given variable.

$$\frac{dy}{dt} = \frac{dy}{dx} \cdot \frac{dx}{dt}$$

e.g. (i) A spherical balloon is being deflated so that the radius decreases at a constant rate of 10 mm/s.
Calculate the rate of change of volume when the radius of the balloon is 100 mm.

$$\frac{dV}{dt} = ? \qquad V = \frac{4}{3}\pi r^3 \qquad \frac{dV}{dt} = \frac{dr}{dt} \cdot \frac{dV}{dr}$$

$$\frac{dr}{dt} = -10 \qquad \frac{dV}{dr} = 4\pi r^2$$

$$= -10\left(4\pi r^2\right)$$

$$= -40\pi r^2$$

$$\text{when } r = 100, \ \frac{dV}{dt} = -40\pi(100)^2$$

$$= -400000\pi$$

$\therefore$ when the radius is 100mm, the volume is decreasing at a rate of $400000\pi \text{mm}^3/\text{s}$

(ii) A spherical bubble is expanding so that its volume increases at a constant rate of $70\text{mm}^3/\text{s}$
What is the rate of increase of its surface area when the radius is 10 mm?

$$\frac{dS}{dt} = ? \qquad \frac{dV}{dt} = 70 \qquad V = \frac{4}{3}\pi r^3 \qquad S = 4\pi r^2$$

$$\frac{dV}{dr} = 4\pi r^2 \qquad \frac{dS}{dr} = 8\pi r$$

$$\frac{dS}{dt} = \frac{dV}{dt} \cdot \frac{dS}{dr} \cdot \frac{dr}{dV}$$

$$= (70)(8\pi r)\left(\frac{1}{4\pi r^2}\right)$$

$$= \frac{140}{r}$$

$$\text{when } r = 10, \frac{dV}{dt} = \frac{140}{10}$$

$$= 14$$

$\therefore$ when radius is 10mm the surface area is increasing at a rate of $14\text{mm}^2/\text{s}$

(iii) 2013 Extension 1 HSC Q13 a)

A spherical raindrop of radius r metres loses water through evaporation at a rate that depends upon its surface area. The rate of change of the volume V of the raindrop is given by

$$\frac{dV}{dt} = -10^{-4}A$$

where t is in seconds and A is the surface area of the rain drop.

a) Show that $\frac{dr}{dt}$ is a constant.

$$\frac{dr}{dt} = ? \qquad \frac{dV}{dt} = -10^{-4}A$$

$$\frac{dr}{dt} = \frac{dV}{dt} \cdot \frac{dr}{dV}$$

$$V = \frac{4}{3}\pi r^3$$

$$\frac{dV}{dr} = 4\pi r^2$$

$$\therefore \frac{dV}{dr} = A$$

$$\frac{dr}{dt} = -10^{-4}A \cdot \frac{1}{A}$$

$$= -10^{-4}$$

$\therefore$ radius decreases at a constant rate of 10^{-4}m/s

b) How long does it take for a raindrop of volume $10^{-6}\,\text{m}^3$ to completely evaporate?

$$V = \frac{4}{3}\pi r^3$$

$$10^{-6} = \frac{4}{3}\pi r^3$$

$$r^3 = \frac{3\times10^{-6}}{4\pi}$$

$$r = \sqrt[3]{\frac{3\times10^{-6}}{4\pi}}$$

$$\frac{dr}{dt} = -10^{-4}$$

$$-10^4 \int_{\sqrt[3]{\frac{3\times10^{-6}}{4\pi}}}^{0} dr = \int_0^t dt$$

$$t = 10^4 \left[r\right]_0^{\sqrt[3]{\frac{3\times10^{-6}}{4\pi}}}$$

$$t = 10^4 \sqrt[3]{\frac{3\times10^{-6}}{4\pi}}$$

$$= 62.03504909\ldots$$

$$= 62 \text{ seconds}$$

$\therefore$ it takes approximately 62 seconds to evaporate

Exercise 7E; 2, 5, 6, 9, 13*

Exercise 7F; 2, 5, 9, 10, 11*