Geometric Properties

(1) The chord of contact from a point on the directrix is a focal chord.

ellipse

As T is on the directrix it has coordinates $\left(\frac{a}{e}, y_0\right)$ i.e. $x_0 = \frac{a}{e}$

$\therefore$ chord of contact will have the equation;

$$\frac{x\left(\frac{a}{e}\right)}{a^2} + \frac{yy_0}{b^2} = 1$$

$$\frac{x}{ae} + \frac{yy_0}{b^2} = 1$$

Substitute in focus $(ae, 0)$

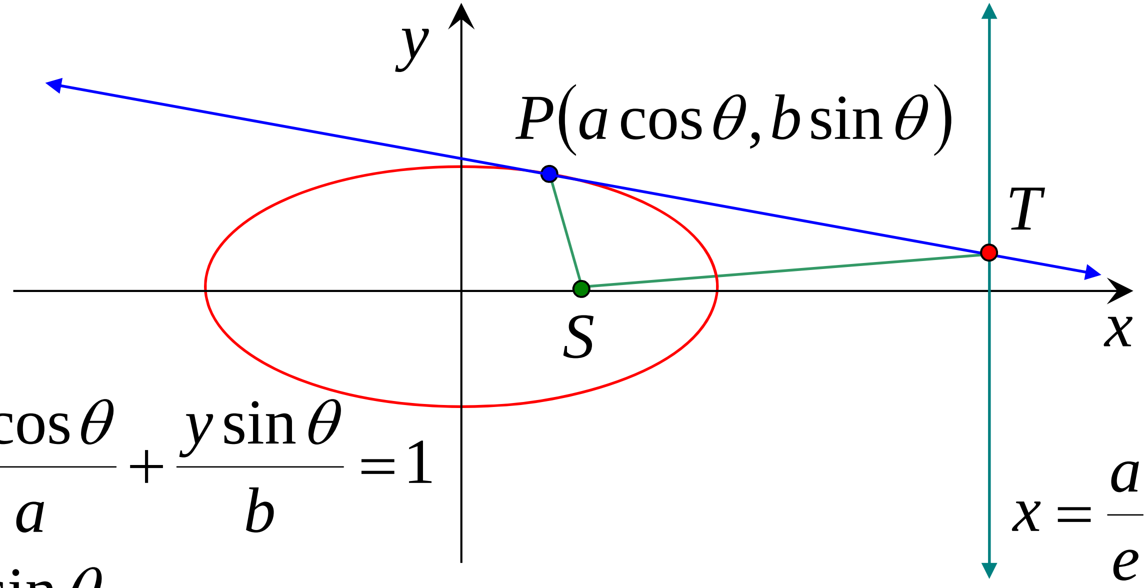
$$\frac{ae}{ae} + 0 = 1 + 0$$

$$= 1$$

$\therefore$ focus lies on chord of contact

i.e. it is a focal chord

(2) That part of the tangent between the point of contact and the directrix subtends a right angle at the corresponding focus.



Prove: $\angle PST = 90^\circ$

equation of tangent is $\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

when $x=\frac{a}{e}$, $\frac{\frac{a}{e}\cos\theta}{a}+\frac{y\sin\theta}{b}=1$

$$\frac{\cos\theta}{e}+\frac{y\sin\theta}{b}=1$$

$$\frac{y\sin\theta}{b}=\frac{e-\cos\theta}{e}$$

$$y=\frac{b(e-\cos\theta)}{e\sin\theta}$$

$$\therefore T\left(\frac{a}{e},\frac{b(e-\cos\theta)}{e\sin\theta}\right)$$

$$m_{PS} = \frac{b\sin\theta - 0}{a\cos\theta - ae}$$

$$= \frac{b\sin\theta}{a(\cos\theta - e)}$$

$$m_{TS} = \frac{\dfrac{b(e-\cos\theta)}{e\sin\theta} - 0}{\dfrac{a}{e} - ae}$$

$$= \frac{b(e-\cos\theta)}{e\sin\theta} \times \frac{e}{a - ae^2}$$

$$= \frac{b(e-\cos\theta)}{a(1-e^2)\sin\theta}$$

$$= \frac{b(e-\cos\theta)}{\dfrac{a^2(1-e^2)}{a}\sin\theta}$$

$$= \frac{a(e-\cos\theta)}{b\sin\theta}$$

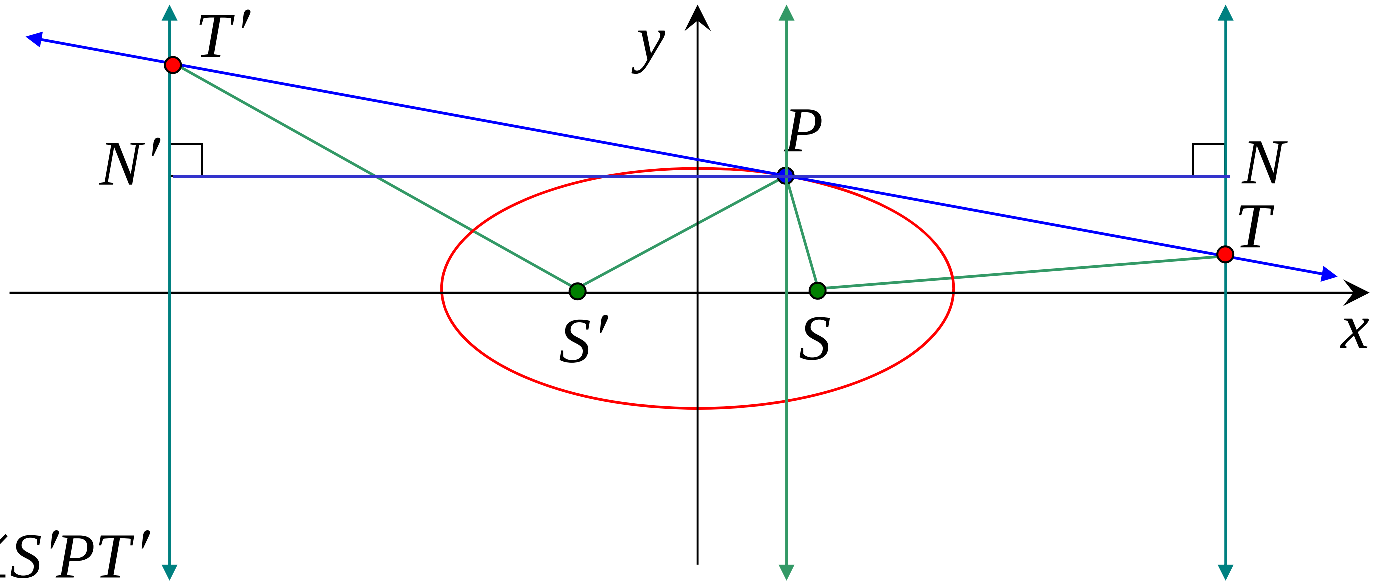
$$m_{PS} \times m_{TS} = \frac{b\sin\theta}{a(\cos\theta - e)} \times \frac{a(e-\cos\theta)}{b\sin\theta}$$

$$= -1$$

$$\therefore \angle PST = 90^\circ$$

(3) Reflection Property

Tangent to an ellipse at a point P on it is equally inclined to the focal chords through P.



Prove: $\angle SPT = \angle S'PT'$

Construct a line || y axis passing through P

$$\frac{PT}{PT'} = \frac{PN}{PN'}$$ (ratio of intercepts of || lines)

$$\therefore \frac{PT}{PN} = \frac{PT'}{PN'}$$

$$ePN = PS \quad \text{and} \quad ePN' = PS'$$

$$\therefore \frac{PT}{\frac{PS}{e}} = \frac{PT'}{\frac{PS'}{e}}$$

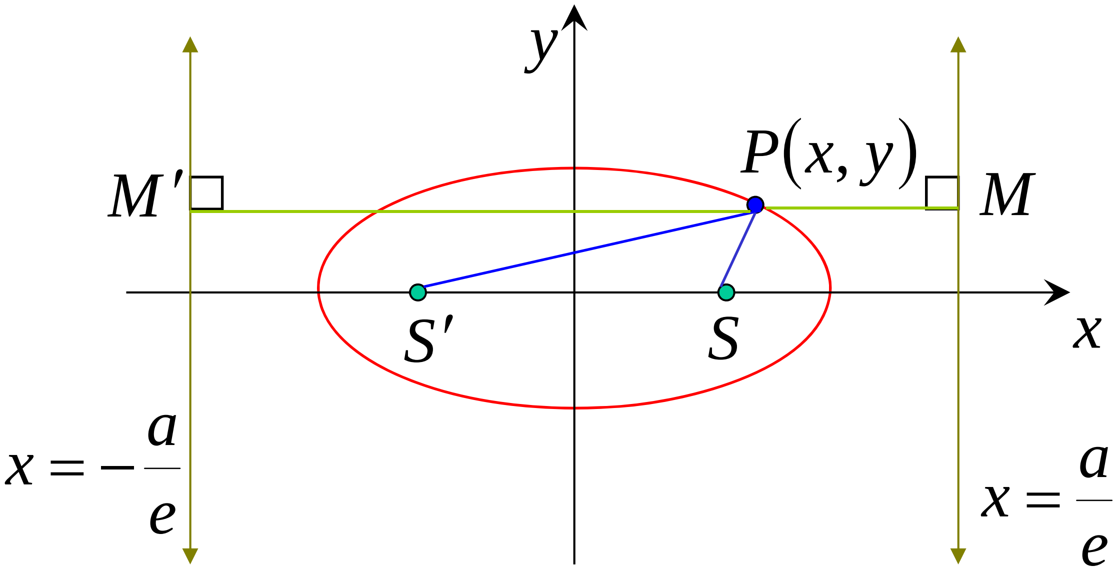
$$\frac{PT}{PS} = \frac{PT'}{PS'}$$

$$\angle PST = \angle PS'T' = 90^\circ \quad \text{(proven in property (2))}$$

$$\therefore \sec \angle SPT = \sec \angle S'PT'$$

$$\angle SPT = \angle S'PT'$$

(ii) Show that $PS + PS' = 2a$



By definition of an ellipse;

$$
\begin{aligned}
PS + PS' &= ePM + ePM' \\
&= e(PM + PM') \\
&= e\left(\frac{2a}{e}\right) \\
&= 2a
\end{aligned}
$$

e.g. Find the cartesian equation of $|z+2|+|z-2|=8$

The sum of the focal lengths of an ellipse is constant

$$2a=8$$
$$a=4$$

$$ae=2$$
$$4e=2$$
$$e=\frac{1}{2}$$

$$b^2=a^2\left(1-e^2\right)$$
$$b^2=16\left(1-\frac{1}{4}\right)$$
$$=12$$

$\therefore$ locus is the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{12}=1$

Patel: **Exercise 6E; 1, 2, 4, 7, 8, 10**
Cambridge: **Exercise 3D; 1 to 17 odds**
Cambridge: **Exercise 3F; 2 to 20 evens**