4a) $(1+x)^3(1+x)^9 = (1+x)^{12}$

<u>coefficients of x^3 in LHS</u>

$(1+x)^3(1+x)^9$

$= \left({}^3C_0 + {}^3C_1 x + {}^3C_2 x^2 + {}^3C_3 x^3\right)\left(1 + 9x + \dots + x^9\right)$

${}^3C_0\left({}^9C_3 x^3\right) + \left({}^3C_1 x\right)\left({}^9C_2 x^2\right) + \left({}^3C_2 x^2\right)\left({}^9C_1 x\right) + \left({}^3C_3 x^3\right)\left({}^9C_0\right)$

coefficient x^3 $= {}^3C_0\,{}^9C_3 + {}^3C_1\,{}^9C_2 + {}^3C_2\,{}^9C_1 + {}^3C_3\,{}^9C_0$

Coefficient of x^3 in RHS

$(1+x)^{12}$

${}^{12}C_3 x^3$

But $(1+x)^3(1+x)^9 = (1+x)^{12}$

$\therefore$ coefficients of x^3 must be $=$

$${}^{12}C_3 = {}^3C_0\,{}^9C_3 + {}^3C_1\,{}^9C_2 + {}^3C_2\,{}^9C_1 + {}^3C_3\,{}^9C_0$$

5a)

$$^nC_k = \frac{n!}{k!(n-k)!}$$

$$^nC_{n-k} = \frac{n!}{(n-k)!(n-(n-k))!}$$

$$= \frac{n!}{(n-k)!\,k!}$$

$$= {^nC_k}$$

5d $(1+x)^n(1+x)^n \equiv (1+x)^{2n}$

$\underline{x^{n+1} \text{ in } (1+x)^{2n}}$

$^{2n}C_{n+1} = \dfrac{(2n)!}{(n+1)!\,(n-1)!}$

$\underline{x^{n+1} \text{ in } (1+x)^n(1+x)^n}$

$^nC_1 x \times {}^nC_n x^n + {}^nC_2 x^2 \times {}^nC_{n-1} x^{n-1} + {}^nC_3 x^3 \times {}^nC_{n-2} x^{n-2} + \dots + {}^nC_n x^n \times {}^nC_1 x$

$\therefore$ ~~se~~ efficient of x^{n+1}

$= {}^nC_1\,{}^nC_n + {}^nC_2\,{}^nC_{n-1} + {}^nC_3\,{}^nC_{n-2} + \dots + {}^nC_n\,{}^nC_1$

$= {}^nC_1\,{}^nC_0 + {}^nC_2\,{}^nC_1 + {}^nC_3\,{}^nC_2 + \dots + {}^nC_n\,{}^nC_{n-1}$

But $(1+x)^n(1+x)^n \equiv (1+x)^{2n}$

$$\therefore\ {}^nC_1\,{}^nC_0 + {}^nC_2\,{}^nC_1 + {}^nC_3\,{}^nC_2 + \dots + {}^nC_n\,{}^nC_{n-1} = \frac{(2n)!}{(n+1)!\,(n-1)!}$$

5e) $(1+x)^n\left(1+\frac{1}{x}\right)^n$

$= (1+x)^n (x+1)^n \left(\frac{1}{x}\right)^n$

$= (1+x)^{2n}\frac{1}{x^n}$

coefficients of $\frac{1}{x} = x^{-1}$

$(1+x)^n\left(1+\frac{1}{x}\right)^n = \left({}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n\right)\left({}^nC_0 + {}^nC_1\frac{1}{x} + {}^nC_2\frac{1}{x^2} + \dots + {}^nC_n\frac{1}{x^n}\right)$

${}^nC_0 \times {}^nC_1\frac{1}{x} + {}^nC_1 x \times {}^nC_2\frac{1}{x^2} + {}^nC_2 x^2 \times {}^nC_3\frac{1}{x^3} + \dots + {}^nC_{n-1}x^{n-1} \times {}^nC_n\frac{1}{x^n}$

$= \left({}^nC_0\,{}^nC_1 + {}^nC_1\,{}^nC_2 + {}^nC_2\,{}^nC_3 + \dots + {}^nC_{n-1}\,{}^nC_n\right)\frac{1}{x}$

$$\underline{(1+x)^{2n}\frac{1}{x^n}} = \left({}^{2n}C_0 + {}^{2n}C_1 x + \dots + {}^{2n}C_n x^n + \dots + {}^{2n}C_{2n} x^{2n}\right)\frac{1}{x^n}$$

$$ {}^{2n}C_{n-1} x^{n-1} \times \frac{1}{x^n} = {}^{2n}C_{n-1}\frac{1}{x}$$

But $(1+x)^n\left(1+\frac{1}{x}\right)^n \equiv (1+x)^{2n}\frac{1}{x^n}$

$$\therefore\ {}^nC_0\,{}^nC_1 + {}^nC_1\,{}^nC_2 + {}^nC_2\,{}^nC_3 + \dots + {}^nC_{n-1}\,{}^nC_n = {}^{2n}C_{n-1}$$

$$= \frac{(2n)!}{(n-1)!\,(n+1)!}$$

6b) $\left({}^{2n}C_0\right)^2 - \left({}^{2n}C_1\right)^2 + \left({}^{2n}C_2\right)^2 - \dots + \left({}^{2n}C_{2n}\right)^2 = (-1)^n \, {}^{2n}C_n$

$$\left(1+\frac{1}{x}\right)^{2n}\left(1-\frac{1}{x}\right)^{2n} = \left(1-\frac{1}{x^2}\right)^{2n}$$

coefficient of $\frac{1}{x^{2n}}$

LHS

$$\left({}^{2n}C_0 + \frac{{}^{2n}C_1}{x} + \frac{{}^{2n}C_2}{x^2} + \dots + \frac{{}^{2n}C_{2n}}{x^{2n}}\right)\left({}^{2n}C_0 - \frac{{}^{2n}C_1}{x} + \frac{{}^{2n}C_2}{x^2} - \dots + \frac{{}^{2n}C_{2n}}{x^{2n}}\right)$$

coefficient of $\frac{1}{x^{2n}}$

$$= \left({}^{2n}C_0\right)\left({}^{2n}C_{2n}\right) + \left({}^{2n}C_1\right)\left(-{}^{2n}C_{2n-1}\right) + \dots + \left({}^{2n}C_{2n}\right)\left({}^{2n}C_0\right)$$

$$= \left({}^{2n}C_0\right)^2 - \left({}^{2n}C_1\right)^2 + \dots +$$

RHS

$$\left(1-\frac{1}{x^2}\right)^{2n} = {}^{2n}C_0 - \frac{{}^{2n}C_1}{x^2} + \frac{{}^{2n}C_2}{x^4} + \dots + (-1)^n \frac{{}^{2n}C_n}{x^{2n}} \quad + \frac{{}^{2n}C_{2n}}{x^{4n}}$$

8b) $\sum_{k=0}^{n} {}^nC_k = 2^n$ $\qquad$ $\sum_{k=1}^{n} k\,{}^nC_k = n2^{n-1}$

$$\sum_{k=0}^{n} (k+1)\,{}^nC_k = \sum_{k=0}^{n} k\,{}^nC_k + \sum_{k=0}^{n} {}^nC_k$$

$$= 0\,{}^nC_0 + \sum_{k=1}^{n} k\,{}^nC_k + \sum_{k=0}^{n} {}^nC_k$$

$$= n2^{n-1} + 2^n$$

$$= 2^{n-1}(n+2)$$

10 a) $\int_0^4 (1+x)^n \, dx$

$= \left[\frac{(1+x)^{n+1}}{n+1} \right]_0^4$

$= \frac{5^{n+1} - 1}{n+1}$

1

1 1

1 2 1

1 3 3 1

1 4 6 4 1

b) $(1+x)^n = \sum_{k=0}^{n} {}^nC_k x^k.$

$\int_0^4 (1+x)^n dx = \sum_{k=0}^{n} \int_0^4 {}^nC_k x^k dx$

$\frac{5^{n+1} - 1}{n+1} = \sum_{k=0}^{n} \left[\frac{{}^nC_k x^{k+1}}{k+1} \right]_0^4$

$= \sum_{k=0}^{n} \frac{{}^nC_k 4^{k+1}}{k+1}$

$\underline{n = 3}$

$$LHS = \frac{5^4 - 1}{4}$$

$$= 156$$

$$RHS = \sum_{k=0}^{3} \frac{{}^3C_k \, 4^{k+1}}{k+1}$$

$$= \frac{{}^3C_0 \, 4}{1} + \frac{{}^3C_1 \, 4^2}{2} + \frac{{}^3C_2 \, 4^3}{3} + \frac{{}^3C_3 \, 4^4}{4}$$

$$= 4 + 24 + 64 + 64$$

$$= 156$$

10b)

$$(1+x)^n = \sum_{k=0}^{n} {}^nC_k \, x^k$$

$$\int_0^4 (1+x)^n = \sum_{k=0}^{n} {}^nC_k \int_0^4 x^k \, dx$$

$$\frac{5^{n+1}-1}{n+1} = \sum_{k=0}^{n} {}^nC_k \left[\frac{x^{k+1}}{k+1}\right]_0^4$$

$$= \sum_{k=0}^{n} {}^nC_k \frac{4}{k+1}$$

a) $\int_0^4 (1+x)^n dx = \frac{5^{n+1}-1}{n+1}$

15

$$\int_0^{\frac{\pi}{2}} (\sin x)^{2k} \cos x \, dx$$

$u = \sin x$

$du = \cos x \, dx$

$$= \int_0^1 u^{2k} \, du$$

$$= \left[\frac{u^{2k+1}}{2k+1} \right]_0^1$$

$$= \frac{1}{2k+1}$$

b)

$$\cos^{2n+1}x = \cos^{2n}x\cos x$$

$$= (1-\sin^2 x)^n \cos x$$

$$= \left({}^nC_0 - {}^nC_1 \sin^2 x + {}^nC_2 \sin^4 x - {}^nC_3 \sin^6 x + \dots + {}^nC_n \sin^{2n} x\right)\cos x$$

$$= \sum_{k=0}^{n} {}^nC_k (-1)^k (\sin^2 x)^k \cos x$$

$$= \underline{\sum_{k=0}^{n} {}^nC_k (-1)^k \sin^{2k} x \cos x}$$

c) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^{2n+1}x \, dx = \int_0^{\frac{\pi}{2}} \sum_{k=0}^{n} {}^nC_k (-1)^k \sin^{2k}x \cos x \, dx.$

$$= \sum_{k=0}^{n} {}^nC_k (-1)^k \int_0^{\frac{\pi}{2}} \sin^{2k}x \cos x \, dx$$

$$= \sum_{k=0}^{n} {}^nC_k (-1)^k \times \frac{1}{2k+1}$$

d) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^5 x \, dx = \sum_{k=0}^{2} \frac{{}^2C_k (-1)^k}{2k+1}$

$$= \frac{{}^2C_0}{1} - \frac{{}^2C_1}{3} + \frac{{}^2C_2}{5}$$

$$= 1 - \frac{2}{3} + \frac{1}{5}$$