

# *Integration Using Substitution*

Try to convert the integral into a “standard integral”

***OLD STANDARD INTEGRALS used prior to 2016***

$$\int x^n dx = \frac{1}{n+1} x^{n+1}, \quad n \neq -1; x \neq 0, \text{ if } n < 0$$

$$\int \frac{1}{x} dx = \ln x, \quad x > 0$$

$$\int e^{ax} dx = \frac{1}{a} e^{ax}, \quad a \neq 0$$

$$\int \cos ax dx = \frac{1}{a} \sin ax, \quad a \neq 0$$

$$\int \sin ax dx = -\frac{1}{a} \cos ax, \quad a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \quad a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \quad a > 0, \quad -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln \left( x + \sqrt{x^2 - a^2} \right), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln \left( x + \sqrt{x^2 + a^2} \right)$$

## ***NEW STANDARD INTEGRALS used from 2016***

### ***Mathematics***

$$\int (ax + b)^n dx = \frac{(ax + b)^{n+1}}{a(n+1)} + c$$

$$\int e^{ax+b} dx = \frac{1}{a} e^{ax+b} + c$$

$$\int \frac{f'(x)}{f(x)} dx = \ln |f(x)| + c$$

$$\int \sin(ax + b) dx = -\frac{1}{a} \cos(ax + b) + c$$

$$\int \cos(ax + b) dx = \frac{1}{a} \sin(ax + b) + c$$

$$\int \sec^2(ax + b) dx = \frac{1}{a} \tan(ax + b) + c$$

## ***Mathematics Extension 1***

$$\int \frac{1}{\sqrt{a^2 - x^2}} dx = \sin^{-1} \frac{x}{a} + c$$

$$\int \frac{1}{a^2 + x^2} dx = \frac{1}{a} \tan^{-1} \frac{x}{a} + c$$

***no longer listed on standard integral sheet***

$$\int \sec ax \tan ax dx = \frac{1}{a} \sec ax, \quad a \neq 0$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} dx = \ln(x + \sqrt{x^2 - a^2}), \quad x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} dx = \ln(x + \sqrt{x^2 + a^2})$$

## **“NON” STANDARD INTEGRALS**

Not listed in the standard integrals, however will save time if you know

$\int \frac{du}{\sqrt{u}}$	$= 2\sqrt{u}$
$\int \frac{du}{u^2}$	$= -\frac{1}{u}$
$\int \ln x dx$	$= x \ln x - x$
$\int \tan x dx$	$= \log \sec x$
$\int$ complementary trig ratio	$= -\text{complement of the answer}$

e.g.

$$\int \cot x dx$$

$$= -\log \operatorname{cosec} x$$

$$= \log \sin x$$

cot is the complement of tan

so the answer is  
minus and the  
complement of sec is  
cosec

e.g. (i)  $\int x\sqrt{x^2+4}dx$   $= \frac{1}{2} \int 2x\sqrt{x^2+4}dx$

$\frac{1}{2} du$   $u$

$= \frac{1}{2} \int u^{\frac{1}{2}} du$

$= \frac{1}{2} \cdot \frac{2}{\frac{3}{2}} u^{\frac{3}{2}} + c$

$= \frac{1}{3} u^{\frac{3}{2}} + c$

$= \frac{1}{3} (x^2 + 4)^{\frac{3}{2}} + c$

$= \frac{1}{3} (x^2 + 4)\sqrt{x^2 + 4} + c$

---

$$u = x^2 + 4$$

$$du = 2x dx$$

when substituting  
 $u = f(x)$   
 make the function  
 causing the  
 problem  $u$

**OR**

$$\begin{aligned} \text{e.g. (i)} \quad \int x\sqrt{x^2+4}dx &= \int 2 \tan \theta \sqrt{4 \tan^2 \theta + 4} \cdot 2 \sec^2 \theta d\theta & x &= 2 \tan \theta \\ &= 8 \int \tan \theta \sec^3 \theta & dx &= 2 \sec^2 \theta d\theta \end{aligned}$$

$$= 8 \int \frac{\sin \theta d\theta}{\cos^4 \theta}$$

$$u = \cos \theta$$

$$du = -\sin \theta d\theta$$

$$= -8 \int \frac{du}{u^4}$$

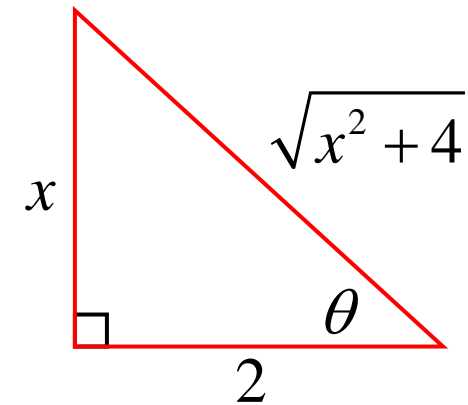
$$= -8 \times -\frac{1}{3} u^{-3} + c$$

$$= \frac{8}{3} \sec^3 \theta + c$$

$$= \frac{1}{3} \left( \sqrt{x^2+4} \right)^3 + c$$

$$= \frac{1}{3} (x^2+4)\sqrt{x^2+4} + c$$

if a root of a sum  
or difference of  
squares is  
involved, could  
try a trig  
substitution



Keep an eye out  
for  
 $f'(x) \times f(x)$

$$\begin{aligned} (ii) \int \frac{x+1}{4x^2+8x-7} dx &= \frac{1}{8} \int \frac{du}{u} & u &= 4x^2+8x-7 \\ &= \frac{1}{8} \log u + c & du &= (8x+8)dx \\ &= \frac{1}{8} \log(4x^2+8x-7) + c \end{aligned}$$

---

$$\begin{aligned} (iii) \int \frac{dx}{x(\log x)^3} &= \int \frac{du}{u^3} & u &= \log x \\ &= \int u^{-3} du & du &= \frac{dx}{x} \\ &= -\frac{1}{2} u^{-2} + c \\ &= -\frac{1}{2u^2} + c \\ &= -\frac{1}{2(\log x)^2} + c \end{aligned}$$

---



$$\begin{aligned}
 (iv) \int \frac{x}{\sqrt{1-x}} dx &= \int \frac{1-u^2}{u} \cdot -2udu && x = 1-u^2 \Rightarrow u = \sqrt{1-x} \\
 &= 2 \int (u^2 - 1) du && dx = -2udu \\
 &= 2 \left( \frac{1}{3} u^3 - u \right) + c \\
 &= \frac{2}{3} (\sqrt{1-x})^3 - 2\sqrt{1-x} + c \\
 &= \frac{2}{3} (1-x)\sqrt{1-x} - 2\sqrt{1-x} + c
 \end{aligned}$$

substituting  
 $u = f(x)$   
 instead of  
 $x = f(u)$   
 avoids domain  
 problems later

$$\begin{aligned}
 (v) \int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin^5 x \cos x dx &= \int_{\frac{\sqrt{3}}{2}}^1 u^5 du && u = \sin x && x = \frac{\pi}{3}, u = \sin \frac{\pi}{3} \\
 &&& du = \cos x dx && \\
 &= \frac{1}{6} [u^6]_{\frac{\sqrt{3}}{2}}^1 && && u = \frac{\sqrt{3}}{2} \\
 &= \frac{1}{6} \left\{ 1^6 - \left( \frac{\sqrt{3}}{2} \right)^6 \right\} && && x = \frac{\pi}{2}, u = \sin \frac{\pi}{2} \\
 &= \frac{37}{384} && && u = 1
 \end{aligned}$$

$$\begin{aligned} (vi) \int_3^4 \frac{xdx}{\sqrt{25-x^2}} &= -\frac{1}{2} \int_{16}^9 \frac{du}{\sqrt{u}} \\ &= \frac{1}{2} \int_9^{16} u^{-\frac{1}{2}} du \\ &= \left[ u^{\frac{1}{2}} \right]_9^{16} \\ &= \sqrt{16} - \sqrt{9} \\ &= \underline{1} \end{aligned}$$

$$\begin{aligned} u &= 25 - x^2 \\ du &= -2xdx \end{aligned}$$

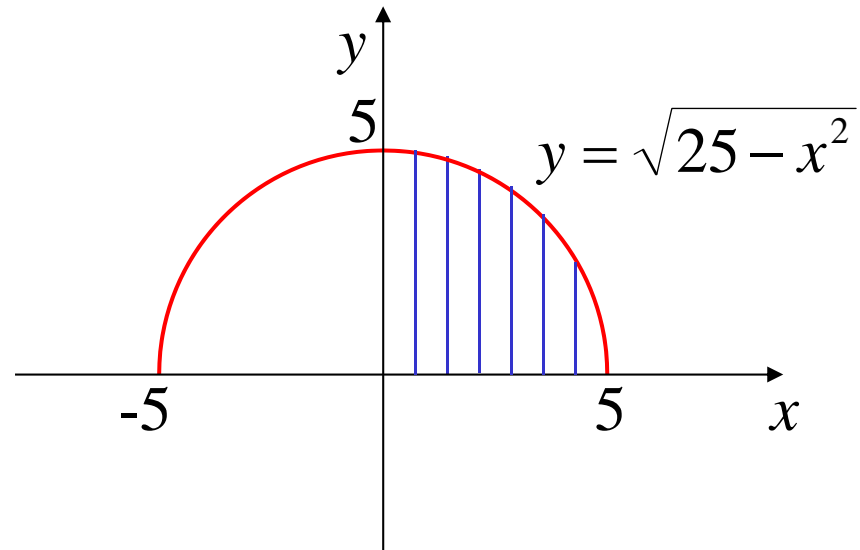
$$x = 3, u = 16$$

$$x = 4, u = 9$$

$$\begin{aligned}
\text{(vii)} \int_0^5 \sqrt{25 - x^2} dx &= \int_0^{\frac{\pi}{2}} \sqrt{25 - 25 \sin^2 u} \cdot 5 \cos u du & x = 5 \sin u \Rightarrow u = \sin^{-1} \frac{x}{5} \\
&= \int_0^{\frac{\pi}{2}} \sqrt{25 \cos^2 u} \cdot 5 \cos u du & dx = 5 \cos u du \\
& & x = 0, u = \sin^{-1} 0 \\
& & u = 0 \\
& & x = 5, u = \sin^{-1} 1 \\
& & u = \frac{\pi}{2} \\
&= 25 \int_0^{\frac{\pi}{2}} \cos^2 u du \\
&= \frac{25}{2} \int_0^{\frac{\pi}{2}} (1 + \cos 2u) du \\
&= \frac{25}{2} \left[ u + \frac{1}{2} \sin 2u \right]_0^{\frac{\pi}{2}} \\
&= \frac{25}{2} \left\{ \frac{\pi}{2} + \frac{1}{2} \sin \pi - 0 \right\} \\
&= \frac{25\pi}{4}
\end{aligned}$$

$$\begin{aligned}
 \text{(vii)} \int_0^5 \sqrt{25 - x^2} dx \\
 &= \frac{1}{4} \pi (5)^2 \\
 &= \frac{25\pi}{4}
 \end{aligned}$$


---



*don't forget about algebraic manipulation*

$$\begin{aligned}
 \text{(viii)} \int x^3 \sqrt{4 - x^2} dx \\
 &= \int \left[ -x(4 - x^2) + 4x \right] \sqrt{4 - x^2} dx \\
 &= \int \left[ -x(4 - x^2)^{\frac{3}{2}} + 4x(4 - x^2)^{\frac{1}{2}} \right] dx \\
 &= \frac{1}{2} \times \frac{2}{5} (4 - x^2)^{\frac{5}{2}} - 2 \times \frac{2}{3} (4 - x^2)^{\frac{3}{2}} + c \\
 &= \frac{1}{5} (4 - x^2)^2 \sqrt{4 - x^2} - \frac{4}{3} (4 - x^2) \sqrt{4 - x^2} + c
 \end{aligned}$$


---

**Exercise 6C; 1, 2ace, 3, 4ace, 5ace etc, 6, 7 & 8ac, 11, 12**

**Exercise 6D; 1, 2a, 3, 4b, 5ac, 6ace etc, 7ab(i,ii)**

**8ab (i,iii,vi), 9ab (ii,iv,vi), 11, 13**

***Patel Exercise 2A ace in all***

***Cambridge Exercise 2B; 1b (iii), 3ce, 4cde, 5cd, 6, 7bc,***

***8ab, 9a, 10bd, 11, 12, 13***

***NOTE: substitution is not given in Extension 2***