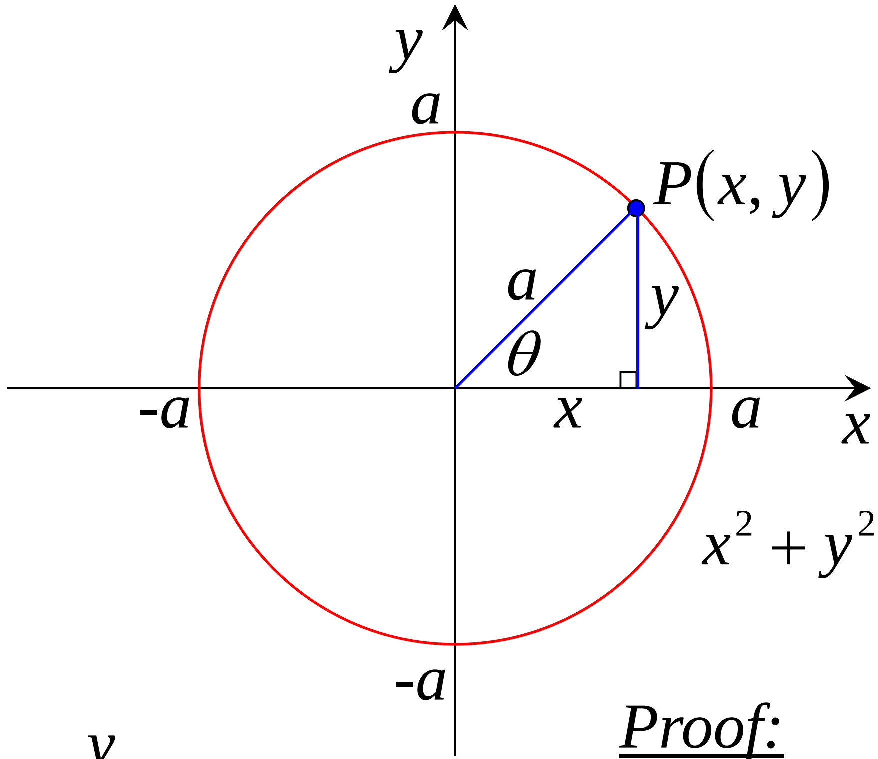
Conics & Parameters

1) **Circle**



$$x^2 + y^2 = a^2$$

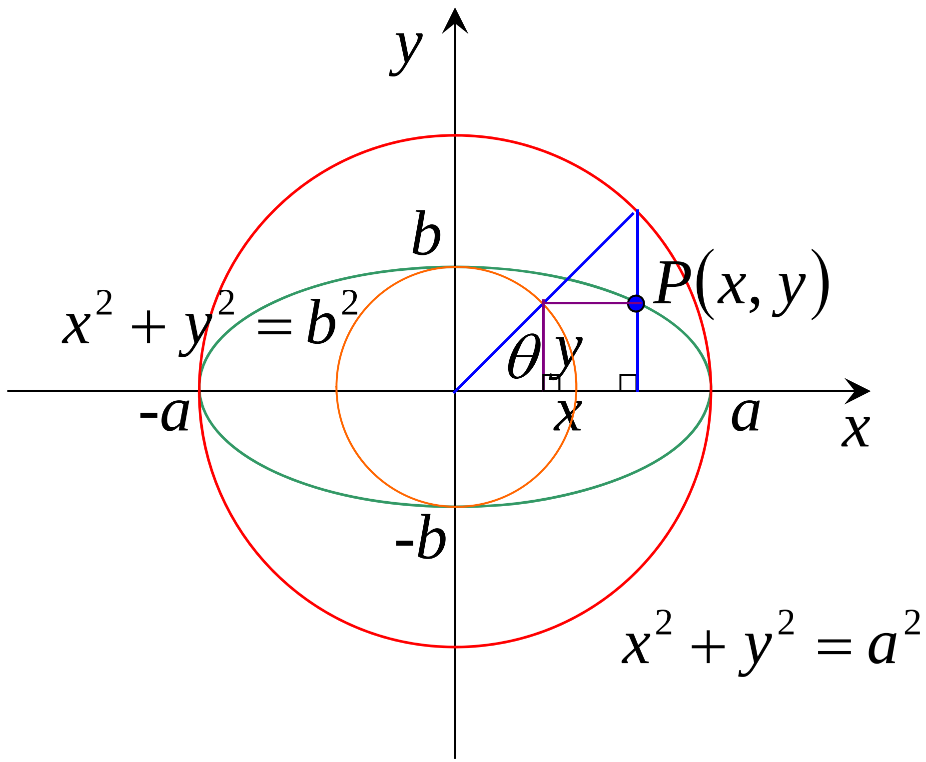
$$\frac{x}{a} = \cos\theta \qquad \frac{y}{a} = \sin\theta$$

$$x = a\cos\theta \qquad y = a\sin\theta$$

Proof:

$$x^2 + y^2 = a^2\cos^2\theta + a^2\sin^2\theta$$
$$= a^2\left(\cos^2\theta + \sin^2\theta\right)$$
$$= a^2$$

2) **Ellipse**



$$\frac{x}{a} = \cos\theta \qquad \frac{y}{b} = \sin\theta$$

$$x = a\cos\theta \qquad y = b\sin\theta$$

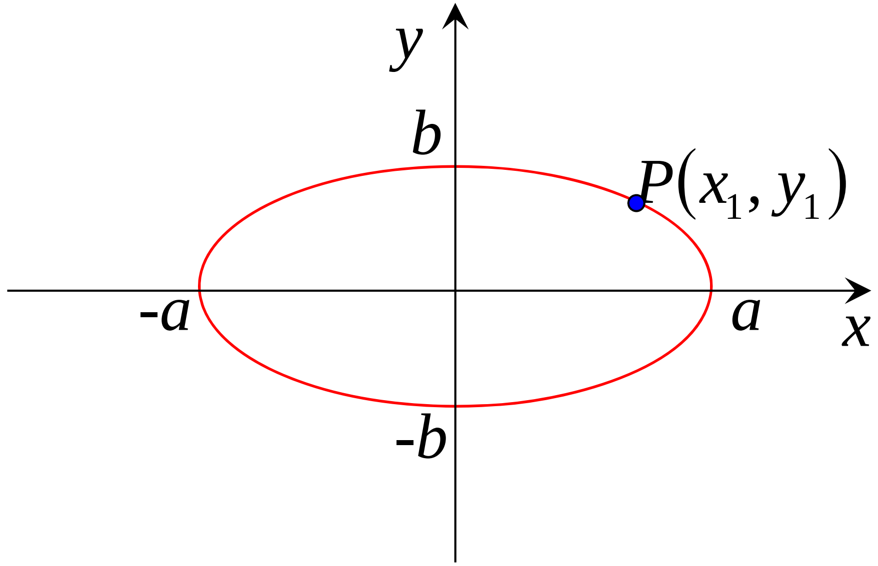
Proof:

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = \frac{a^2\cos^2\theta}{a^2} + \frac{b^2\sin^2\theta}{b^2}$$

$$= \cos^2\theta + \sin^2\theta$$

$$= 1$$

Equation of Tangent and Normal



$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} + \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

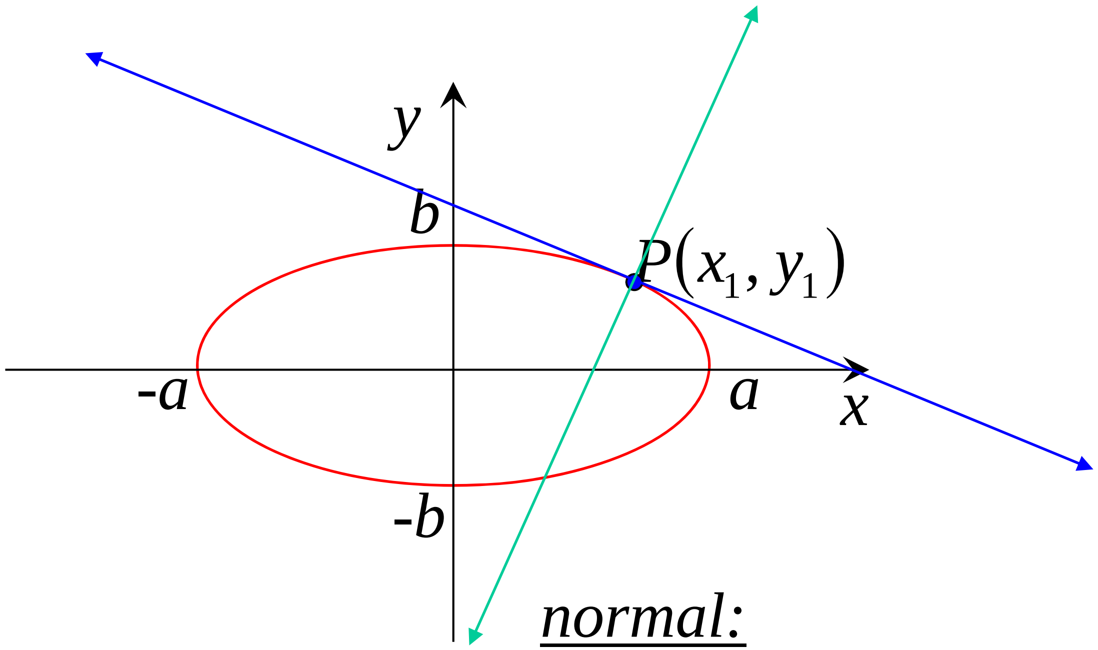
$$\left[\frac{df}{dx} = \frac{df}{dy} \cdot \frac{dy}{dx}\right]$$

$$\frac{2y}{b^2} \cdot \frac{dy}{dx} = -\frac{2x}{a^2}$$

$$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$$

at $P(x_1, y_1)$

$$\frac{dy}{dx} = -\frac{b^2 x_1}{a^2 y_1}$$



tangent:

$$y - y_1 = -\frac{b^2 x_1}{a^2 y_1}(x - x_1)$$

$$a^2 y_1 y - a^2 y_1^2 = -b^2 x_1 x + b^2 x_1^2$$

$$b^2 x_1 x + a^2 y_1 y = b^2 x_1^2 + a^2 y_1^2$$

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$$

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$$

normal:

$$y - y_1 = \frac{a^2 y_1}{b^2 x_1}(x - x_1)$$

$$b^2 x_1 y - b^2 x_1 y_1 = a^2 y_1 x - a^2 x_1 y_1$$

$$a^2 y_1 x - b^2 x_1 y = a^2 x_1 y_1 - b^2 x_1 y_1$$

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 - b^2$$

$$\frac{a^2 x}{x_1} - \frac{b^2 y}{y_1} = a^2 e^2$$

Using Parametric Coordinates;

at $P(a\cos\theta, b\sin\theta)$ $$\frac{dy}{dx} = -\frac{ab^2\cos\theta}{a^2 b\sin\theta}$$

$$= -\frac{b\cos\theta}{a\sin\theta}$$

tangent:

$$y - b\sin\theta = -\frac{b\cos\theta}{a\sin\theta}(x - a\cos\theta)$$

$$ay\sin\theta - ab\sin^2\theta = -bx\cos\theta + ab\cos^2\theta$$

$$bx\cos\theta + ay\sin\theta = ab\sin^2\theta + ab\cos^2\theta$$

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = \sin^2\theta + \cos^2\theta$$

$$\frac{x\cos\theta}{a} + \frac{y\sin\theta}{b} = 1$$

normal:

$$y - b\sin\theta = \frac{a\sin\theta}{b\cos\theta}(x - a\cos\theta)$$

$$by\cos\theta - b^2\sin\theta\cos\theta$$

$$= ax\sin\theta - a^2\sin\theta\cos\theta$$

$$ax\sin\theta - by\cos\theta$$

$$= \left(a^2 - b^2\right)\sin\theta\cos\theta$$

$$\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2 \quad \left(= a^2e^2\right)$$

For ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

tangent at (x_1, y_1) $\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$	normal at (x_1, y_1) $\frac{a^2x}{x_1}-\frac{b^2y}{y_1}=a^2-b^2\left(=a^2e^2\right)$
tangent at $(a\cos\theta, b\sin\theta)$ $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$	normal at $(a\cos\theta, b\sin\theta)$ $\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2\left(=a^2e^2\right)$

e.g. (i) Find the equation of the tangent to the ellipse $\frac{x^2}{16} + y^2 = 1$

at the point $\left(2, \frac{-\sqrt{3}}{2}\right)$

$$\frac{x}{8} + 2y\frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{-x}{16y}$$

at $\left(2, \frac{-\sqrt{3}}{2}\right)$, $\frac{dy}{dx} = \frac{-2}{16\left(\frac{-\sqrt{3}}{2}\right)}$

$$\frac{dy}{dx} = \frac{1}{4\sqrt{3}}$$

$$y + \frac{\sqrt{3}}{2} = \frac{1}{4\sqrt{3}}(x-2)$$

$$4\sqrt{3}y + 6 = x - 2$$

$$x - 4\sqrt{3}y - 8 = 0$$

(*ii*) Find the equation of the normal to the ellipse $x = 2\cos\theta,\ y = \sin\theta$ at the point $\theta = \frac{\pi}{6}$

$$y = \sin f(x) \qquad\qquad y = \cos f(x)$$
$$\frac{dy}{dx} = f'(x)\cos f(x) \qquad\qquad \frac{dy}{dx} = -f'(x)\sin f(x)$$

$$x = 2\cos\theta \qquad y = \sin\theta \qquad \frac{dy}{dx} = \frac{dy}{d\theta} \times \frac{d\theta}{dx}$$
$$\frac{dx}{d\theta} = -2\sin\theta \qquad \frac{dy}{d\theta} = \cos\theta \qquad = \frac{\cos\theta}{-2\sin\theta}$$

$$\text{at } \theta = \frac{\pi}{6}, \frac{dy}{dx} = \frac{\frac{\sqrt{3}}{2}}{-1}$$
$$= \frac{-\sqrt{3}}{2}$$

$$y - \frac{1}{2} = \frac{2}{\sqrt{3}}\left(x - \sqrt{3}\right)$$
$$2\sqrt{3}y - \sqrt{3} = 4x - 4\sqrt{3}$$
$$4x - 2\sqrt{3}y - 3\sqrt{3} = 0$$

(iii) Show that if $y = mx + k$ is a tangent to the ellipse

$$\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1, \text{ then } a^2m^2 + b^2 = k^2$$

$$x = a\cos\theta \qquad y = b\sin\theta \qquad \frac{dy}{dx} = \frac{-b\cos\theta}{a\sin\theta}$$

$$\frac{dx}{d\theta} = -a\sin\theta \qquad \frac{dy}{d\theta} = b\cos\theta$$

If $y = mx + k$ is a tangent then $\frac{dy}{dx} = m$

$$\text{i.e. } m = \frac{-b\cos\theta}{a\sin\theta}$$

$$am\sin\theta = -b\cos\theta$$

$$am\sin\theta + b\cos\theta = 0 \ldots\ldots(1)$$

If tangent meets ellipse at $(a\cos\theta, b\sin\theta)$ then;

$$b\sin\theta = am\cos\theta + k$$

$$am\cos\theta - b\sin\theta = -k \ldots\ldots(2)$$

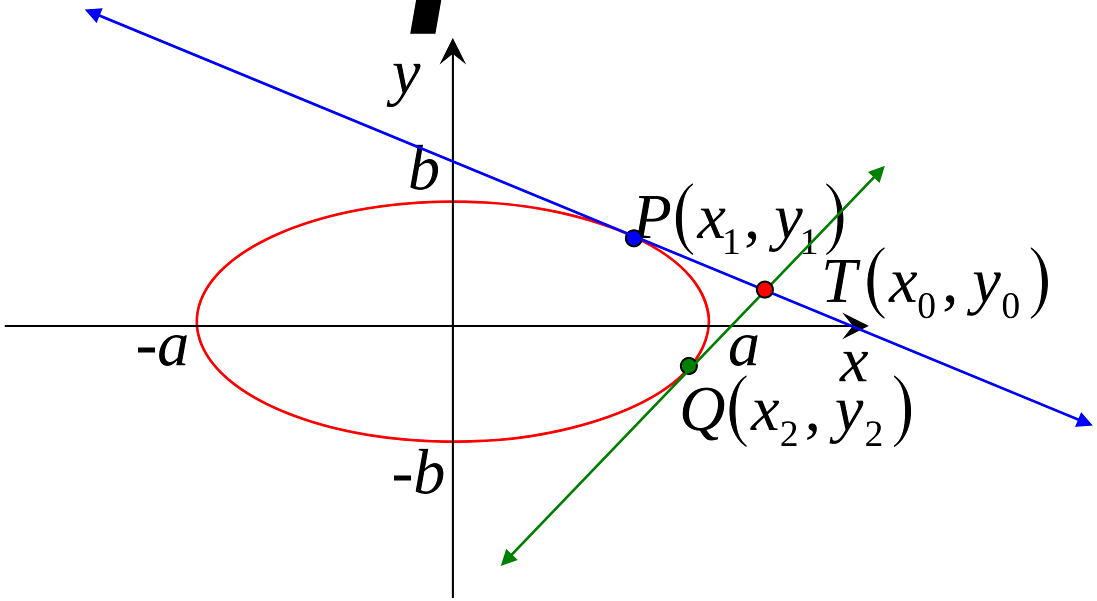
$$(1)^2 : a^2m^2\sin^2\theta + 2abm\sin\theta\cos\theta + b^2\cos^2\theta = 0 \qquad (+)$$

$$(2)^2 : a^2m^2\cos^2\theta - 2abm\sin\theta\cos\theta + b^2\sin^2\theta = k^2$$

$$a^2m^2\left(\sin^2\theta + \cos^2\theta\right) + b^2\left(\sin^2\theta + \cos^2\theta\right) = k^2$$

$$\underline{a^2m^2 + b^2 = k^2}$$

Chord of Contact



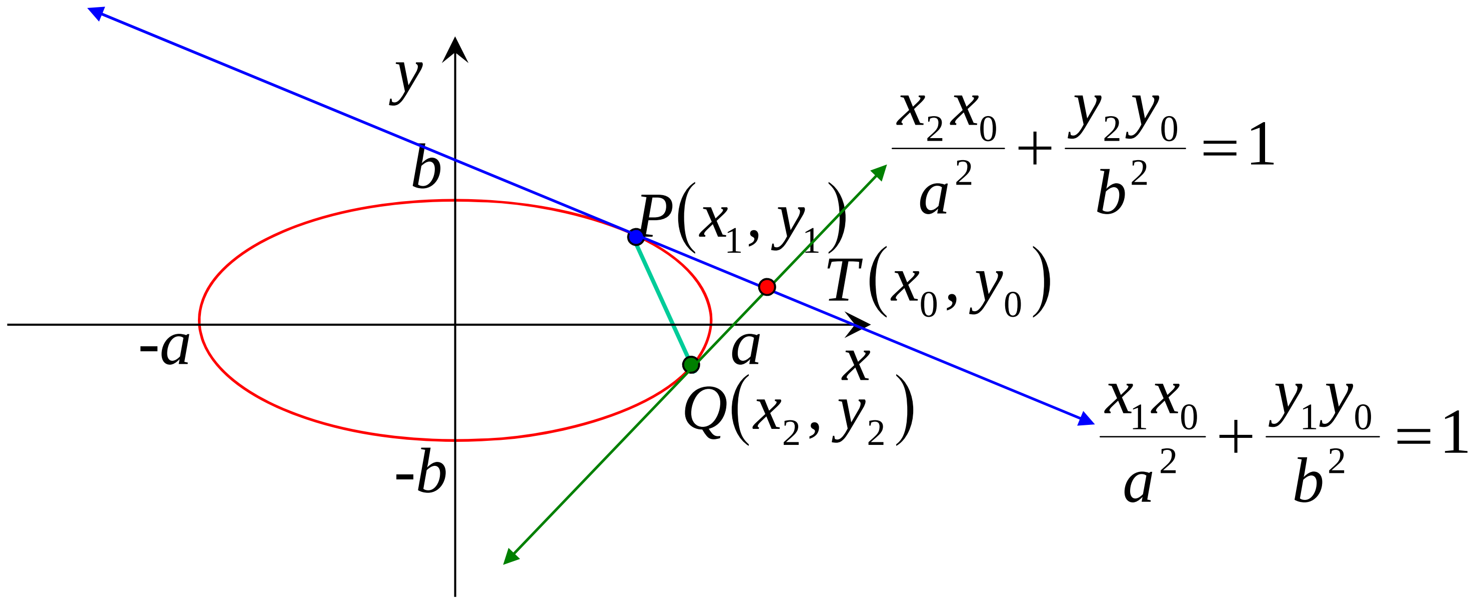
From an external point *T*, two tangents may be drawn.

tangent at *P* has equation $\dfrac{x_1x}{a^2}+\dfrac{y_1y}{b^2}=1$

tangent at *Q* has equation $\dfrac{x_2x}{a^2}+\dfrac{y_2y}{b^2}=1$

Now *T* lies on both lines,

$$\therefore \frac{x_1x_0}{a^2}+\frac{y_1y_0}{b^2}=1 \quad \text{and} \quad \frac{x_2x_0}{a^2}+\frac{y_2y_0}{b^2}=1$$



Thus *P* and *Q* both must lie on a line with equation;

$$\frac{x_0 x}{a^2}+\frac{y_0 y}{b^2}=1$$

which must be the line *PQ* i.e. chord of contact

Patel: **Exercise 6C; 1, 3, 11, 15, 18a**

Cambridge: **Exercise 3C; 1, 3, 6, 8, 9, 10a, 11ac, 12, 13a, 14b, 16,17, 18, 19,20**