Simple Harmonic Motion

A particle that moves back and forward in such a way that its acceleration at any instant is directly proportional to its distance from a fixed point, is said to undergo **Simple Harmonic Motion (SHM)**

$$\ddot{x} \alpha\ x$$

$$\ddot{x} = kx$$

$$\ddot{x} = -n^2 x \quad \text{(constant needs to be negative)}$$

If a particle undergoes SHM, then it obeys;

$$\ddot{x} = -n^2 x$$

$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = -n^2 x$$

$$\frac{1}{2}v^2 = -\frac{1}{2}n^2 x^2 + c$$

$$v^2 = -n^2 x^2 + c$$

when $x = a$, $v = 0$ (a = amplitude)

$$\text{i.e. } 0^2 = -n^2a^2 + c$$

$$c = n^2a^2$$

$$v^2 = -n^2x^2 + n^2a^2$$

$$v^2 = n^2\left(a^2 - x^2\right)$$

$$v = \pm n\sqrt{a^2 - x^2}$$

NOTE:

$$v^2 \geq 0$$

$$a^2 - x^2 \geq 0$$

$$-a \leq x \leq a$$

$\therefore$ Particle travels back and forward between x = -a and x = a

$$\frac{dx}{dt} = -n\sqrt{a^2 - x^2}$$

$$\frac{dt}{dx} = \frac{-1}{n\sqrt{a^2 - x^2}}$$

$$\int_0^t dt = \frac{1}{n}\int_a^x \frac{-1}{\sqrt{a^2 - x^2}}dx$$

$$t = \frac{1}{n}\left[\cos^{-1}\frac{x}{a}\right]_a^x$$

$$= \frac{1}{n}\left\{\cos^{-1}\frac{x}{a} - \cos^{-1}1\right\}$$

$$= \frac{1}{n}\cos^{-1}\frac{x}{a}$$

$$nt = \cos^{-1}\frac{x}{a}$$

$$\frac{x}{a} = \cos nt$$

$$x = a\cos nt$$

If when $t = 0$;

$x = \pm a$, choose - ve and $\cos^{-1}$

$x = 0$, choose + ve and $\sin^{-1}$

In general;

A particle undergoing SHM obeys

$$\ddot{x} = -n^2 x$$

$v^2 = n^2\left(a^2 - x^2\right) \Rightarrow$ allows us to find path of the particle

$$x = a\cos nt \qquad \text{OR } x = a\sin nt$$

where $a =$ amplitude

the particle has;

$\text{period}: T = \dfrac{2\pi}{n}$ (time for one oscillation)

$\text{frequency}: f = \dfrac{1}{T}$ (number of oscillations per time period) e.g. (i) A particle, P, moves on the x axis according to the law $x = 4\sin 3t$.

a) Show that P is moving in SHM and state the period of motion.

$$x = 4\sin 3t$$

$$\dot{x} = 12\cos 3t$$

$$\ddot{x} = -36\sin 3t$$

$$= -9x$$

$$\therefore \text{particle moves in SHM}$$

$$T = \frac{2\pi}{3}$$

$$\therefore \text{period of motion is } \frac{2\pi}{3} \text{ seconds}$$

b) Find the interval in which the particle moves and determine the greatest speed.

$$\therefore \text{particle moves along the interval } -4 \leq x \leq 4$$

$$\text{and the greatest speed is } 12 \text{ units/s}$$

(ii) A particle moves so that its acceleration is given by $\ddot{x} = -4x$
Initially the particle is 3cm to the right of O and traveling with a velocity of 6cm/s.
Find the interval in which the particle moves and determine the greatest acceleration.

$$v\frac{dv}{dx} = -4x$$

$$\int_6^v v\,dv = \int_3^x -4x\,dx$$

$$\left[v^2\right]_6^v = -4\left[x^2\right]_3^x$$

$$v^2 - 36 = -4x^2 + 36$$

$$v^2 = -4x^2 + 72$$

But $v^2 \geq 0$

$$-4x^2 + 72 \geq 0$$

$$x^2 \leq 18$$

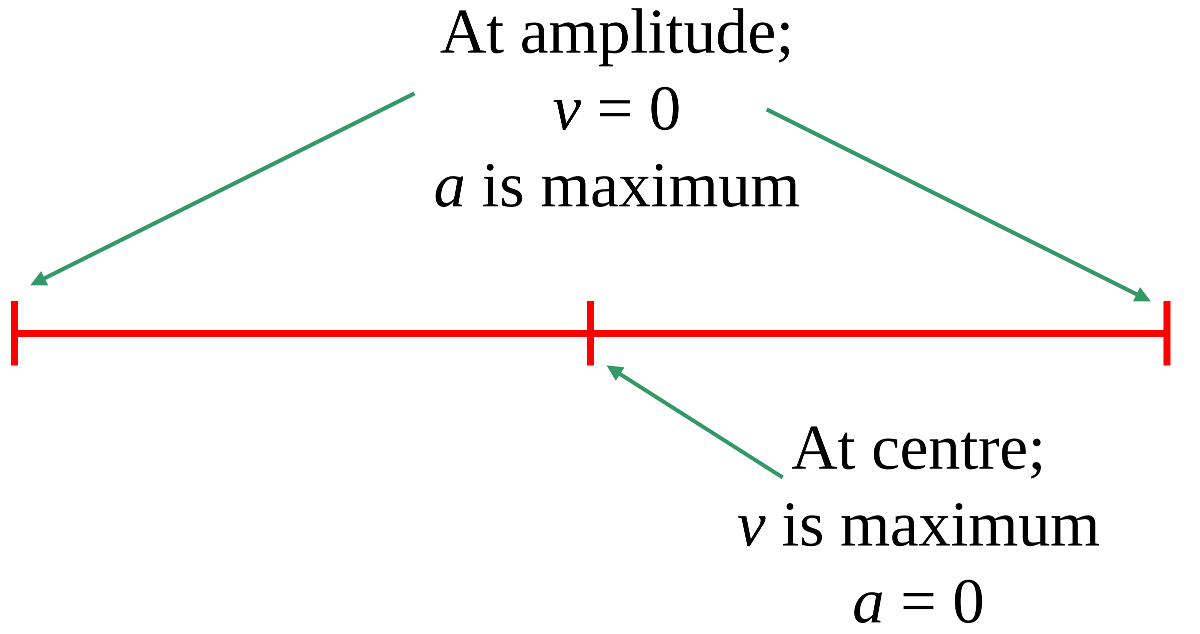
$$-3\sqrt{2} \leq x \leq 3\sqrt{2}$$

when $x = 3\sqrt{2}, \ddot{x} = -4\left(3\sqrt{2}\right)$

$$= -12\sqrt{2}$$

$\therefore$ greatest acceleration is $12\sqrt{2}$ cm/s^2

NOTE:



Exercise 3D; 1, 6, 7, 10, 12, 14ab, 15ab, 18, 19, 22, 24, 25

(start with trig, prove SHM or are told)

Exercise 3F; 1, 4, 5b, 6b, 8, 9a, 10a, 13, 14 a, b(*ii,iv*), 16, 18, 19

(start with $\ddot{x} = -n^2 x$)