Gosford High School

2015

TRIAL
HIGHER SCHOOL CERTIFICATE
EXAMINATION

Mathematics Extension 2

- **General Instructions**
- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total Marks – 100

Section I Pages 2 – 5

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section
- Answer on the response sheet provided

Section II Pages 6 – 12

90 marks

- Attempt Questions 11 – 16
- Start a **new booklet** for **each** question
- Answer Question 14(a) on the answer sheet provided
- Allow about 2 hours and 45 minutes for section II

Marks

Section I

10 Marks
Attempt Questions 1-10.
Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for questions 1-10.

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1 What is the value of $\lim_{h\to 0}\frac{\sqrt{h+1}-1}{h}$? **1**

(A) 0

(B) $\frac{1}{2}$

(C) 1

(D) 2

2 A, B, C are three consecutive terms in an arithmetic progression. **1**

Which of the following is a simplification of $\frac{\sin(A+C)}{\sin B}$?

(A) $2\cos B$

(B) $\sin 2B$

(C) $\cot B$

(D) 1

3 What is the number of asymptotes on the graph of the curve $y=\frac{x^2}{x^2-1}$? **1**

(A) 1

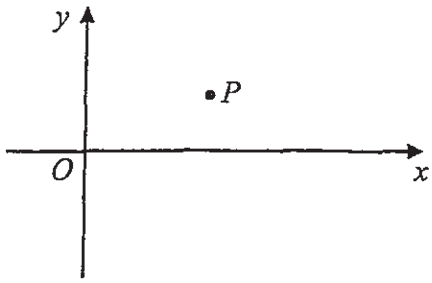
(B) 2

(C) 3

(D) 4

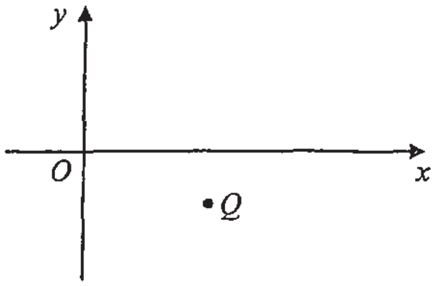
Marks

4 On the Argand diagram below, P represents the complex number z. **1**

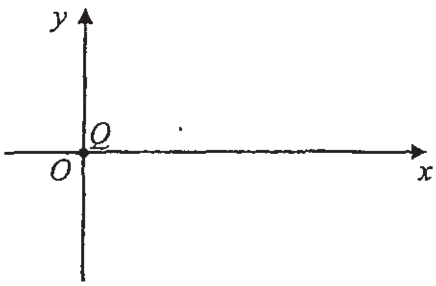


Which of the following Argand diagrams shows the point Q representing $z+\overline{z}$?

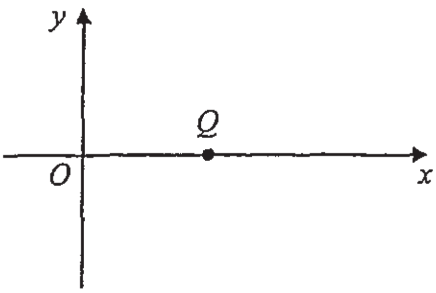
(A)



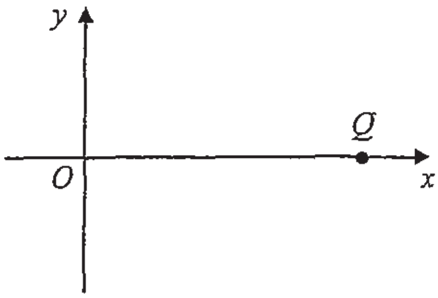
(B)



(C)



(D)



Marks

5 What is the acute angle between the asymptotes of the hyperbola $\dfrac{x^2}{3} - y^2 = 1$? **1**

(A) $\frac{\pi}{6}$

(B) $\frac{\pi}{4}$

(C) $\frac{\pi}{3}$

(D) $\frac{\pi}{2}$

6 Which of the following is an expression for $\displaystyle\int_0^{\frac{\pi}{2}} \frac{1}{1+\sin x}\,dx$ after the substitution $t = \tan\frac{x}{2}$? **1**

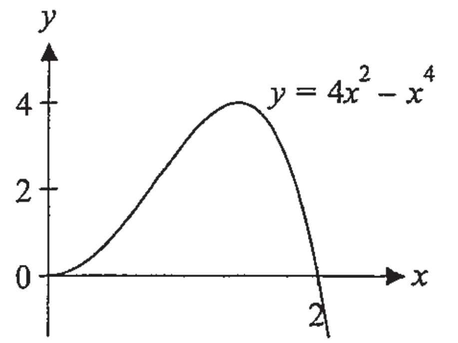
(A) $\displaystyle\int_0^1 \frac{1}{1+2t}\,dt$

(B) $\displaystyle\int_0^1 \frac{2}{1+2t}\,dt$

(C) $\displaystyle\int_0^1 \frac{1}{(1+t)^2}\,dt$

(D) $\displaystyle\int_0^1 \frac{2}{(1+t)^2}\,dt$

7



The region in the first quadrant bounded by the curve $y = 4x^2 - x^4$ and the x axis between $x = 0$ and $x = 2$ is rotated through 2π radians about the y axis. Which of the following is an expression for the volume V of the solid formed ? **1**

(A) $V = 2\pi\displaystyle\int_0^4 \sqrt{4-y}\,dy$

(B) $V = 4\pi\displaystyle\int_0^4 \sqrt{4-y}\,dy$

(C) $V = 8\pi\displaystyle\int_0^4 \sqrt{4-y}\,dy$

(D) $V = 16\pi\displaystyle\int_0^4 \sqrt{4-y}\,dy$

Marks

8 The equation $x^4 + px + q = 0$, where $p \neq 0$ and $q \neq 0$, has roots α, β, γ and δ. **1**
What is the value of $\alpha^4 + \beta^4 + \gamma^4 + \delta^4$?

(A) $-4q$

(B) $p^2 - 2q$

(C) $p^4 - 2q$

(D) p^4

9 Which of the following is the range of the function $f(x) = \sin^{-1} x + \tan^{-1} x$? **1**

(A) $-\pi < y < \pi$

(B) $-\pi \leq y \leq \pi$

(C) $-\frac{3\pi}{4} \leq y \leq \frac{3\pi}{4}$

(D) $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

10 If $e^x + e^y = 1$, which of the following is an expression for $\frac{dy}{dx}$? **1**

(A) $-e^{x-y}$

(B) e^{x-y}

(C) e^{y-x}

(D) $-e^{y-x}$

Marks

Section II

90 Marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer the questions on your own paper, or in writing booklets if provided.
Start each question on a new page.
All necessary working should be shown in every question.

Question 11 (15 marks) **Use a SEPARATE writing booklet**

(a) If $z = 1+3i$ and $w = 2-i$ find in the form $a+ib$ (for real a and b) the values of

(i) $\bar{z} - w$ **1**

(ii) zw **1**

(b)(i) Express $-1+\sqrt{3}\,i$ in modulus/argument form. **2**

(ii) Hence find the value of $z^8 - 16z^4$ in the form $a+ib$ where a and b are real. **2**

(c) In the Argand diagram $OABC$ is a square, where O, A, B, C are in anti-clockwise cyclic order. The complex number z is represented by the vector $\overline{OA}$.

(i) Find in terms of z the complex numbers represented by the vectors $\overline{OC}$ and $\overline{OB}$. **2**

(ii) If the vector $\overline{OB}$ represents the complex number $4+2i$, find z in the form $a+ib$ where a and b are real. **2**

(d) The polynomial $P(x) = x^6 + ax^3 + bx^2$ has a factor $(x+1)^2$. **3**

Find the values of the real numbers a and b.

(e) The equation $x^4 + bx^3 + cx^2 + dx + 1 = 0$ has roots $\alpha, \frac{1}{\alpha}, \beta, \frac{1}{\beta}$. **2**

Show that $b = d$

End of Question 11

Question 12 (15 marks) **Start a new booklet** **Marks**

a) Find the following

(i)

$$\int \frac{e^{-x}}{1+e^{x}}\,dx$$ **3**

(ii)

$$\int \frac{x^2}{x+1}\,dx$$ **2**

b) Find the exact value of the following definite integral:

$$\int_0^{\frac{\pi}{6}} \sec^3 2\theta\, d\theta$$ **4**

c) By using the substitution of $x = tan\ \theta$, show that

$$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} cosec\theta cot\theta\, d\theta$$ **3**

d) The area bounded by the line $y = (2 - x)$ and the x axis, is rotated about the y axis. By using the method of cylindrical shells, find the volume generated. **3**

End of Question 12

Question 13 (15 marks) **Start a new booklet** **Marks**

(a) (i) If $z = cos\theta + isin\theta$, show that $z^n + \frac{1}{z^n} = 2cosn\theta$ **1**

(ii) For $z = r(cos\theta + isin\theta)$, find r and the smallest positive value θ which satisfies $2z^3 = 9 + 3\sqrt{3}\,i$ **2**

(b) (i) Find the values of A, B, and C such that: **3**

$$\frac{4x^2 - 3x - 4}{x^3 + x^2 - 2x} = \frac{A}{x} + \frac{B}{x-1} + \frac{C}{x+2}$$

(ii) Hence evaluate

$$\int \frac{4x^2-3x-4}{x^3+x^2-2x}\;dx$$ **1**

(c) Solve the equation $x^4 - 7x^3 + 17x^2 - x - 26 = 0$, given that $x = (3 - 2i)$ is a root of the equation. **3**

(d) (i) Derive the equation of the tangent at the point $P\left(ct, \frac{c}{t}\right)$ on the rectangular hyperbola $xy = c^2$. **2**

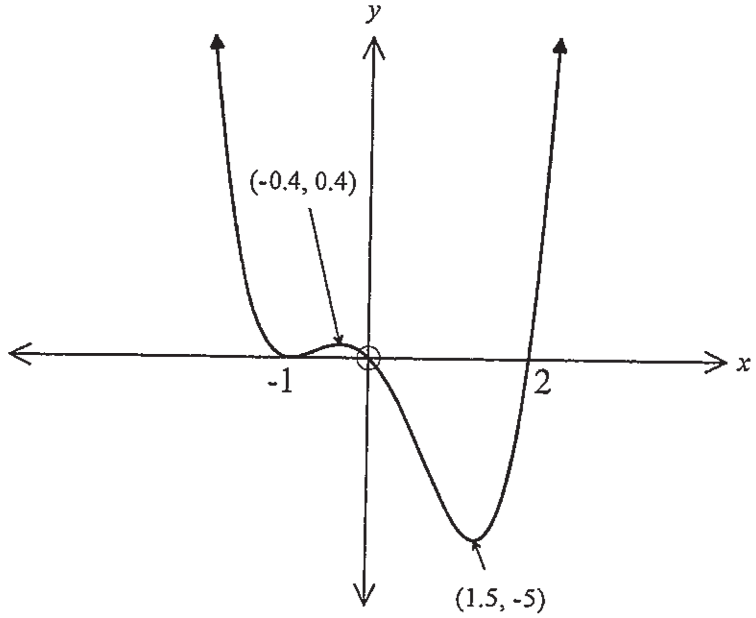
(ii) Find the coordinates of A and B where this tangent cuts the x and y axis respectively. **2**

(iii) Prove that the area of the triangle OAB is a constant. (Where O is the origin). **1**

End of Question 13

Question 14 **(15 marks)** **Start a new booklet** **Marks**

(a) The graph of $y = f(x)$ is shown below.



Draw neat, **separate** sketches for each of the following, showing all important features.

(i) $y = |f(x)|$ **1**

(ii) $y = \frac{1}{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

(b) Show that the equation of the normal to the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ **3**

at the point $P(x_1, \ y_1)$ is given by the equation: $\frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$

(c) A particle is projected vertically upwards. The resistance to the motion is proportional to the square of the velocity. The velocity of projection is V

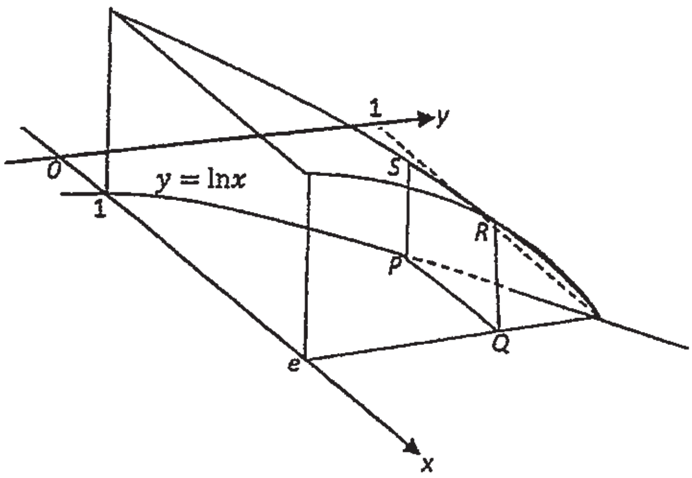
(i) Show that the acceleration is given by: $\ddot{x} = -(g + kv^2)$ **1**

(ii) Find the maximum height reached and the time taken to reach this height, expressing your answer in terms of V and k. **4**

End of Question 14

Question 15 (15 marks) **Start a new booklet** **Marks**

(a)



The base of a solid is the region bounded by the curve $y = \ln(x)$, the x-axis, and the lines $x = 1$ and $x = e$, as shown in the diagram.
Vertical cross-sections taken through this solid in a direction parallel to the x-axis are squares. A typical cross-section PQRS is shown.
Find the volume of the solid. **3**

(b) A particle of mass m kg is dropped from rest in a medium where the resistance to motion has magnitude $\frac{1}{40}mv^2$ when the speed of the particle is v ms^{-1}. After t seconds the particle has fallen x metres. The acceleration due to gravity is 10 ms^{-2}.

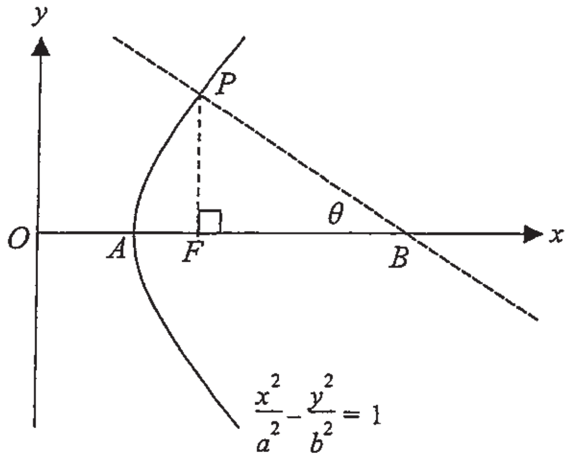
(i) Explain why $\ddot{x} = \frac{1}{40}\left(400 - v^2\right)$. **1**

(ii) Find an expression for t in terms of v by integration. **2**

(iii) Show that $v = 20\left(1 - \dfrac{2}{1+e^t}\right)$. **1**

(iv) Find x as a function of t. **2**

(c)



In the diagram, F is a focus of the hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$ with eccentricity e.
This branch of the hyperbola cuts the x axis at A where $AF = h$. P is the point on the hyperbola vertically above F and the normal at P cuts the x axis at B making an acute angle θ with the x axis.

(i) Show that $\tan\theta = \frac{1}{e}$. **3**

(ii) Show that $PF = h(e+1)$ **1**

(d) The equation $x^3 + 3x^2 + 2x + 1 = 0$ has roots α, β and γ.

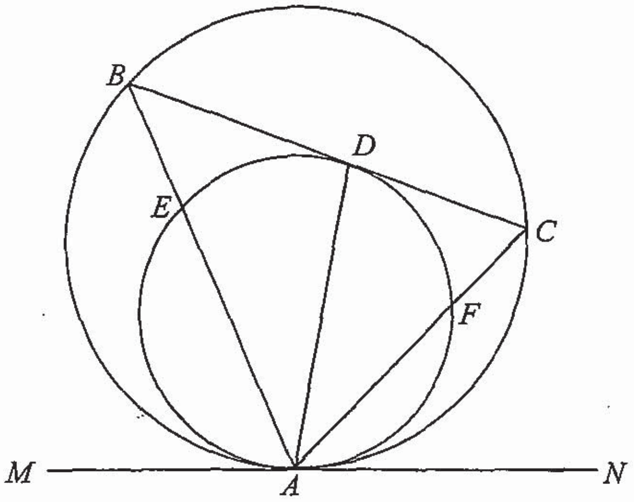
Find the monic cubic equation with roots α^2, β^2 and γ^2. **2**

End of Question 15

Question 16 (15 marks) **Start a new booklet** **Marks**

(a) If $A(x)$ and $B(x)$ are odd polynomial functions show that the product $P(x) = A(x).B(x)$ is an even polynomial function. **2**

(b)



In the diagram, MAN is the common tangent to two circles touching internally at A. B and C are two points on the larger circle such that BC is a tangent to the smaller circle with point of contact D. AB and AC cut the smaller circle at E and F respectively. Copy the diagram. Show that AD bisects $\angle BAC$. **4**

(c) Derive the reduction formula:

$$\int x^n e^{-x^2} dx = -\frac{1}{2} x^{n-1} e^{-x^2} + \frac{n-1}{2} \int x^{n-2} e^{-x^2} dx$$

and use this reduction formula to evaluate $\displaystyle\int_0^1 x^5 e^{-x^2} dx$ **4**

(d) Use Mathematical Induction to prove that $5^n > 4n + 12$ for all integers $n > 1$. **3**

(e) Five letters are chosen from the word CHRISTMAS. These five letters are then placed alongside one another to form a five letter arrangement. Find the number of distinct arrangements that are possible, considering all choices. **2**

End of Question 16

END OF EXAMINATION

Ext 2 (Solutions) TRIAL 2015

Q1/ $\lim_{h \to 0} \frac{\sqrt{h+1} - 1}{h} \times \frac{\sqrt{h+1} + 1}{\sqrt{h+1} + 1}$

$= \lim_{h \to 0} \frac{h+1-1}{h(\sqrt{h+1}+1)}$

$= \lim_{h \to 0} \frac{1}{\sqrt{h+1}+1}$

$= \frac{1}{2}$ (B)

Q2/ $B - A = C - B$

$2B = A + C$

$B = \frac{A+C}{2}$

$\therefore 2B = A + C$ $\nearrow$ $= \frac{\sin 2B}{\sin B}$

$= \frac{2 \cancel{\sin B} \cos B}{\cancel{\sin B}}$

$= 2\cos B$ (A)

Q3/ $x = 1$, $x = -1$ (vertical asymptotes)

$\lim_{x \to \infty} \frac{\frac{x^2}{x^2}}{\frac{x^2}{x^2} - \frac{1}{x^2}}$

$\lim_{x \to \infty} \frac{1}{1 - \frac{1}{x^2}}$ $\therefore y = 1$ (horizontal asymptote)

$= 1$

(C)

4/ $z + \overline{z} = 2\,\text{Re}\,z$ (D)

5/ Asymptotes are $y = \pm \frac{1}{\sqrt{3}} x$

$\therefore \tan\theta = \left| \dfrac{\frac{1}{\sqrt{3}} + \frac{1}{\sqrt{3}}}{1 - \frac{1}{3}} \right|$

$\tan\theta = \left| \frac{2}{\sqrt{3}} \div \frac{2}{3} \right|$

$\tan\theta = \frac{2}{\sqrt{3}} \times \frac{3}{2}$

$\tan\theta = \frac{3}{\sqrt{3}}$

$\therefore \tan\theta = \sqrt{3}$

$\theta = \frac{\pi}{3}$ (C)

6/ $t = \tan\frac{x}{2}$

$\frac{dt}{dx} = \frac{1}{2}\sec^2\frac{x}{2}$

$\frac{2\,dt}{\sec^2\frac{x}{2}} = dx$

$\frac{2\,dt}{1+t^2} = dx$

when $x = \frac{\pi}{2}$

$t = \tan\frac{\pi}{4}$

$\therefore t = 1$

when $x = 0$

$t = \tan 0$

$\therefore t = 0$

$1 + \sin x = 1 + \frac{2t}{1+t^2}$

$$\frac{1+2t}{1+t^2} = \frac{t^2+2t+1}{1+t^2}$$

$$= \frac{(t+1)^2}{1+t^2}$$

$$\int_0^1 \frac{\cancel{t^2+1}}{(1+t)^2} \times \frac{2\,dt}{\cancel{t^2+1}}$$

$$= \int_0^1 \frac{2}{(1+t)^2}\,dt \qquad (D)$$

7/ $y = 4x^2 - x^4$

$x^4 - 4x^2 + 4 = 4 - y$

$(x^2-2)^2 = 4-y$

$(x^2-2) = \pm\sqrt{4-y}$

$A = \pi\left(x_2^2 - x_1^2\right)$ (Radius outer − Radius inner)

$\Delta V = \pi\left(x_2^2 - x_1^2\right)\Delta y$

$= \pi\left\{\left(2+\sqrt{4-y}\right) - \left(2-\sqrt{4-y}\right)\right\}\Delta y$

$= 2\pi\sqrt{4-y}\ \Delta y$

$$\therefore V = 2\pi \int_0^4 \sqrt{4-y}\, dy \quad (A)$$

8/ $x^4 + px + q = 0$

$$\left.\begin{aligned} \alpha^4 + p\alpha + q &= 0 \\ \beta^4 + p\beta + q &= 0 \\ \gamma^4 + p\gamma + q &= 0 \\ \delta^4 + p\delta + q &= 0 \end{aligned}\right\} \text{ADD}$$

$$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 + p(\alpha + \beta + \gamma + \delta) + 4q = 0$$

$$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 + 0 + 4q = 0$$

$$\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = -4q \quad (A)$$

9/ Domain is $-1 \le x \le 1$

$$\therefore \sin^{-1}(-1) + \tan^{-1}(-1) \le y \le \sin^{-1}(1) + \tan^{-1}(1)$$

$$-\frac{\pi}{2} + -\frac{\pi}{4} \le y \le \frac{\pi}{2} + \frac{\pi}{4}$$

$$-\frac{3\pi}{4} \le y \le \frac{3\pi}{4} \quad (C)$$

10/ $e^x + \frac{dy}{dx} e^y = 0$

$$\frac{dy}{dx} = \frac{-e^x}{e^y} \quad (A)$$

$$= -e^{x-y}$$

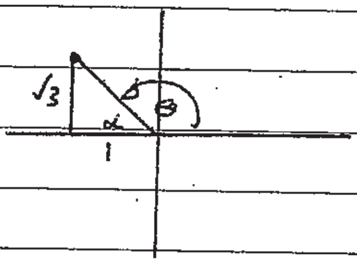
Q11/ (a) $z = 1 + 3i \qquad \bar{z} = 1 - 3i$

$w = 2 - i \qquad \bar{w} = 2 + i$

(i) $1 - 3i - 2 + i = -1 - 2i$

(ii) $(1+3i)(2-i) = 2 - i + 6i + 3$
$= 5 + 5i$

(b) (i) $R = \sqrt{3+1}$
$= 2$



$\tan\alpha = \sqrt{3}$

$\alpha = \frac{\pi}{3}$

$\therefore \theta = \frac{2\pi}{3}$

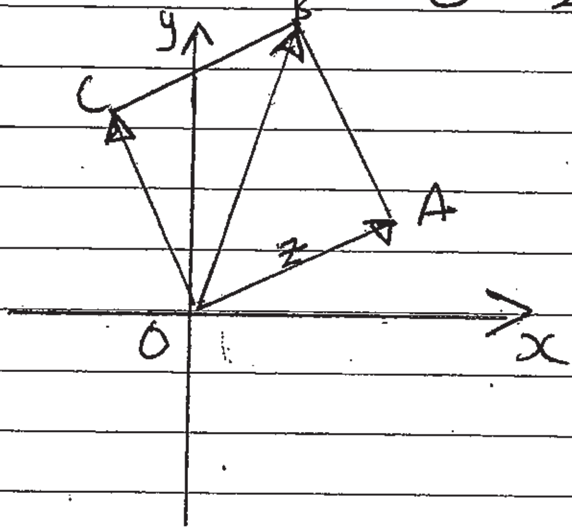
$\therefore -1 + \sqrt{3}i = 2\left(\cos\frac{2\pi}{3} + i\sin\frac{2\pi}{3}\right)$
$= 2\,\text{cis}\,\frac{2\pi}{3}$

(ii) $z^8 = 2^8\left(\cos\frac{16\pi}{3} + i\sin\frac{16\pi}{3}\right)$
$= 2^8\left(-\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$
$= 2^8\left(-\frac{1}{2} - \frac{\sqrt{3}i}{2}\right)$
$= 2^8\frac{(-1 - \sqrt{3}i)}{2}$
$= 2^7(-1 - \sqrt{3}i)$

$16z^4 = 16 \cdot 2^4\left(\cos\frac{8\pi}{3} + i\sin\frac{8\pi}{3}\right)$
$= 2^8\left(-\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$
$= 2^8\left(-\frac{1}{2} + \frac{\sqrt{3}i}{2}\right)$
$= 2^7(-1 + \sqrt{3}i)$

$$\therefore z^8 - 16z^4 = 2^7(-1-\sqrt{3}i) - 2^7(-1+\sqrt{3}i)$$
$$= -2^7 - 2^7\sqrt{3}i + 2^7 - 2^7\sqrt{3}i$$
$$= -2 \cdot 2^7\sqrt{3}i$$
$$= -2^8\sqrt{3}i$$
$$= 0 - 256\sqrt{3}i$$

(e)



$\overrightarrow{OA} = z \quad \therefore \overrightarrow{OC} = iz$

(i) $\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{OC}$
$$= z + iz$$
$$= z(1+i)$$

(ii) $4 + 2i = z(1+i)$
$$\frac{4+2i}{1+i} = z$$
$$\frac{(4+2i)}{(1+i)} \times \frac{(1-i)}{(1-i)} = z$$
$$6 - 2i = 2z$$
$$3 - i = z$$

(d) $P(x) = x^6 + ax^3 + bx^2$

$P(-1) = 0$

$\therefore 0 = (-1)^6 + a(-1)^3 + b(-1)^2$

$0 = 1 - a + b$

$\boxed{a - b = 1}$

$P'(x) = 6x^5 + 3ax^2 + 2bx$

$P'(-1) = 0$

$\therefore 0 = 6(-1)^5 + 3a(-1)^2 + 2b(-1)$

$0 = -6 + 3a - 2b$

$\boxed{3a - 2b = 6}$

$$\left.\begin{array}{l} 3a - 2b = 6 \\ 2a - 2b = 2 \end{array}\right\} \text{subtract}$$

$a = 4$

$b = 3$

(e) sum of roots 1 at a time are

$$\alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta} = \frac{-b}{a}$$

$$\therefore \alpha + \frac{1}{\alpha} + \beta + \frac{1}{\beta} = -b$$

Sum of roots 3 at a time

$$\left(\alpha \times \frac{1}{\alpha} \times \beta\right) + \left(\alpha \times \beta \times \frac{1}{\beta}\right) + \left(\frac{1}{\alpha} \times \beta \times \frac{1}{\beta}\right) + \left(\alpha \times \frac{1}{\alpha} \times \frac{1}{\beta}\right) = \frac{-d}{a}$$

$$\beta + \alpha + \frac{1}{\alpha} + \frac{1}{\beta} = -d$$

$\therefore b = d$

Q12/(a) $\displaystyle\int \frac{e^{-x}}{1+e^{x}}\,dx$

$$= \int \frac{1}{e^{x}(1+e^{x})}\,dx$$

For partial fractions let $u = e^{x}$

$$\frac{1}{u(1+u)} = \frac{a}{u} + \frac{b}{(1+u)}$$

$\therefore 1 \equiv a(1+u) + bu$

Let $u = 0$

$a = 1$

Let $u = -1$

$b = -1$

$$\therefore \int \frac{1}{e^{x}(1+e^{x})}\,dx = \int \frac{1}{e^{x}} - \frac{1}{1+e^{x}}\,dx$$

$$= \int e^{-x} - \left(\frac{1+e^{x}-e^{x}}{1+e^{x}}\right)dx$$

$$= \int e^{-x} - \left(1 - \frac{e^{x}}{1+e^{x}}\right)dx$$

$$= -e^{-x} - x + \ln(1+e^{x}) + C$$

(ii) $\displaystyle\int \frac{x^2}{x+1}\,dx$

$$\begin{array}{r l} & x-1 \\ x+1 \,\big)\!\!\! & \overline{x^2+0x+0} \\ & \underline{x^2+x} \\ & \quad -x+0 \\ & \quad \underline{-x-1} \\ & \qquad\quad 1 \end{array}$$

$$\therefore \int \frac{x^2}{x+1}\,dx = \int x-1+\frac{1}{x+1}\,dx$$

$$= \frac{x^2}{2} - x + \ln|x+1| + C$$

(b) $\displaystyle\int_0^{\frac{\pi}{6}} \sec^3 2\theta \, d\theta$

$$= \int_0^{\frac{\pi}{6}} \sec^2 2\theta \cdot \sec 2\theta \, d\theta$$

Let $u = \sec 2\theta$

$u = (\cos 2\theta)^{-1}$

$$\frac{du}{d\theta} = -(\cos 2\theta)^{-2} \cdot -2\sin 2\theta$$

$$= \frac{2\sin 2\theta}{\cos 2\theta \cdot \cos 2\theta}$$

$$= 2\sec 2\theta \tan 2\theta$$

$$\frac{dV}{d\theta} = \sec^2\theta$$

$$V = \frac{\tan 2\theta}{2}$$

$$= \left[\frac{\tan 2\theta}{2} \cdot \sec 2\theta\right]_0^{\frac{\pi}{6}} - \int_0^{\frac{\pi}{6}} \sec 2\theta \cdot \tan^2 2\theta \, d\theta$$

$$= \left[\frac{\tan\frac{\pi}{3}}{2} \cdot \sec\frac{\pi}{3}\right] - \int_0^{\frac{\pi}{6}} \sec 2\theta \left(\sec^2 2\theta - 1\right) d\theta$$

$$= \frac{\sqrt{3}}{2} \cdot 2 - \int_0^{\frac{\pi}{6}} \sec^3 2\theta - \sec 2\theta \, d\theta$$

$$\therefore 2\int_0^{\frac{\pi}{6}} \sec^3 2\theta \, d\theta = \sqrt{3} + \int_0^{\frac{\pi}{6}} \sec 2\theta \, d\theta$$

$$= \sqrt{3} + \int_0^{\frac{\pi}{6}} \frac{\sec 2\theta (\sec 2\theta + \tan 2\theta)}{(\sec 2\theta + \tan 2\theta)}$$

$$= \sqrt{3} + \int_0^{\frac{\pi}{6}} \frac{\sec^2 2\theta + \sec 2\theta \tan 2\theta}{(\sec 2\theta + \tan 2\theta)}$$

$$= \sqrt{3} + \frac{1}{2}\int_0^{\frac{\pi}{6}} \frac{2\sec^2 2\theta + 2\sec 2\theta \tan 2\theta}{(\sec 2\theta + \tan 2\theta)}$$

$$= \sqrt{3} + \frac{1}{2}\ln\Big[\sec 2\theta + \tan 2\theta\Big]_0^{\frac{\pi}{6}}$$

$$= \sqrt{3} + \frac{1}{2}\ln\Big[\sec\frac{\pi}{3} + \tan\frac{\pi}{3}\Big] - 0$$

$$= \sqrt{3} + \frac{1}{2}\ln(2+\sqrt{3})$$

$$\therefore \int_0^{\frac{\pi}{6}} \sec^3 2\theta \, d\theta = \frac{\sqrt{3}}{2} + \frac{1}{4}\ln(2+\sqrt{3})$$

(c) $x = \tan\theta$ when $x = \sqrt{3}$

$\frac{dx}{d\theta} = \sec^2\theta$ $\sqrt{3} = \tan\theta$

$dx = \sec^2\theta \, d\theta$ $\theta = \frac{\pi}{3}$

when $x = 1$

$1 = \tan\theta$

$\theta = \frac{\pi}{4}$

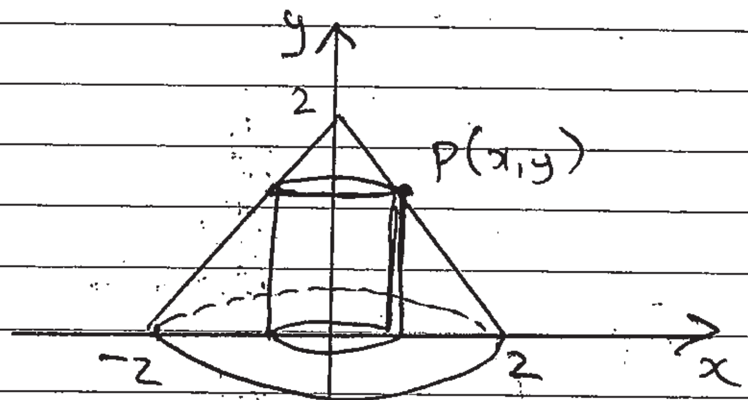
$$\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\tan^2\theta \cdot \sec\theta} \cdot \sec^2\theta \, d\theta$$

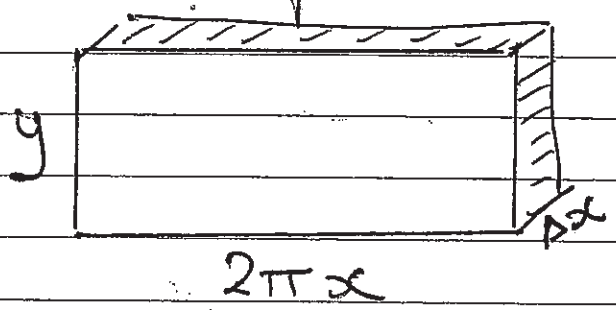
$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{\sec\theta}{\tan^2\theta} \, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\cancel{\cos}\theta \times \frac{\sin\theta}{\cancel{\cos}\theta} \times \tan\theta} \, d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cosec}\theta \cot\theta \, d\theta$$

(d)





$$\Delta V = 2\pi x y \, \Delta x$$

$$\text{But } y = (2-x)$$

$$\Delta V = 2\pi(2x - x^2)\,\Delta x$$

$$\text{Total Volume} = \lim_{\Delta x \to 0} \sum_{x=0}^{2} 2\pi(2x - x^2)\,\Delta x$$

$$= 2\pi \int_0^2 2x - x^2 \cdot dx$$

$$= 2\pi \left[x^2 - \frac{x^3}{3} \right]_0^2$$

$$= 2\pi \left[4 - \frac{8}{3} - (0-0) \right]$$

$$= 2\pi \left[\frac{4}{3} \right]$$

$$= \frac{8\pi}{3} u^2$$

(13) $z = \cos\theta + i\sin\theta$

$$z^n = \cos(n\theta) + i\sin(n\theta)$$

$$\frac{1}{z^n} - z^{-n} = \cos(-n\theta) + i\sin(-n\theta)$$

$$= \cos(n\theta) - i\sin(n\theta)$$

$\therefore$

$$z^n + \frac{1}{z^n} = \cos(n\theta) + i\sin(n\theta) + \cos(n\theta) - i\sin(n\theta)$$

$$= 2\cos(n\theta)$$

(ii) $z^3 = r^3(\cos 3\theta + i\sin 3\theta)$

$2r^3(\cos 3\theta + i\sin 3\theta) = 9 + 3\sqrt{3}i$

$\therefore 2r^3\cos 3\theta = 9$ and $2r^3\sin 3\theta = 3\sqrt{3}$

by division

$\tan 3\theta = \frac{3\sqrt{3}}{9}$

$= \frac{\sqrt{3}}{3}$ OR $\frac{1}{\sqrt{3}}$

$\therefore 3\theta = \frac{\pi}{6}$ (smallest positive value)

$\theta = \frac{\pi}{18}$

Sub into $2r^3\cos 3\theta = 9$

$2r^3\cos\frac{\pi}{6} = 9$

$\cancel{2}r^3\frac{\sqrt{3}}{\cancel{2}} = 9$

$r^3 = \frac{9}{\sqrt{3}}$

$r^3 = \frac{9}{\sqrt{3}} \times \frac{\sqrt{3}}{\sqrt{3}}$

$r^3 = 3\sqrt{3}$

$r = \sqrt{3}$

(b) (i) $A(x-1)(x+2) + Bx(x+2) + C(x-1)x \equiv 4x^2-3x-4$

Let $x=0$ $\quad -2A = -4$

$A = 2$

Let $x = 1$ $\quad 3B = -3$

$B = -1$

Let $x = -2$ $\quad 6C = 18$

$C = 3$

(ii) $\displaystyle\int \frac{2}{x} + \frac{-1}{x-1} + \frac{3}{x+2}\, dx$

$= 2\ln x - \ln(x-1) + 3\ln(x+2) + C$

(c) As $(3-2i)$ is a factor then $3+2i$ is also a factor since the coefficients are real

$\therefore x^2 - (3-2i+3+2i)x + (3-2i)(3+2i)$

$= x^2 - 6x + 13$ is a factor

$$\begin{array}{r}
x^2 - x - 2 \\
x^2-6x+13 \overline{\smash{\big)}\, x^4 - 7x^3 + 17x^2 - x - 26} \\
\underline{x^4 - 6x^3 + 13x^2} \\
-x^3 + 4x^2 - x \\
\underline{-x^3 + 6x^2 - 13x} \\
-2x^2 + 12x - 26 \\
\underline{-2x^2 + 12x - 26} \\
0
\end{array}$$

$\therefore$ $x^2 - x - 2$ is a factor

ie $(x+1)(x-2)$ are factors

$\therefore$ solutions to $x^4 - 7x^3 + 17x^2 - x - 26 = 0$

are $x = 3 \pm 2i$, -1 and 2

(d) $xy = c^2$ $\quad y = c^2 x^{-1}$

$$\frac{dy}{dx} = \frac{-c^2}{x^2} \quad \text{at } x = ct$$

$$m_T = \frac{-c^2}{c^2 t^2}$$

$$= \frac{-1}{t^2}$$

$$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$$

$$t^2 y - ct = -x + ct$$

$$x + t^2 y - 2ct = 0$$

(ii) When $x = 0$ $\quad t^2 y = 2ct$

$$y = \frac{2ct}{t^2}$$

$$y = \frac{2c}{t}$$

$\therefore$ B is $\left(0, \frac{2c}{t}\right)$

When $y=0$ $x=2ct$

$\therefore$ A is $(2ct, 0)$

(iii) $OA = 2ct$

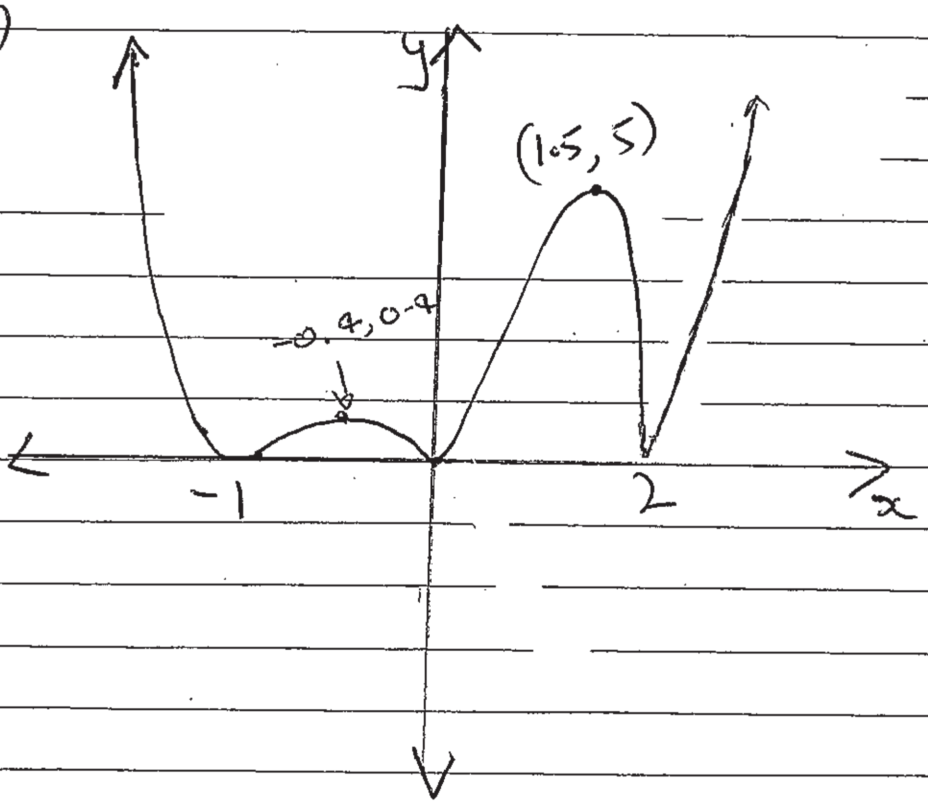
$OB = \frac{2c}{t}$

$$\text{Area}_{\Delta OAB} = \frac{1}{2} \times 2ct \times \frac{2c}{t}$$

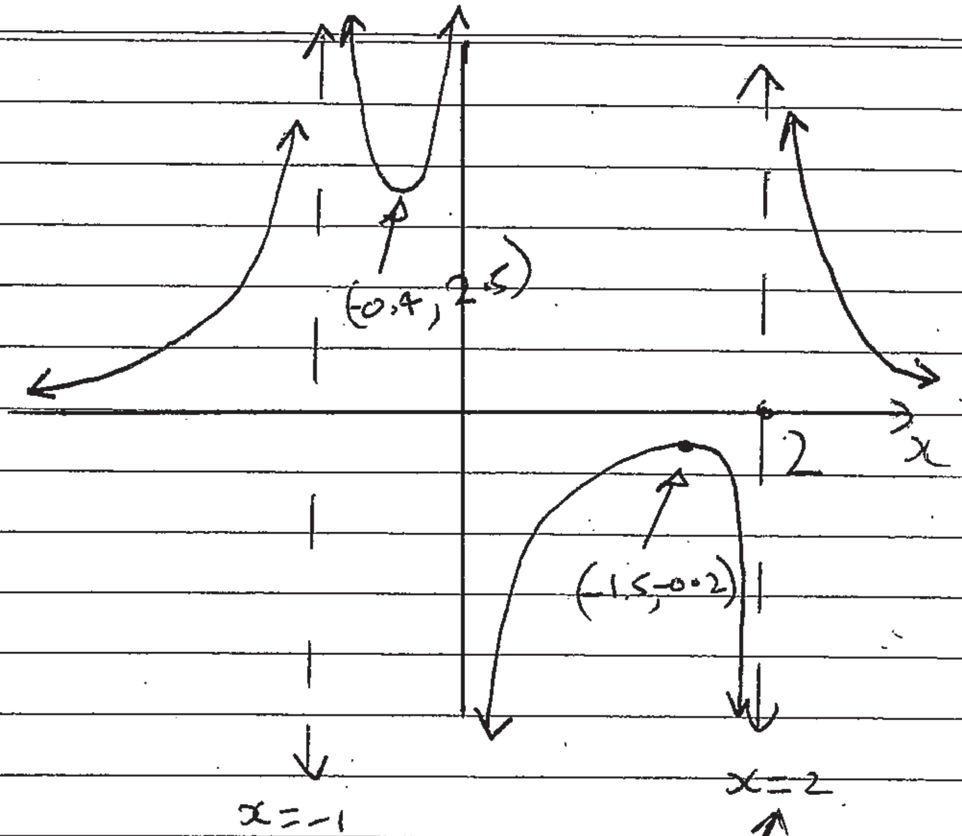
$$= 2c^2$$

which is constant as c is a constant.

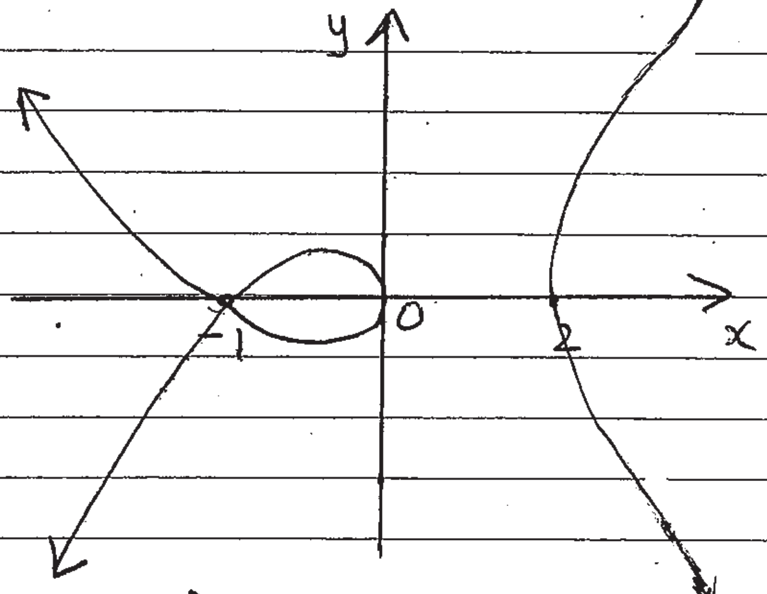
Q14/(a)



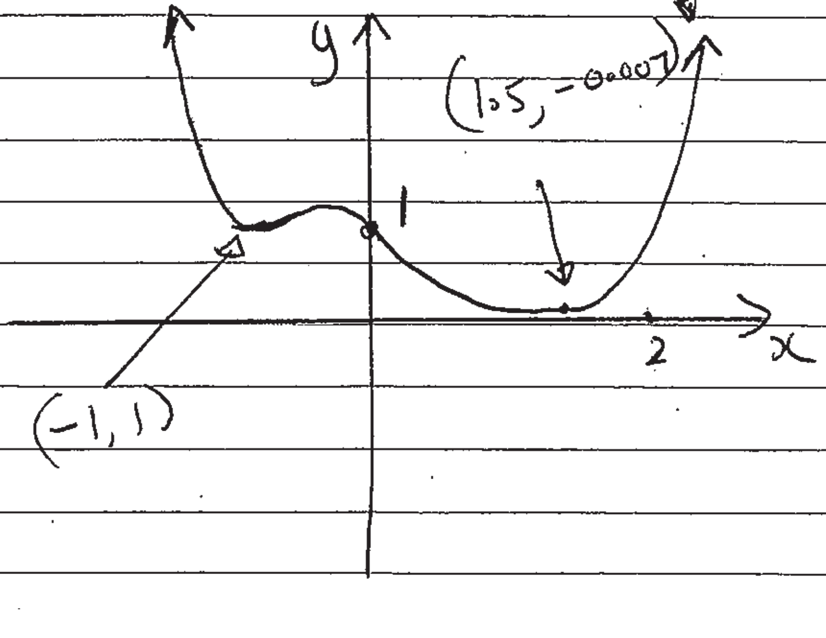
(ii)



iii



(iv)



(b) $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$

$$\frac{2x}{a^2} + \frac{dy}{dx}\frac{2y}{b^2} = 0$$

$$\frac{dy}{dx} = \frac{-2x}{a^2} \times \frac{b^2}{2y}$$

$$= -\frac{b^2x}{a^2y}$$

At (x_1, y_1) $m_n = \frac{a^2y_1}{b^2x_1}$

$$\therefore y - y_1 = \frac{a^2y_1}{b^2x_1}(x - x_1)$$

$$b^2x_1y - b^2x_1y_1 = a^2y_1x - a^2x_1y_1$$

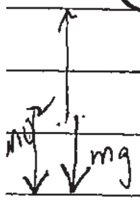
Divide by x_1y_1

$$\frac{b^2y}{y_1} - b^2 = \frac{a^2x}{x_1} - a^2$$

$$\therefore \frac{a^2x}{x_1} - \frac{b^2y}{y_1} = a^2 - b^2$$

(on rearranging)

(c) $m\ddot{x} = -mg - mkv^2$



$$\ddot{x} = -g - kv^2$$

Taking upwards as positive

$$\therefore \ddot{x} = -(g + kv^2)$$

(ii) $v\dfrac{dv}{dx} = -(g+kv^2)$

$$\frac{dv}{dx} = \frac{-(g+kv^2)}{v}$$

$$\frac{dx}{dv} = \frac{v}{-(g+kv^2)}$$

$$x = -\int \frac{v}{g+kv^2}\,dv$$

$$x = -\frac{1}{2k}\int \frac{2kv}{g+kv^2}\,dv$$

$$x = -\frac{1}{2k}\ln(g+kv^2) + C$$

when $x = 0$ $v = V$

$$0 = -\frac{1}{2k}\ln(g+kV^2) + C$$

$$\therefore C = \frac{1}{2k}\ln(g+kV^2)$$

$$x = -\frac{1}{2k}\ln(g+kv^2) + \frac{1}{2K}\ln(g+kV^2)$$

$$x = \frac{1}{2k}\ln\left\{\frac{g+kV^2}{g+kv^2}\right\}$$

Maximum height when $v = 0$

$$x_{max} = \frac{1}{2K}\ln\left(\frac{g+kV^2}{g}\right)$$

$$\frac{dv}{dt} = -(g+kv^2)$$

$$\frac{dt}{dv} = \frac{-1}{g+kv^2}$$

$$t = -\int \frac{1}{g+kv^2}\,dv$$

$$t = -\frac{1}{K}\int \frac{1}{\frac{g}{K}+v^2}\,dv$$

$$\therefore\ t = -\frac{1}{K} \times \frac{1}{\sqrt{\frac{g}{K}}}\tan^{-1}\left(\frac{\sqrt{K}v}{\sqrt{g}}\right) + C$$

$$t = -\frac{1}{K}\frac{\sqrt{K}}{\sqrt{g}}\tan^{-1}\left(\frac{\sqrt{K}v}{\sqrt{g}}\right) + C$$

when $t=0$ $v=V$

$$0 = -\frac{1}{\sqrt{Kg}} \tan^{-1}\left(\frac{\sqrt{K}}{\sqrt{g}} V\right) + C$$

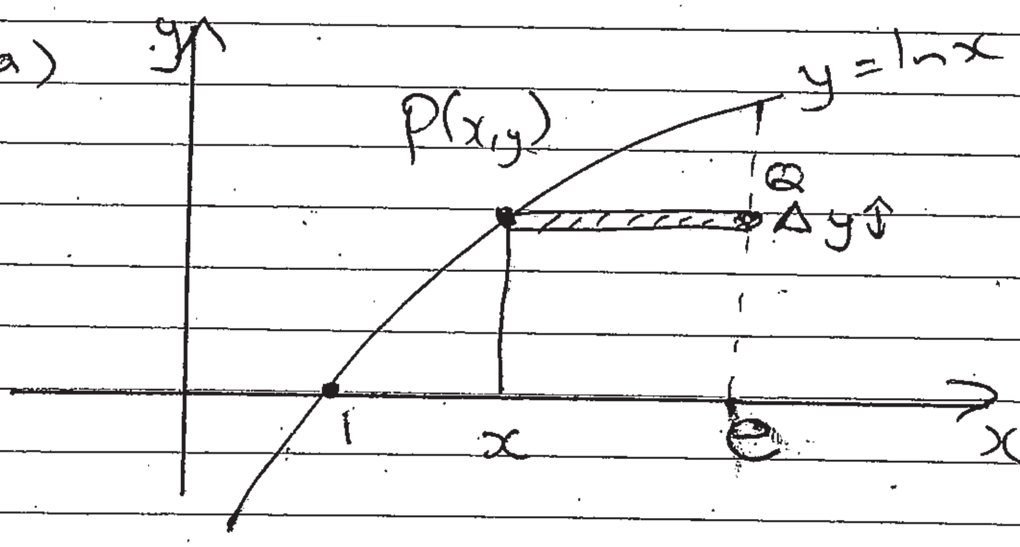
$$C = \frac{1}{\sqrt{Kg}} \tan^{-1}\left(\frac{\sqrt{K}}{\sqrt{g}}\right) V$$

$$\therefore \quad t = -\frac{1}{\sqrt{Kg}} \tan^{-1}\left(\frac{\sqrt{K}}{\sqrt{g}}\right) v + \frac{1}{\sqrt{Kg}} \tan^{-1}\left(\frac{\sqrt{K}}{\sqrt{g}}\right) V$$

Max height when $v = 0$

ie $$t = \frac{1}{\sqrt{Kg}} \tan^{-1}\left(\frac{\sqrt{K}}{\sqrt{g}}\right) V$$

15 (a)



Take a slice through $P(x, y)$ parallel to the x axis with thickness Δy

$PQ = e - x$

$\therefore$ Area of square $PQRS = (e-x)^2$

$\therefore \Delta V = (e-x)^2 \Delta y$

Total Volume $= \lim_{\Delta y \to 0} \sum_{x=1}^{e} (e-x)^2 \Delta y$

but $y = \ln x$

$\frac{dy}{dx} = \frac{1}{x}$

$dy = \frac{1}{x} dx$

$\therefore V = \int_1^e (e-x)^2 \cdot \frac{1}{x} dx$

$V = \int_1^e (e^2 - 2ex + x^2) \times \frac{1}{x} dx$

$= \int_1^e \frac{e^2}{x} - 2e + x \quad dx$

$= \left[e^2 \ln x - 2ex + \frac{x^2}{2} \right]_1^e$

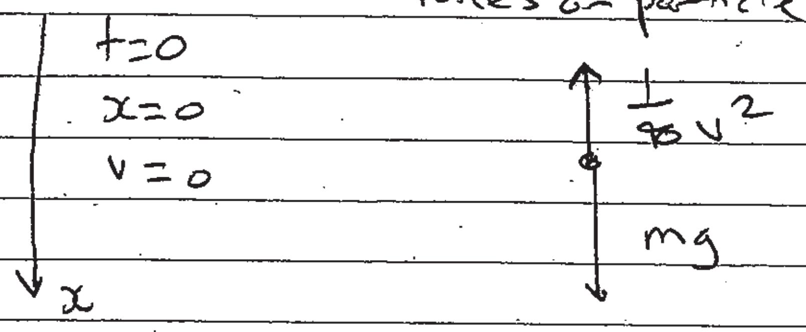
$= e^2 - 2e^2 + \frac{e^2}{2} - (0 - 2e + \frac{1}{2})$

$= -\frac{e^2}{2} + 2e - \frac{1}{2} \quad u^3$

Taking Downwards as +

Q13(b)

(i) Forces on particle



$$m\ddot{x} = mg - \frac{1}{40}mv^2$$

$$\ddot{x} = \frac{1}{40}(400 - v^2) \quad \text{as } g = 10$$

(ii) $$\frac{dv}{dt} = \frac{1}{40}(400 - v^2)$$

$$\frac{dt}{dv} = \frac{40}{20^2 - v^2}$$

$$= \frac{1}{20+v} + \frac{1}{20-v}$$

$$t = \ln\left(\frac{20+v}{20-v}\right) + c$$

when $v = 0$, $t = 0$ $\therefore c = 0$

$$t = \ln\left(\frac{20+v}{20-v}\right)$$

(iii) as $t = \ln\left(\frac{20+v}{20-v}\right)$

then $e^t = \frac{20+v}{20-v}$

$$(20-v)e^t = 20+v$$

$$20e^t - ve^t = 20+v$$
$$20e^t - 20 = v + ve^t$$
$$20(e^t-1) = v(1+e^t)$$

$$\frac{20(e^t-1)}{1+e^t} = v$$

$$20\left(\frac{1+e^t}{1+e^t} - \frac{2}{1+e^t}\right) = v$$

$$\therefore \quad v = 20\left(1 - \frac{2}{1+e^t}\right)$$

(iv) $$\frac{dx}{dt} = 20\left(1 - \frac{\frac{2}{e^t}}{\frac{1}{e^t}+\frac{e^t}{e^t}}\right)$$
$$= 20\left(1 - \frac{2e^{-t}}{e^{-t}+1}\right)$$

$$x = 20\left(t - 2\ln(1+e^{-t})\right)$$

when $x=0$, $t=0$
$$0 = 20(0 - 2\ln 2)$$
$$= -40\ln 2$$

$$\therefore \quad x = 20\left(t - 2\ln\left(\frac{1+e^{-t}}{2}\right)\right)$$

(c) F is $(ae, 0)$ A is $(a, 0)$

$$b^2 = a^2(e^2-1)$$

$$PF = e . PM$$
$$= e\left(ae - \frac{a}{e}\right)$$
$$= ae^2 - a$$
$$= a(e^2-1)$$ $\therefore$ P is $(ae, a(e^2-1))$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{dy}{dx}\frac{2y}{b^2} = 0$$

$$\frac{dy}{dx} = \frac{2x}{a^2} \times \frac{b^2}{2y}$$
$$= \frac{b^2x}{a^2y}$$

$\therefore m_n = \frac{-a^2y}{b^2x}$ so at P

$$m_n = -\frac{a^2 . a(e^2-1)}{b^2 ae}$$

but $b^2 = a^2(e^2-1)$

$$m_n = -\frac{1}{e}$$

then $\tan(180-\theta) = m_n$

$\therefore \tan\theta = \frac{1}{e}$

(ii) $h = AF = a(e-1)$

and $PF = a(e^2-1)$

$\therefore h(e+1) = a(e-1)(e+1)$

$= a(e^2-1)$

$\therefore PF = h(e+1)$

(d) $x^3 + 3x^2 + 2x + 1 = 0$

$\therefore$ with roots α^2, B^2 and Y^2 we get

$(x^{\frac{1}{2}})^3 + 3(x^{\frac{1}{2}})^2 + 2(x^{\frac{1}{2}}) + 1 = 0$

$x^{\frac{3}{2}} + 3x + 2x^{\frac{1}{2}} + 1 = 0$

$x^{\frac{3}{2}} + 2x^{\frac{1}{2}} = -(3x+1)$

$x^{\frac{1}{2}}(x+2) = -(3x+1)$

Square both sides

$x(x+2)^2 = (3x+1)^2$

$x(x^2+4x+4) = 9x^2+6x+1$

$x^3+4x^2+4x = 9x^2+6x+1$

$\therefore$ monic equation is

$x^3 - 5x^2 - 2x - 1 = 0$

Q16/(a) As both $A(x)$ and $B(x)$ are odd

then $A(-x) = -A(x)$
$B(-x) = -B(x)$

Now $P(-x) = A(-x) \cdot B(-x)$

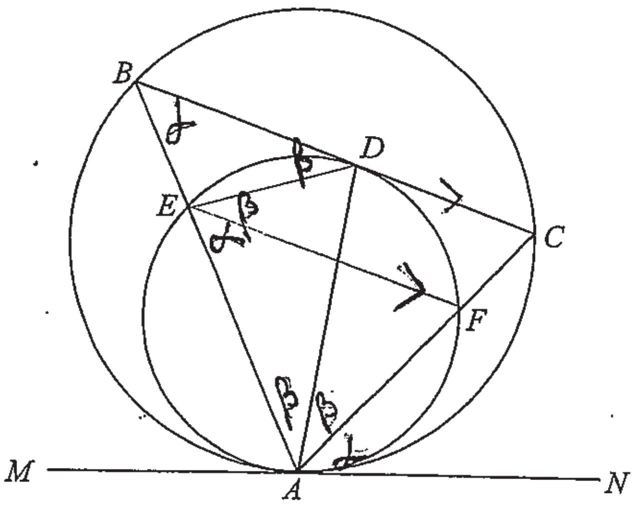
for $P(x) = A(x) \cdot B(x)$

$\therefore P(-x) = -A(x) \cdot -B(x)$

$= A(x) \cdot B(x)$

$\therefore P(-x) = P(x)$ $\therefore$ even.

(b)



Construct ED and EF

$\angle CAN = \angle ABC$ (alternate segment theorem large circle)

$\angle FAN = \angle AEF$ (alternate segment theorem smaller circle)

$\therefore \angle ABC = AEF$

$\therefore BC \parallel EF$ (Corresponding $\angle$'s equal)

$\angle BDE = \angle DAE = \beta$ (alternate segment theorem large circle).

$\angle BDE = \angle DEF = \beta$ (alternate $\angle$'s $BC \parallel EF$)

$\angle DEF = \angle DAC = \beta$ (equal $\angle$'s on circumference standing on arc DF)

$\therefore$ $DAE = \angle DAC = \beta$

Hence AD bisects $\angle BAC$

(c) $\int x^n e^{-x^2}\,dx$

Let $u = x^{n-1}$ $\qquad v' = xe^{-x^2}$

$\frac{du}{dx} = (n-1)x^{n-2}$ $\qquad v = -\frac{1}{2}e^{-x^2}$

$$\therefore \int x^n e^{-x^2}\,dx = -\frac{1}{2}x^{n-1}e^{-x^2} + \frac{n-1}{2}\int x^{n-2}e^{-x^2}\,dx$$

$$\int_0^1 x^5 e^{-x^2}\,dx = \left[-\frac{1}{2}x^4 e^{-x^2}\right]_0^1 + 2\int_0^1 x^3 e^{-x^2}\,dx$$

$$= \frac{-1}{2e} + 2\left[-\frac{1}{2}x^2 e^{-x^2}\right]_0^1 + 1\int_0^1 xe^{-x^2}\,dx$$

$$= -\frac{1}{2e} + 2\times -\frac{1}{2e} + 2\left[-\frac{1}{2}x^0 e^{-x^2}\right]_0^1$$

$$= -\frac{1}{2e} - \frac{1}{e} + 2\left[-\frac{1}{2e} - \frac{1}{2}\right]$$

$$= -\frac{1}{2e} - \frac{1}{e} - \frac{1}{e} - 1$$

$$= -\frac{1}{2e} - \frac{2}{e} - 1$$

$$= -\frac{1}{2e} - \frac{4}{2e} - 1$$

$$= -1 - \frac{5}{2e}$$

(d) $5^n > 4n + 12$ for $n > 1$

Step 1 Prove true for $n = 2$

$5^2 > 4(2) + 12$

$25 > 20$ $\therefore$ true for $n = 2$

Step 2. Assume true for $n = k$ where k is a positive integer

ie $5^k > 4k + 12$

Step 3 Prove true for $n = k+1$

ie $5^{k+1} > 4k + 16$

LHS $= 5.5^k$

$> 5(4k+12)$

from assumption

$> 20k + 60 > 4k + 16$ which is true for k being a positive integer.

Step 4 As, it is true for $n=2$ and if true for $n=k$, it is true for $n=k+1$, therefore true for all positive integers of n, $n>1$

(e) 2 S's

$\boxed{7}\ \boxed{6}\ \boxed{5} \times \frac{20}{2!}$ ways $= 2100$ arrangements

1 S

$\boxed{7}\ \boxed{6}\ \boxed{5}\ \boxed{4} \times 5$ ways $= 4200$ arrangements

No S's

$\boxed{7}\ \boxed{6}\ \boxed{5}\ \boxed{4}\ \boxed{3} = 2520$ arrangements.

$\therefore$ Total arrangements $= 2100 + 4200 + 2520$

$= 8820$ arrangements

Alternative approach

3 Types	No of Selections × No of arrangements	
2 S's	$(1 \times {}^7C_3) \times \frac{5!}{2!}$	$= 2100$
1 S	$(1 \times {}^7C_4) \times 5!$	$= 4200$
0 S's	${}^7C_5 \times 5!$	$= 2520$
		8820