Ext 1

NORTH SYDNEY BOYS HIGH SCHOOL

2015 HSC ASSESSMENT TASK 3

Mathematics

Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write on one side of the paper (with lines) in the booklet provided
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**
- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Berry
- O Mr Ireland
- O Mr Lin
- O Mr Weiss
- O Ms Ziaziaris
- O Mr Zuber

Student Number:

(To be used by the exam markers only.)

Question No	1-10	11	12	13	14	Total	Total
Mark	$\frac{}{10}$	$\frac{}{15}$	$\frac{}{15}$	$\frac{}{15}$	$\frac{}{15}$	$\frac{}{70}$	$\frac{}{100}$

Section I

10 marks
Attempt Questions $1-10$
Allow about 15 minutes for this section

Use the multiple choice answer sheet for Questions $1-10$

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1. What is the value of

$$\lim_{x\to 0}\frac{\sin 4x}{5x}\ ?$$

(A) 0

(B) $\frac{4}{5}$

(C) 1

(D) $\frac{5}{4}$

2. $y = f(x)$ is a linear function with gradient $\frac{1}{4}$, find the gradient of $y = f^{-1}(x)$.

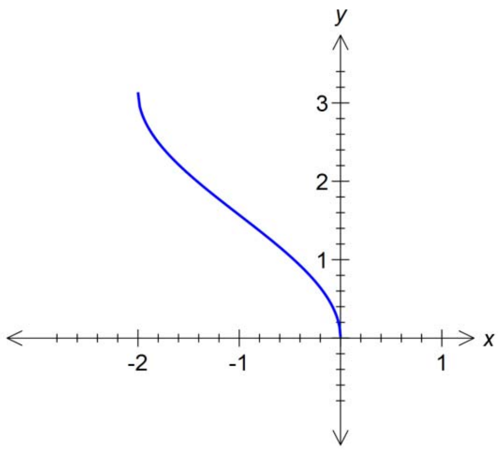
(A) 4

(B) $\frac{1}{4}$

(C) -4

(D) $-\frac{1}{4}$

3.



Which of the following best describes the above function?

(A) $y = \sin^{-1}(x+1)$

(B) $y = \sin^{-1}(x) + 1$

(C) $y = \cos^{-1}(x+1)$

(D) $y = \cos^{-1}(x) + 1$

4. What are the coordinates of the point that divides the interval joining the points A$(-6,4)$ and B$(-2,-10)$ externally in the ratio 1:3?

(A) $(-8,8)$

(B) $(-8,11)$

(C) $(2,8)$

(D) $(2,11)$

5. Which of the following is the solution to $\frac{2}{x-2} < 2$?

(A) $x < 2 \text{ or } x > 3$

(B) $2 < x < 3$

(C) $-2 < x < 3$

(D) $-3 < x < 2$

6. The polynomial $P(x) = 2x^3 - 8x^2 + 7x - 14$ has roots α, $-\alpha$ and β. What is the value of β?

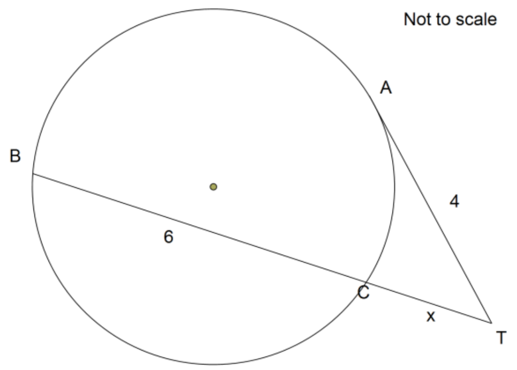
(A) 2

(B) -2

(C) 4

(D) -4

7. The line TA is a tangent to the circle at A and TB is a secant meeting the circle at B and C.



Given that TA = 4, CB = 6 and TC = *x* , what is the value of *x*?

(A) 2

(B) 4

(C) 6

(D)8

8. Given that $\log_a 4 = x$, find an expression for $a^{\frac{3x}{2}}$

(A) 2

(B) 4

(C) 8

(D) 16

9. Find the gradient of the normal to the parabola $x = 6t$, $y = 3t^2$ at the point where $t = -2$.

(A) -2

(B) $-\frac{1}{2}$

(C) $\frac{1}{2}$

(D) 2

10. An approximate solution to the equation $f(x) = x + 2\log_e x$ is $x = 0.5$. Using one application of Newton's method, a more accurate approximation is given by:

(A) $0.5 - \frac{0.5+\log_e 0.25}{5}$

(B) $0.5 + \frac{0.5+\log_e 0.25}{5}$

(C) $0.5 - \frac{5}{0.5+\log_e 0.25}$

(D) $0.5 + \frac{5}{0.5+\log_e 0.25}$

Section II

60 Marks
Attempt Questions $11-14$
Allow about 1 hour and 45 minutes for this section

Answer each question on a NEW page. Extra writing booklets are available.

In Questions $11-14$, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 Marks) Start a NEW page.

(a) When the polynomial $P(x) = 2x^3 - 3x^2 + ax - 2$ is divided by $(x+1)$ the remainder is 7. What is the value of a? **2**

(b) (i)

$$\int \frac{1}{(x+4)^2}dx$$ **1**

(ii)

$$\int \frac{1}{x^2+4}dx$$ **1**

(iii)

$$\int \frac{x}{x^2+4}dx$$ **2**

(iv)

$$\int \frac{x}{(x^2+4)^2}dx$$ **2**

(c) Find the acute angle between the lines $y = 2x + 1$ and $2x + 5y - 2 = 0$ **2**

(d) Evaluate

$$\int_0^{\frac{\pi}{6}} \sin^2 2x \, dx$$ **3**

(e) Find the general solution to $2cos^2x = 1$ **2**

Question 12 (15 Marks) Start a NEW page.

(a) (i) Without using calculus, sketch the graph of $P(x) = x(x+2)(1-x)^2$ **2**

(ii) Hence solve $x(x+2)(1-x)^2 < 0$ **1**

(b) Using the substitution $u = \frac{1}{x}$ find the exact value of:

$$\int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx$$ **3**

(c) (i) A chef takes an onion tart out of the fridge at $4°C$ into a room where the air temperature is $25°C$. The rate at which the onion tart warms follows Newton's law, that is:

$$\frac{dT}{dt} = -k(T-25)$$

where *k* is a positive value, time *t* is measured in minutes and temperature *T* is measured in degrees Celsius.

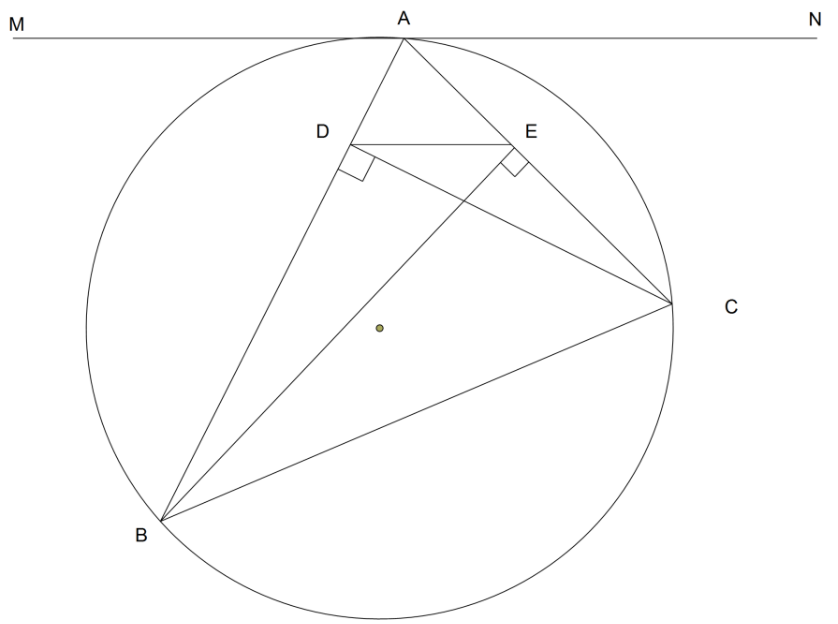
Show that $T = 25 - Ae^{-kt}$ is a solution to $\frac{dT}{dt} = -k(T-25)$ and find the value of *A*. **2**

(ii) The temperature of the onion tart reaches $15°C$ in 45 minutes. Find the exact value of *k*. **2**

(iii) Find the temperature of the onion tart 90 minutes after being removed from the fridge. **1**

Question 12 continues on page 8

(d) (i)



ABC is a triangle inscribed in a circle. *MAN* is the tangent at *A* to the circle *ABC*.
CD and *BE* are altitudes of the triangle.

Copy the diagram into your answer booklet.

(ii) Give a reason why *BCED* is a cyclic quadrilateral **1**

(iii) Hence show that *DE* is parallel to *MAN*. **3**

End of Question 12

Question 13 (15 Marks) Start a NEW page

(a) Is the graph of $y = \log_e x^2$ identical to $y = 2\log_e x$? Give a reason for your answer. **1**

(b) (i) A particle is moving in a straight line. At time *t* seconds it has displacement *x* metres from a fixed point *O* on the line, velocity $v\ ms^{-1}$ and acceleration $a\ ms^{-2}$ given by $a = x + \frac{3}{2}$. Initially the particle is 5*m* to the right of *O* and moving towards *O* with a speed of 6 ms^{-1}.

Explain whether the particle is initially speeding up or slowing down. **1**

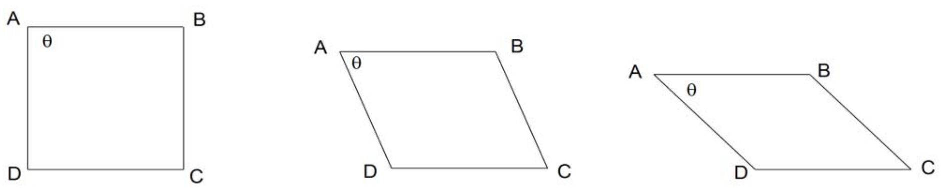
(ii) Find an expression for v^2 in terms of x. **2**

(iii) Find where the particle changes direction. **1**

(c) (i) Express $3\cos\theta - \sqrt{3}\sin\theta$ in the form $A\cos(\theta + \alpha)$ **2**

(ii) Hence, or otherwise, solve $3\cos\theta - \sqrt{3}\sin\theta + 3 = 0$ for $0 \le \theta \le 2\pi$ **2**

(d) (i)



A square *ABCD* of side 1 unit is gradually 'pushed over' to become a rhombus. The angle at *A* (θ) decreases at a constant rate of 0.1 radian per second.

At what rate is the area of rhombus ABCD decreasing when $\theta = \frac{\pi}{6}$? **3**

(ii) At what rate is the shorter diagonal of the rhombus *ABCD* decreasing when $\theta = \frac{\pi}{3}$ **3**

Question 14 (15 Marks) Start a NEW page.

(a) Prove that $11^{2n} + 11^n + 8$ is a multiple of 10 for all positive integers n **3**

(b) (i) Show that $\frac{d}{dx}(x\sin^{-1}x + \sqrt{1-x^2}) = \sin^{-1}x$ **2**

(ii) Hence, using a similar expression, find a primitive for $\cos^{-1}x$ **1**

(iii) The curves $y = \sin^{-1}x$ and $y = \cos^{-1}x$ intersect at $P\left(\frac{1}{\sqrt{2}}, \frac{\pi}{4}\right)$.

The curve $y = \cos^{-1}x$ also intersects with the x axis at Q. **3**

Find the area enclosed by the x-axis and the arcs OP and PQ.

(c) (i) A parabola has parametric equations

$x = t^2 + 1$

$y = 2(2t + 1)$

Sketch the parabola showing its orientation and vertex. **1**

(ii) Point P is the point on the parabola where $t = p$

Point P' is the point on the parabola where $t = -p$

Find the equation of the locus of the midpoint of PP' and state its geometrical significance **2**

(iii) A line with gradient *m* passes through $(0,5)$ and cuts the parabola at distinct points Q and R.

Find the range of possible values for *m*. **3**

End of Examination.

Multiple Choice Answers

1./ B

2./ A

3./ C

4./ B

5./ A

6./ C

7./ A

8./ C

9./ C

10./ A

Question 11

Marks

(a) $P(x) = 2x^3 - 3x^2 + ax - 2$

$P(-1) = 2(-1)^3 - 3(-1)^2 + a(-1) - 2$ ✓

$= -a - 7$

$\therefore 7 = -a - 7$

$\therefore a = -14$ ✓

(b) (i) $\int \frac{1}{(x+4)^2}\, dx$

$= \int (x+4)^{-2}\, dx$

$= (x+4)^{-1} + c$

$= \frac{-1}{x+4} + c$ ✓

(ii) $\int \frac{1}{x^2+4}\, dx$

$= \frac{1}{2} \tan^{-1}\left(\frac{x}{2}\right) + c$ ✓

(iii) $\int \frac{x}{x^2+4}\, dx$

$= \frac{1}{2} \int \frac{2x}{x^2+4}$ ✓

$= \frac{1}{2} \log_e (x^2+4) + c$ ✓

Marks

(iv) $\int \frac{x}{(x^2+4)^2}\, dx$

$= \int x(x^2+4)^{-2}\, dx$

$= \left(-\frac{1}{2}\right)(x^2+4)^{-1} + C$ ✓✓

$= -\frac{1}{2(x^2+4)} + C$

(c) $y = 2x + 1$

$\therefore m_1 = 2$

$2x + 5y - 2 = 0$

$5y = -2x + 2$

$y = -\frac{2}{5}x + \frac{2}{5}$

$\therefore m_2 = -\frac{2}{5}$

$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right|$

$= \left|\frac{2 - (-2/5)}{1 + (2)(-2/5)}\right|$ ✓

$= \left|\frac{2\,2/5}{1/5}\right|$

$\therefore = |12|$

$\therefore \theta = 85°14'11''$ (nearest second) ✓

Marks

(d) $\int_0^{\frac{\pi}{6}} \sin^2(2x)\,dx$

$= \int_0^{\frac{\pi}{6}} \left(\frac{1}{2} - \frac{1}{2}\cos 4x\right) dx$ ✓

$= \left[\frac{x}{2} - \frac{1}{8}\sin 4x\right]_0^{\frac{\pi}{6}}$ ✓

$= \left(\pi/12 - \frac{1}{8}\sin\left(\frac{4\pi}{6}\right)\right) - \left(\frac{0}{2} - \frac{1}{8}\sin(4\times 0)\right)$

$= \frac{\pi}{12} - \frac{1}{8} \times \frac{\sqrt{3}}{2}$

$= \frac{\pi}{12} - \frac{\sqrt{3}}{16}$ ✓

(e) $2\cos^2 x = 1$

$\cos^2 x = \frac{1}{2}$

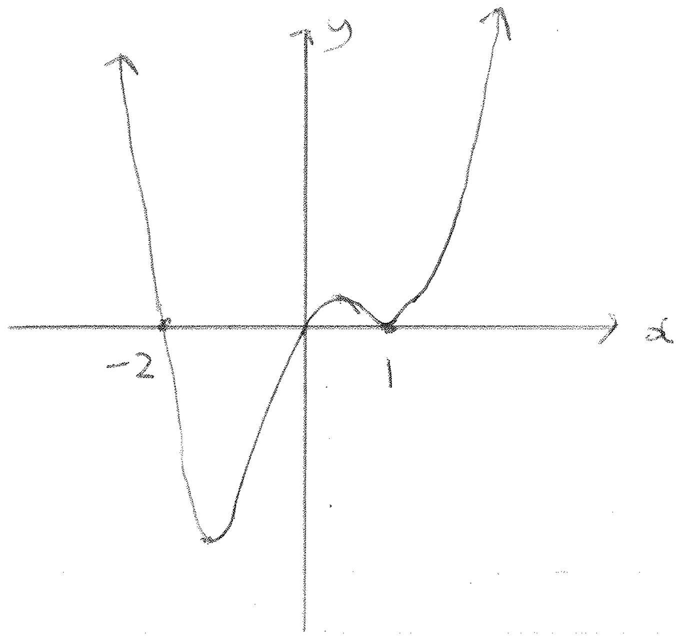
$\cos x = \pm\frac{1}{\sqrt{2}}$ ✓

$\therefore\ x = n\pi \pm \frac{\pi}{4}$ (where n is an integer) ✓

Question 12

Marks

(a)



✓✓

(ii) $-2 < x < 0$ ✓

(b) $u = \frac{1}{x}$

$\therefore \frac{du}{dx} = -\frac{1}{x^2}$

$du = -\frac{1}{x^2}\,dx$ ✓

when $x = 2$
$u = \frac{1}{2}$

when $x = 1$
$u = 1$ ✓

$\therefore \int_1^2 \frac{e^{\frac{1}{x}}}{x^2}\,dx = \int_1^{\frac{1}{2}} (-e^u)\,du$

$= \int_{\frac{1}{2}}^1 e^u$

Marks

$$= \left[e^{u}\right]_{\frac{1}{2}}^{1}$$

$$= e - e^{1/2}$$ ✓

$$= e - \sqrt{e}$$

(c) (i) $T = 25 - Ae^{-kt}$

$$\frac{dT}{dt} = kAe^{-kt}$$

$$= -k\left(-Ae^{-kt}\right)$$

$$= -k\left(\underbrace{25 - Ae^{-kt}}_{T} - 25\right)$$

$$= -k\left(T - 25\right)$$, as required ✓

when $t = 0$ $T = 4$

$$4 = 25 - Ae^{-0k}$$

$$4 = 25 - A$$

$$\therefore A = 21$$ ✓

(ii) when $t = 45$ $T = 15$

$$15 = 25 - 21e^{-45k}$$ ✓

$$-10 = -21e^{-45k}$$

Marks

$$21e^{-45k} = 10$$

$$e^{-45k} = 10/21$$

$$-45k = \log_e(10/21)$$

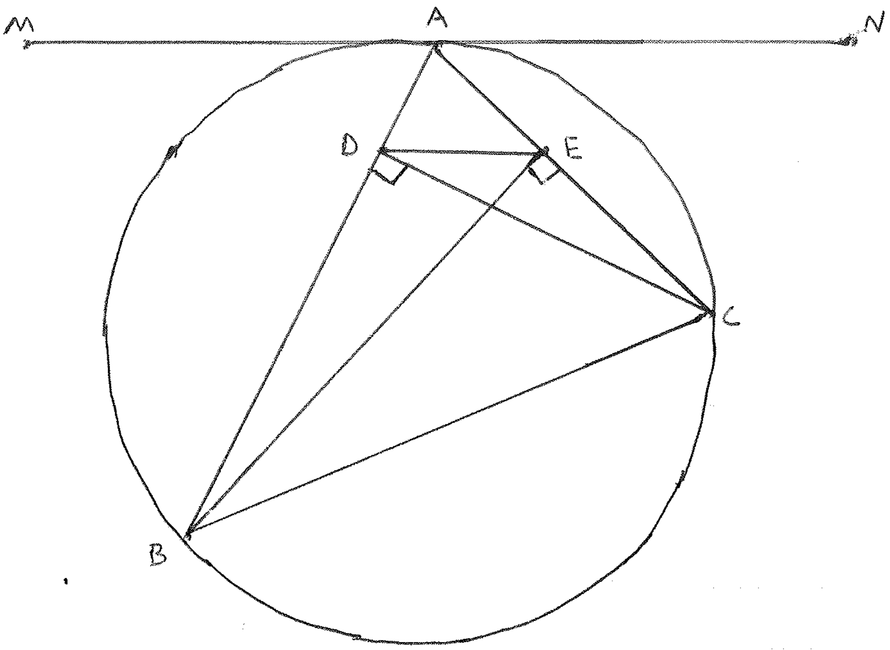
$$\therefore \quad k = -\frac{\log_e(10/21)}{45}$$ ✓

(iii) $T = 25 - 21e^{-90k}$

$= 20.238^\circ$ (3 d.p.) ✓

Marks

(d) (i)



(ii) BC subtends equal angles at D and E ✓

(iii) $\angle ABC = \angle AED$ (exterior angle of a cyclic quadrilateral is equal to the opposite interior angle) ✓

$\angle ABC = \angle NAC$ (angle between a chord and tangent is equal to the angle subtended by the chord at the circumference in the alternate segment) ✓

$\therefore \angle AED = \angle NAC$ (both equal $\angle ABC$)

$\therefore MAN \parallel DE$ (alternate angles are equal) ✓

Question 13

Marks

(a) No. $y = \log_e x^2$ has domain all real x, $x \neq 0$ while $y = 2\log_e x$ has domain $x > 0$.

✓

(b) (i) It is slowing down since velocity is negative while acceleration is positive.

✓

(ii) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = a$

$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = x + \frac{3}{2}$

$\frac{d}{dx}v^2 = 2x + 3$

$v^2 = \int (2x + 3)\, dx$

$v^2 = x^2 + 3x + c$ ✓

when $x = 5$, $v = -6$

$(-6)^2 = 5^2 + 3 \times 5 + c$

$36 = 25 + 15 + c$

$\therefore c = 36 - 40$

$c = -4$

$\therefore v^2 = x^2 + 3x - 4$ ✓

(iii) when $v^2 = 0$

$\therefore\ x^2 + 3x - 4 = 0$

$(x+4)(x-1) = 0$

$\therefore\ x = -4$ or $x = 1$

$\therefore$ the particle changes direction at $\dot{x} = 1$ (not at $x = -4$ since it turns back around at $x = 1$ and continues in the positive direction indefinitely)

✓

(c) (i) $A\cos(\theta + \alpha) = A\cos\theta\cos\alpha - A\sin\theta\sin\alpha$

$\therefore\ A\cos\alpha = 3 \quad \ldots (1)$

$A\sin\alpha = \sqrt{3} \quad \ldots (2)$

$\therefore\ A^2\sin^2\alpha + A^2\cos^2\alpha = (\sqrt{3})^2 + 3^2$

$\therefore\ A^2(\sin^2\alpha + \cos^2\alpha) = 3 + 9$

$\therefore\ A^2 = 12$

$A = \sqrt{12}$ (take $A > 0$)

$A = 2\sqrt{3}$

✓

sub into (2)

$\sqrt{12}\sin\alpha = \sqrt{3}$

$\sin\alpha = \dfrac{\sqrt{3}}{2\sqrt{3}}$

Marks

$\therefore \sin\alpha = \frac{1}{2}$

$\therefore \alpha = \frac{\pi}{6}$

$\therefore 3\cos\theta - \sqrt{3}\sin\theta = \sqrt{12}\cos\left(\theta + \frac{\pi}{6}\right)$ ✓

(ii) $3\cos\theta - \sqrt{3}\sin\theta + 3 = 0$

$\therefore 3\cos\theta - \sqrt{3}\sin\theta = -3$

$\therefore \sqrt{12}\cos(\theta + \pi/6) = -3.$ ✓

$\cos(\theta + \pi/6) = -3/\sqrt{12}$

$\cos\left(\theta + \frac{\pi}{6}\right) = \frac{-3 \times \sqrt{12}}{12}$

$\cos(\theta + \pi/6) = -6\sqrt{3}/12$

$\therefore \cos(\theta + \pi/6) = -\sqrt{3}/2$

since $0 \leq \theta \leq 2\pi$, $\pi/6 \leq \theta + \pi/6 \leq \frac{13\pi}{6}$

$\therefore \theta + \pi/6 = 5\pi/6$ or $7\pi/6$

$\therefore \theta = 4\pi/6$ or $6\pi/6$

$\therefore \theta = 2\pi/3$ or π ✓

Marks

(d) (i) $A = 2 \times \frac{1}{2} \times 1 \times 1 \times \sin\theta$

(area of 2 congruent isosceles triangles)

$$\therefore A = \sin\theta \quad ✓$$

$$\therefore \frac{dA}{d\theta} = \cos\theta$$

$$\frac{dA}{dt} = \frac{dA}{d\theta} \times \frac{d\theta}{dt} \quad ✓$$

$$= \cos\left(\pi/6\right) \times 0.1$$

$$= \sqrt{3}/20 \text{ units}^2/\text{second}. \quad ✓$$

(ii) let the shorter diagonal be l.

$$l^2 = 1^2 + 1^2 - 2 \times 1 \times 1 \times \cos\theta$$

$$l^2 = 2 - 2\cos\theta$$

$$\therefore l = \sqrt{2 - 2\cos\theta} \quad (l > 0) \quad ✓$$

$$\frac{dl}{d\theta} = \cancel{2}\sin\theta \times (2 - 2\cos\theta)^{-\frac{1}{2}} \times \frac{1}{\cancel{2}}$$

$$= \frac{\sin\theta}{\sqrt{2 - 2\cos\theta}} \quad ✓$$

Marks.

$$\frac{dl}{dt} = \frac{dt}{d\theta} \times \frac{d\theta}{dt}$$

$$= \frac{\sin(\pi/3)}{\sqrt{2 - 2\cos \pi/3}} \times 0.1$$

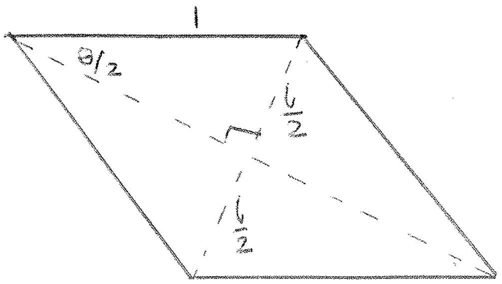
$$= \frac{\sqrt{3}/2}{\sqrt{2 - 2 \times (\frac{1}{2})}} \times 0.1$$

$$= \frac{\sqrt{3}/2}{1} \times 0.1$$

$$= \frac{\sqrt{3}}{20} \text{ units/second.}$$

✓

alternative solution for (d)(ii)



since the diagonals of a rhombus meet at right angles..

$$\sin\left(\frac{\theta}{2}\right) = \left(\frac{l}{2}\right)/1$$

$$\therefore l = 2\sin\frac{\theta}{2}$$

$$\frac{dl}{d\theta} = 2 \times \frac{1}{2} \times \cos\frac{\theta}{2}$$

$$= \cos\frac{\theta}{2}$$

$$\frac{dl}{dt} = \frac{dl}{d\theta} \times \frac{d\theta}{dt}$$

$$= \cos\left(\frac{\pi/3}{2}\right) \times 0.1$$

$$= \sqrt{3}/2 \times 0.1$$

$$= \frac{\sqrt{3}}{20} \text{ units/second}$$

Question 14 Marks

(a) Base Case $(n=1)$

$$11^{2\times 1} + 11^1 + 8 = 121 + 11 + 8$$
$$= 140$$
$$= 14 \times 10$$

which is divisible by 10 ✓

Assume true for $n = k$

i.e assume $11^{2k} + 11^k + 8 = 10M$ where M is an integer

$$\therefore \quad 11^{2k} = 10M - 11^k - 8 \quad \ldots \text{(1)}$$

Prove true for $n = k+1$

i.e prove $11^{2(k+1)} + 11^{k+1} + 8$ is divisible by 10

$$11^{2(k+1)} + 11^{k+1} + 8 = 11^{2k+2} + 11^{k+1} + 8$$

$$= 11^2 \times 11^{2k} + 11^{k+1} + 8$$

$$= 121 \times (10M - 11^k - 8) + 11 \times 11^k + 8$$ ✓
(from (1))

$$= 1210M - 121 \times 11^k - 968 + 11 \times 11^k + 8$$

$$= 1210M - 110 \times 11^k - 960$$

$$= 10(121M - 11 \times 11^k - 96)$$ ✓

Marks

which is divisible by 10

$\therefore$ the proposition is true by the process of mathematical induction.

(b) (i) $\frac{d}{dx}\left(x\sin^{-1}(x) + \sqrt{1-x^2}\right)$

$= \left(\frac{d}{dx}(x)\right)\sin^{-1}(x) + x\left(\frac{d}{dx}\sin^{-1}(x)\right) + \left(\frac{1}{2}\right)(-2x)(1-x^2)^{-\frac{1}{2}}$ ✓

$= 1\sin^{-1}(x) + \frac{x}{\sqrt{1-x^2}} + \frac{-2x}{2\sqrt{1-x^2}}$

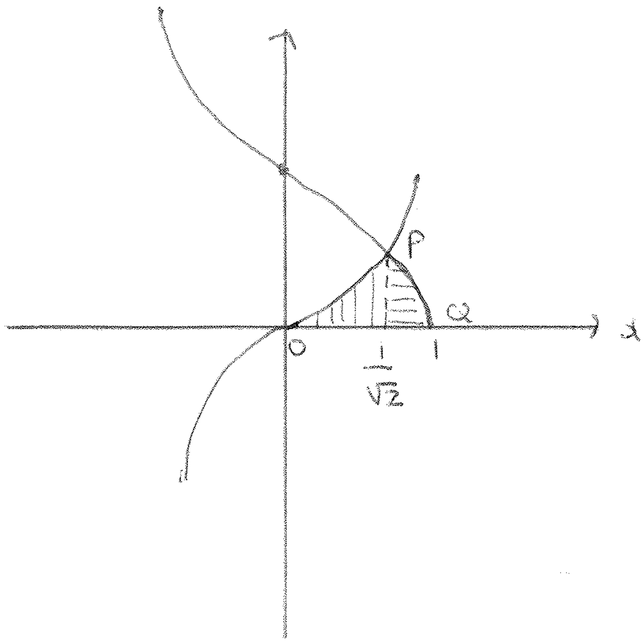
$= \sin^{-1}(x) + \frac{x}{\sqrt{1-x^2}} - \frac{x}{\sqrt{1-x^2}}$ ✓

$= \sin^{-1}(x)$, as required

(ii) $x\cos^{-1}(x) - \sqrt{1-x^2}$ ✓

Marks

(iii)



$$A = \int_{\frac{1}{\sqrt{2}}}^{1} (\cos^{-1} x)\, dx + \int_0^{\frac{1}{\sqrt{2}}} \sin^{-1} x \, dx$$ ✓

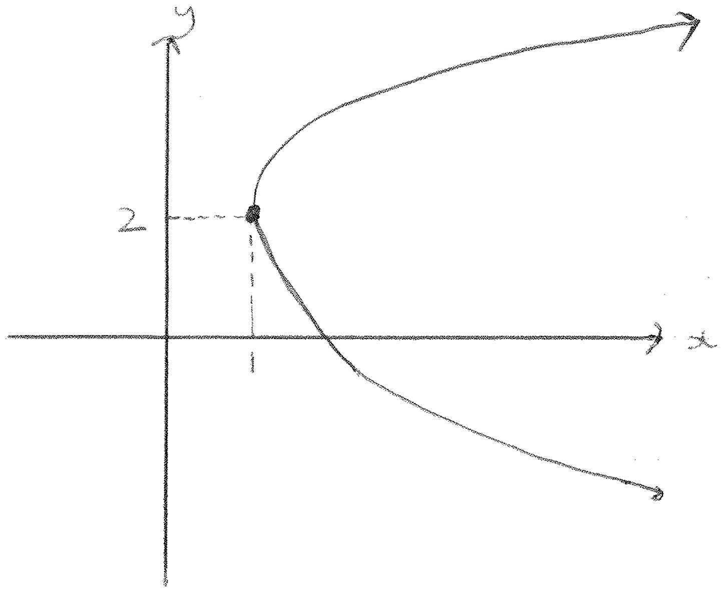
$$= \left[x \cos^{-1} x - \sqrt{1-x^2} \right]_{\frac{1}{\sqrt{2}}}^{1} + \left[x \sin^{-1} x + \sqrt{1-x^2} \right]_0^{\frac{1}{\sqrt{2}}}$$

$$= \left(1 \times 0 - \sqrt{1-1}\right) - \left(\frac{1}{\sqrt{2}} \times \frac{\pi}{4} - \sqrt{1-\tfrac{1}{2}}\right) + \left(\frac{1}{\sqrt{2}} \times \frac{\pi}{4} + \sqrt{1-\tfrac{1}{2}}\right) - \left(0 \sin^{-1}(0) + \sqrt{1-0^2}\right)$$ ✓

$$= -\cancel{\frac{\pi}{4\sqrt{2}}} + \frac{1}{\sqrt{2}} + \cancel{\frac{\pi}{4\sqrt{2}}} + \frac{1}{\sqrt{2}} - 1$$

$$= (\sqrt{2} - 1) \text{ units}^2$$ ✓

(c)(i)



(ii) $M = \left(\frac{(p^2+1) + ((-p)^2+1)}{2}, \frac{2(2p+1) + 2(2(-p)+1)}{2} \right)$

$= \left(\frac{2p^2+2}{2} , \frac{4p+2 \; -4p+2}{2} \right)$

$= ((p^2+1), \; 2)$

$y = 2, \quad x \geq 1$

this is the axis of the parabola

(iii) equation of line is $y = mx + 5$

substituting in $(t^2+1, 2(2t+1))$

$2(2t+1) = m(t^2+1) + 5$

$4t + 2 = mt^2 + m + 5$

$\therefore \; mt^2 - 4t + m + 3 = 0$

Marks

$\Delta > 0$

$\therefore \quad (-4)^2 - (4)(m)(m+3) > 0$ ✓

$16 - 4m^2 - 12m > 0$

$m^2 + 3m - 4 < 0$

$(m+4)(m-1) < 0$

$\therefore \quad -4 < m < 1$

but $m \neq 0$ ✓