PENRITH HIGH SCHOOL

2015
HSC TRIAL EXAMINATION

Mathematics Extension 2

General Instructions:

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In questions 11 – 16, show relevant mathematical reasoning and/or calculations
- Answer all Questions on the writing sheets provided

Total marks–100

SECTION I **Pages 3–7**

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

SECTION II **Pages 9–15**

90 marks

- Attempt Questions 11–16
- Allow about 2 hours 45 minutes for this section

Student Name: ______________________ **Teacher Name:** ______________________

This paper MUST NOT be removed from the examination room

Assessor: Mr Ferguson

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Which pair of coordinates gives the foci of $4x^2 - 25y^2 = 100$?

(A) $\left(\pm\sqrt{29}, 0\right)$

(B) $\left(\pm\dfrac{\sqrt{29}}{5}, 0\right)$

(C) $\left(\pm\dfrac{\sqrt{21}}{5}, 0\right)$

(D) $\left(\pm\sqrt{21}, 0\right)$

2 What are the values of a and b for which the following identity is true?

$$\frac{3x^2+7}{\left(x^2+9\right)\left(x^2+4\right)} = \frac{a}{x^2+9} + \frac{b}{x^2+4}$$

(A) $a = 1$ and $b = 2$

(B) $a = 4$ and $b = -1$

(C) $a = 1$ and $b = -2$

(D) $a = 4$ and $b = 1$

3 The region in the first quadrant between the x-axis and $y = 6x - x^2$ is rotated about the y-axis. The volume of this solid of revolution is.

(A) $\pi\int_0^6 \left(6x - x^2\right) dx$

(B) $\pi\int_0^6 x\left(6x - x^2\right)^2 dx$

(C) $2\pi\int_0^6 x\left(6x - x^2\right) dx$

(D) $\pi\int_0^6 \left(3 + \sqrt{9-y}\right)^2 dx$

4 Which expression is equal to $\int \frac{dx}{\sqrt{4x-x^2}}$?

(A) $\ln\left[(x-2)+\sqrt{6-x}\right]+c$

(B) $\ln\left[(x-2)+\sqrt{6+x}\right]+c$

(C) $\sin^{-1}\frac{x-2}{2}+c$

(D) $\cos^{-1}\frac{x-2}{2}+c$

5 The polynomial $4x^3+x^2-3x+5=0$ has roots α, β and γ. Which polynomial equation has roots $\frac{\alpha}{2}, \frac{\beta}{2}$ and $\frac{\gamma}{2}$?

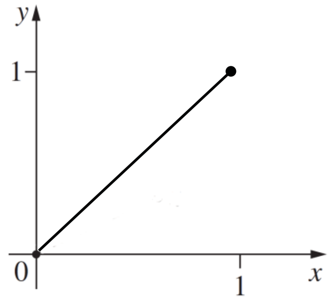
(A) $8x^3+2x^2-6x+10=0$

(B) $2x^3+x^2-6x+5=0$

(C) $32x^3+4x^2-6x+5=0$

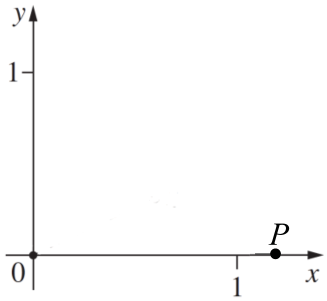
(D) $5x^3-3x^2+x+4=0$

6 The Argand diagram below shows the complex number z .

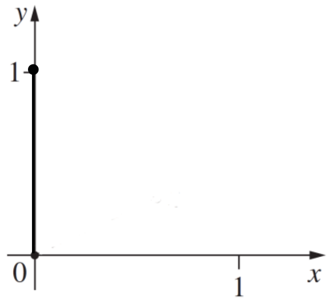


Which diagram best represents the locus of P such that $P = |z|$?

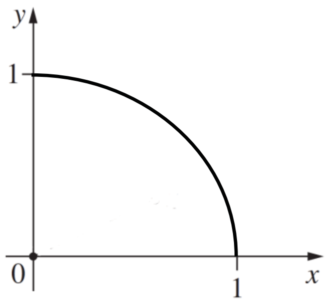
(A)



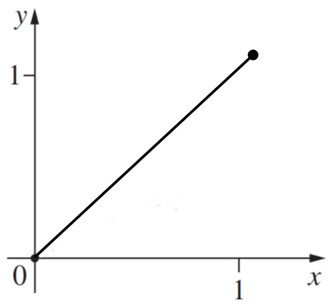
(B)



(C)

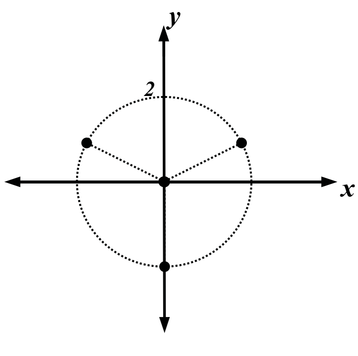


(D)

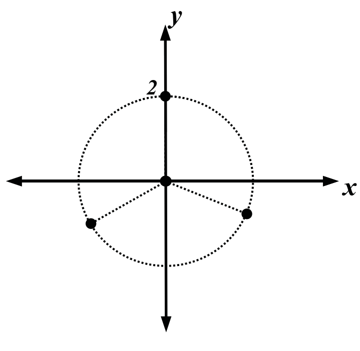


7 Which diagram best represent the cube roots of $8i$?

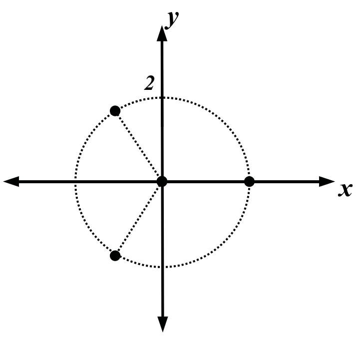
(A)



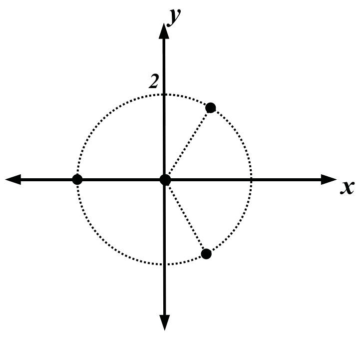
(B)



(C)

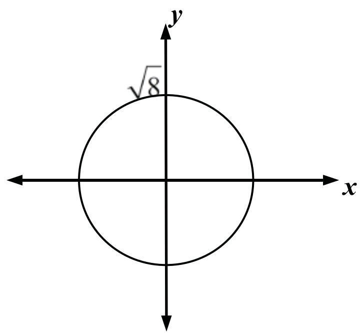


(D)

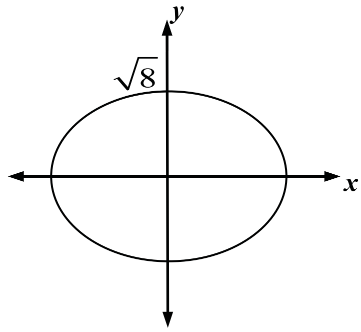


8 Which diagram best represents $z^2 + \overline{z}^2 = 16$

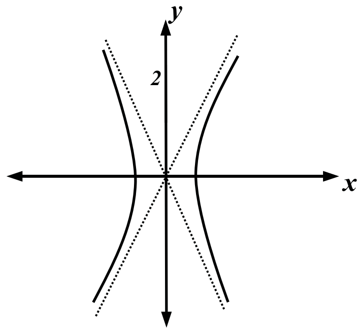
(A)



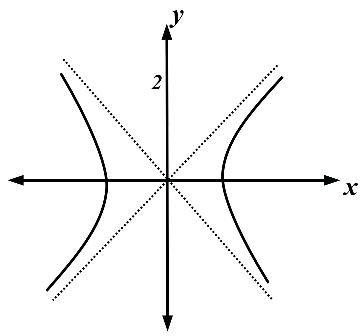
(B)



(C)



(D)



9 A particle of mass m falls from rest under gravity and the resistance to its motion is mkv^2, where v is its speed and k is a positive constant. Which of the following is the correct expression for square of the velocity where x is the distance fallen?

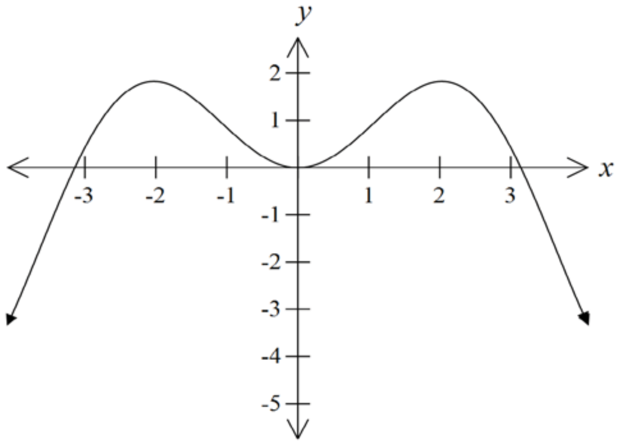
(A) $v^2 = \frac{g}{k}\left(1 - e^{-2kx}\right)$

(B) $v^2 = \frac{g}{k}\left(1 + e^{-2kx}\right)$

(C) $v^2 = \frac{g}{k}\left(1 - e^{2kx}\right)$

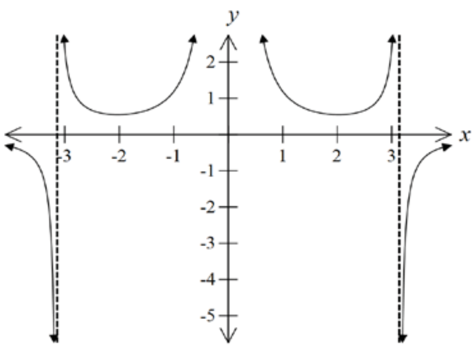
(D) $v^2 = \frac{g}{k}\left(1 + e^{2kx}\right)$

10 The diagram shows the graph of the function $y = f(x)$.

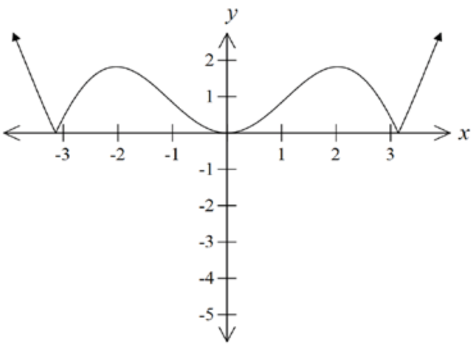


Which of the following is the graph of $y = |f(x)|$?

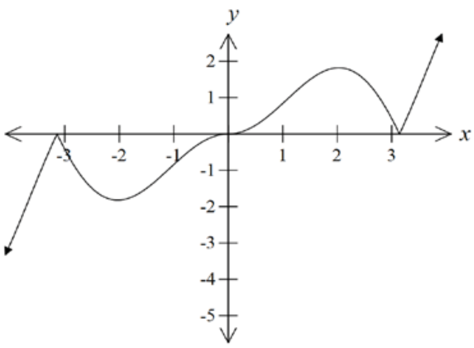
(A)



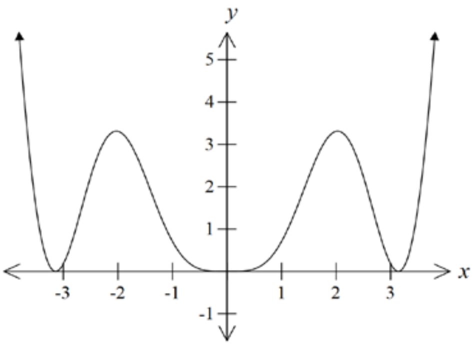
(B)



(C)



(D)



Section II

90 marks
Attempt Questions 11–16
Allow about 2 hours and 45 minutes for this section

Answer each question on a new writing sheet. Extra writing sheets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing sheet.

a) Given that $z = \sqrt{3} + \dfrac{1+i}{1-i}$ find:

(i) $\operatorname{Im}(z)$ **1**

(ii) $\overline{z}$ **1**

(iii) z in modulus argument form **2**

b) Sketch separately the following loci in an Argand plane.

(i) $2\left|z-(1+i)\right| = \left|z-(4+i)\right|$ **2**

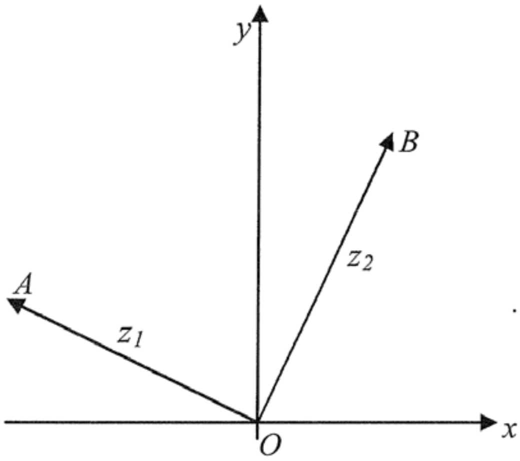
(ii) $\left\{ z : 0 \le \arg(z+4+i) \le \dfrac{2\pi}{3} \text{ and } \left|z+4+i\right| \le 4 \right\}$ **3**

c) Find $\displaystyle\int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x}$ **3**

d) For $x > 0, y > 0, z > 0$ show that $x + y + z + \dfrac{1}{x} + \dfrac{1}{y} + \dfrac{1}{z} \ge 6$ **3**

Question 12 (15 marks) Use a SEPARATE writing sheet.

a)



In the Argand diagram $\overrightarrow{OA}$ and $\overrightarrow{OB}$ represent complex numbers $z_1 = 2\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)$ and $z_2 = 2\left(\cos\frac{5\pi}{12} + i\sin\frac{5\pi}{12}\right)$ respectively.

i) Show that ΔOAB is equilateral **3**

ii) Explain why $z_2 - z_1$ is equal to z_2 rotated by $\frac{\pi}{3}$ radians **1**

iii) Express $z_2 - z_1$ in modulus-argument form. **3**

b) Solve $z^2 = -8 - 6i$ **2**

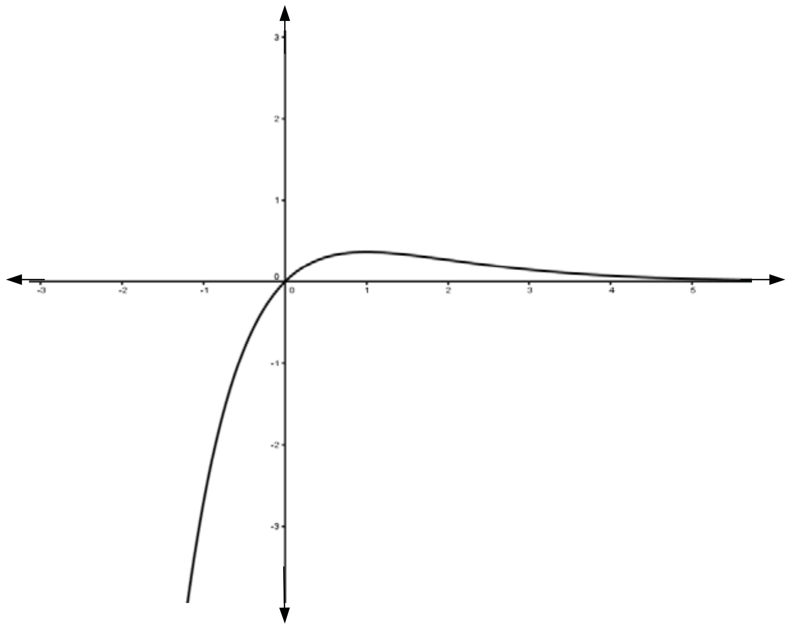
c) (i) Show that the recurrence (reduction) formula for $I_n = \int tan^n x \, dx$ is **3**

$$I_n = \frac{1}{n-1} \tan^{n-1} x - I_{n-2}.$$

(ii) Hence evaluate $\int_0^{\frac{\pi}{4}} tan^3 x \, dx$ **3**

Question 13 (15 marks) Use a SEPARATE writing sheet.

a) The diagram shows the graph of $y = f(x)$. The graph has a horizontal asymptote at $y = 0$.



Draw separate one-third page sketches of the graphs of the following:

(i) $y = (f(x))^2$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = x\,f(x)$ **2**

(iv) $y = f(|x|)$ **2**

b) The ellipse, E, has equation $9x^2 + 25y^2 = 225$.

P is any point on the ellipse and A and B are the points $(5,0)$ and $(-5,0)$ respectively. AP, produced if necessary, meets the y axis in Q, and BP, also produced if necessary, meets the y axis in R

The tangent at P meets the y axis in T

(i) Find the eccentricity **1**

(ii) Sketch the ellipse, E, showing the coordinates of its foci. **2**

(iii) Given that the equation of the tangent at P is $\dfrac{x\cos\theta}{a} + \dfrac{y\sin\theta}{b} = 1$.

Prove that T is the midpoint of QR **4**

Question 14 (15 marks) Use a SEPARATE writing sheet.

a) For any non-zero real number t, the point $\left(t, \dfrac{1}{t}\right)$ lies on the graph of $y = \dfrac{1}{x}$.

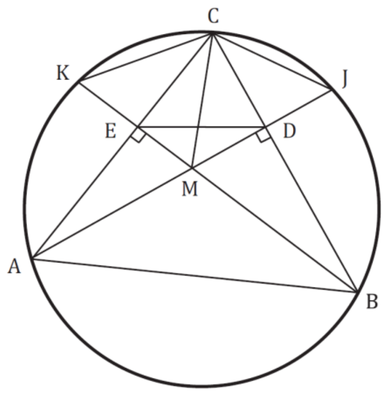
(i) Show that $9xy = 1$ is the equation of the locus of the point that divides the straight line joining $\left(t, \dfrac{1}{t}\right)$ and $\left(-t, \dfrac{-1}{t}\right)$ in the ratio of $1:2$ respectively, as t varies. **2**

(ii) Show that the equation of the tangent to $y = \dfrac{1}{x}$ at the point $\left(t, \dfrac{1}{t}\right)$ may be written in the form $t^2 y - 2t + x = 0$ **2**

(iii) $R(0, h)$ is a point on the y axis. Show that there is exactly one point on the hyperbola $y = \dfrac{1}{x}$ with tangents that pass through R **2**

b) Find $\displaystyle\int \frac{dx}{a^2 \cos^2 x + b^2 \sin^2 x}$ **3**

c)



In the diagram ABC is a triangle inscribed in a circle. The altitude AD is produced to meet the circle at J. The altitude BE is produced to meet the circle at K and the two altitudes intersect at M.

i) Copy the diagram onto your answer sheet

ii) Show that $ABDE$ and $CEMD$ are cyclic **2**

iii) Prove that AC bisects $\angle KCM$ **2**

iv) Prove that $KC = JC$ **2**

Question 15 (15 marks) Use a SEPARATE writing sheet.

a) A particle of mass m kg is set in motion, with speed u ms^{-1} and moves in a straight line before coming to rest. At time t seconds the particle has displacement x metres from its starting point O, velocity v ms^{-1} and acceleration a ms^{-2}

The resultant force acting on the particle directly opposes its motion and has magnitude $m(1+v)$ Newtons.

(i) Show that $a=-(1+v)$ **1**

(ii) Find expressions for

1. x in terms of v **2**

2. v in terms of t **2**

3. x in terms of t **2**

(iii) Show that $x+v+t=u$ **2**

(iv) Find the distance travelled and time taken by the particle in coming to rest. **2**

b) Given that $z=1-2i$ is a factor of the equation $P(z)=z^4-z^3+6z^2-z+15$

(i) Factorise $P(z)$ into real quadratic factors **2**

(ii) Solve for $P(z)=0$ for z **2**

Question 16 (15 marks) Use a SEPARATE writing sheet.

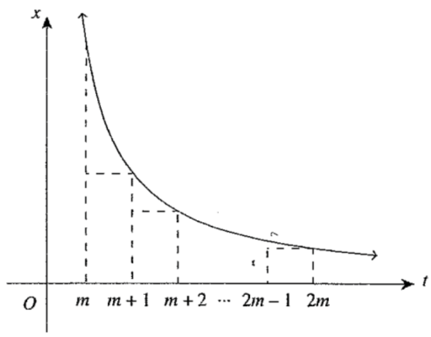
a) (i) Prove that $\dfrac{1}{2p+1}+\dfrac{1}{2p+2}>\dfrac{1}{p+1}$, for all $p>0$ **2**

(ii) Consider the statement

$$\psi(m):\frac{1}{m+1}+\frac{1}{m+2}+............+\frac{1}{2m}\geq\frac{37}{60}$$ **3**

Show that by mathematical induction that $\psi(m)$ is true for all integers $m\geq 3$.

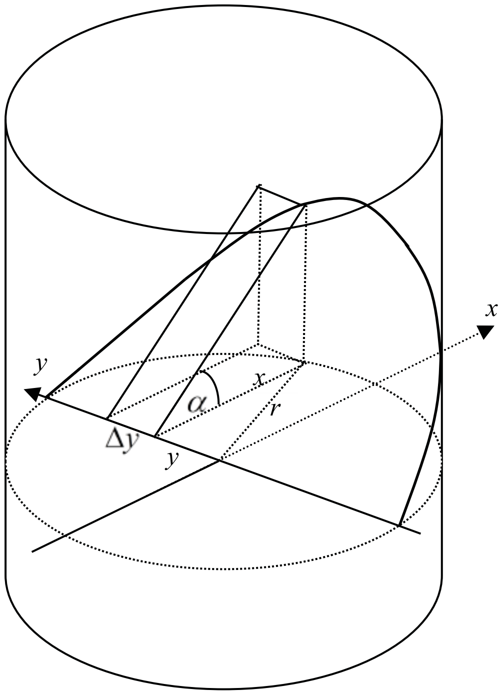
(iii) The diagram below shows the graph of $x=\dfrac{1}{t}$, for $t>0$



(iv) By comparing areas, show that $\displaystyle\int_{m}^{m+1}\frac{1}{t}\,dt>\frac{1}{m+1}$ **2**

(v) Hence, without using a calculator, show that $\log_e 2>\dfrac{37}{60}$ **3**

b) A wedge is cut from a right circular cylinder of radius r by two planes, one perpendicular to the axis of the cylinder while the second makes an angle α with the first and intersects it at the centre of the cylinder.



A is the area of the triangle that forms one face of the slice.

i) Show that $A = \frac{1}{2}(r^2 - y^2)\tan\alpha$. **2**

ii) Hence show that the volume of the wedge is $\frac{2}{3}r^3 \tan\alpha$ **3**

End of Exam

STANDARD INTEGRALS

$$\int x^n \, dx = \frac{1}{n+1} x^{n+1}, \; n \neq -1; \; x \neq 0, \; if \; n < 0$$

$$\int \frac{1}{x} \, dx = \ln x, \; x > 0$$

$$\int e^{ax} \, dx = \frac{1}{a} e^{ax}, \; a \neq 0$$

$$\int \cos ax \, dx = \frac{1}{a} \sin ax, \; a \neq 0$$

$$\int \sin ax \, dx = -\frac{1}{a} \cos ax, \; a \neq 0$$

$$\int \sec^2 ax \, dx = \frac{1}{a} \tan ax, \; a \neq 0$$

$$\int \sec ax \tan ax \, dx = \frac{1}{a} \sec ax, \; a \neq 0$$

$$\int \frac{1}{a^2 + x^2} \, dx = \frac{1}{a} \tan^{-1} \frac{x}{a}, \; a \neq 0$$

$$\int \frac{1}{\sqrt{a^2 - x^2}} \, dx = \sin^{-1} \frac{x}{a}, \; a \neq 0, \; -a < x < a$$

$$\int \frac{1}{\sqrt{x^2 - a^2}} \, dx = \ln(x + \sqrt{x^2 - a^2}), \; x > a > 0$$

$$\int \frac{1}{\sqrt{x^2 + a^2}} \, dx = \ln(x + \sqrt{x^2 + a^2})$$

NOTE: $\ln x = \log_e x, \; x > 0$

Student Number:_____________ Teacher Name:________________

Section I - Multiple Choice

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample: 2 + 4 = (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ⊗ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate the correct answer by writing the word correct and drawing an arrow as follows.

correct

A ⊗ B ⊗ C ○ D ○

Start Here →

1. A ○ B ○ C ○ D ○
2. A ○ B ○ C ○ D ○
3. A ○ B ○ C ○ D ○
4. A ○ B ○ C ○ D ○
5. A ○ B ○ C ○ D ○
6. A ○ B ○ C ○ D ○
7. A ○ B ○ C ○ D ○
8. A ○ B ○ C ○ D ○
9. A ○ B ○ C ○ D ○
10. A ○ B ○ C ○ D ○

Multiple Choice

1) $\frac{x^2}{25} - \frac{y^2}{4} = 1$

$a^2 = 25 \quad b^2 = 4$

focie $(\pm ae, 0)$

$b^2 = a^2(e^2 - 1)$

$4 = 25(e^2 - 1)$

$e^2 - 1 = \frac{4}{25}$

$e^2 = \frac{29}{25}$

$e = \pm\frac{\sqrt{29}}{5}$

$(ae, 0) = (\pm\sqrt{29}, 0)$

$=$ A

2) $a(x^2+4) + b(x^2+9) \equiv 3x^2+7$

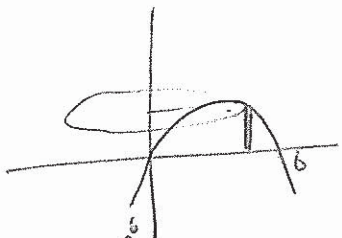
when $a = 4 \quad b = -1$

$4x^2 + 16 - x^2 - 9$

$= 3x^2 + 7$

$\therefore$ B

3)



$2\pi\int_0^6 xy\,dx$

$2\pi\int_0^6 x(6x - x^2)\,dx$

C

4) $\int \frac{dx}{\sqrt{4-(x-2)^2}}$

$= \sin^{-1}\frac{x-2}{2} + C$ (standard integral)

$=$ C

5) $4(2x)^3 + (2x)^2 - 3(2x) + 5 = 0$

$32x^3 + 4x^2 - 6x + 5 = 0$

C

6) A $\quad \rho = |1+i|$

$\rho = \sqrt{2}$

7) A

8) $(x+iy)^2 + (x-iy)^2 = 16$

$x^2 + 2xyi - y^2 + x^2 - 2xyi - y^2 = 16$

$2x^2 - 2y^2 = 16$

$x^2 - y^2 = 8$ (rectangular hyperbola.

D

9) A $\quad m\ddot{x} = mg - mkv^2$

$\ddot{x} = g - kv^2$

$v\frac{dv}{dx} = g - kv^2$

$\frac{dv}{dx} = \frac{g}{v} - kv$

$\frac{dx}{dv} = \frac{v}{g - kv^2}$

$x = -\frac{1}{2k}\ln(g - kv^2) + C$ when $x = 0$ $v = 0$

$x = -\frac{1}{2k}\ln\left(\frac{g - kv^2}{g}\right)$

$\therefore v^2 = \frac{g}{k}(1 - e^{-2kx})$

10) B

Question 11

a) $z = \sqrt{3} + \frac{1+i}{1-i}$

$\frac{1+i}{1-i} = \frac{1+i}{1-i} \times \frac{1+i}{1+i}$

$= \frac{1+i^2+2i}{1-i^2}$

$= \frac{2i}{2}$

$= i$

$\therefore z = \sqrt{3}+i$

(i) $\text{Im}(z) = 1$

(ii) $\bar{z} = \sqrt{3} - i$

(iii) $z = \sqrt{3} + i$

$r^2 = (\sqrt{3})^2 + 1^2$

$= 4$

$\therefore r = 2$

$z = 2\left(\frac{\sqrt{3}}{2} + \frac{i}{2}\right)$

$= 2\left(\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}\right)$

$= 2 \operatorname{cis} \frac{\pi}{6}.$

b) (i) $2|z-(1+i)| = |z-(4+i)|$.

squaring both sides

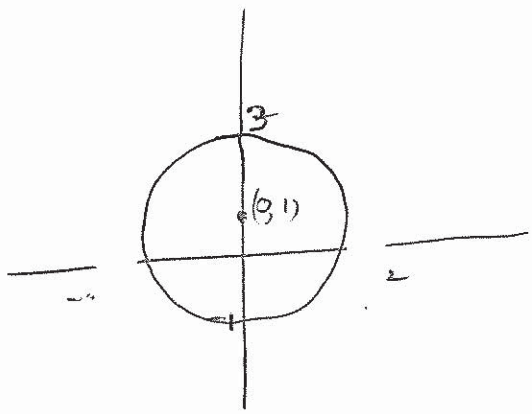
$4[(x-1)^2+(y-1)^2] = (x-4)^2+(y-1)^2$

$4(x^2-2x+1+y^2-2y+1) - (x^2-8x+16+y^2-2y+1) = 0$

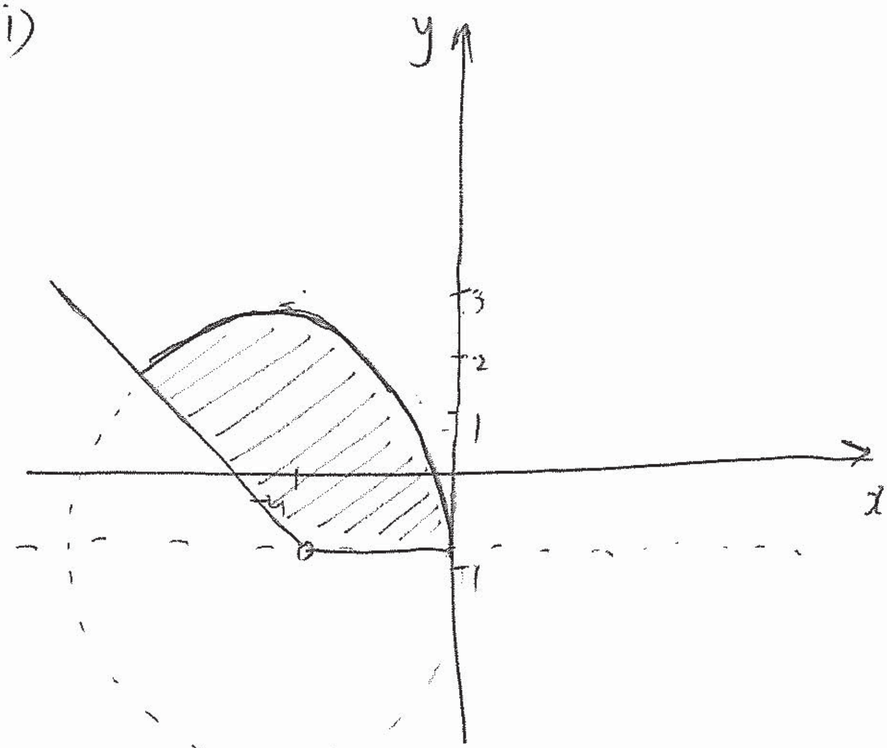
$3x^2-9+3y^2-6y=0$

$x^2+(y-1)^2=4$

$\therefore$ circle centre $(0,1)$ radius 2.



(ii)



c)

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan x}$$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{1+\tan\left(\frac{\pi}{2}-x\right)} \qquad \therefore \int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{dx}{1+\cot x}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\tan x}{1+\tan x}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{1+\tan x-1}{1+\tan x}\,dx.$$

$$= \int_0^{\frac{\pi}{2}} 1 - I \, dx.$$

$$2I = \int_0^{\frac{\pi}{2}} 1\,dx.$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4}.$$

alternative solution

$$I = \int_0^{\frac{\pi}{2}} \frac{dx}{1+\frac{\sin x}{\cos x}}$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos x}{\cos x+\sin x}\,dx \quad — (1)$$

$$= \int_0^{\frac{\pi}{2}} \frac{\cos\left(\frac{\pi}{2}-x\right)}{\cos\left(\frac{\pi}{2}-x\right)+\sin\left(\frac{\pi}{2}-x\right)}\,dx$$

$$= \int_0^{\frac{\pi}{2}} \frac{\sin x}{\sin x+\cos x}\,dx \quad — (2)$$

$$(1)+(2) = \int_0^{\frac{\pi}{2}} \frac{\sin x+\cos x}{\sin x+\cos x}$$

$$2I = \int_0^{\frac{\pi}{2}} 1\,dx$$

$$2I = \left[x\right]_0^{\frac{\pi}{2}}$$

$$2I = \frac{\pi}{2}$$

$$I = \frac{\pi}{4},$$

d) For $x>0,\ y>0,\ z>0$

$$x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z} \geqslant 6$$

$$(a-1)^2 \geqslant 0$$

$$a^2-2a+1 \geqslant 0$$

$$a^2+1 \geqslant 2a$$

$$a+\frac{1}{a} \geqslant 2 \quad \text{as } a \neq 0.$$

$$\therefore\ x+y+z+\frac{1}{x}+\frac{1}{y}+\frac{1}{z}$$

$$= \left(x+\frac{1}{x}\right)+\left(y+\frac{1}{y}\right)+\left(z+\frac{1}{z}\right)$$

$$\geqslant 2+2+2$$

$$\geqslant 6.$$

Question 12

(i) $z_2 \times cis\frac{\pi}{3} = 2cis\frac{5\pi}{12}\, cis\frac{\pi}{3}$

$= 2cis\left(\frac{5\pi}{12}+\frac{\pi}{3}\right)$

$= 2cis\frac{3\pi}{4}$

$\therefore z_1$ is z_2 rotated by $\frac{\pi}{3}$ clockwise

$\therefore \angle AOB = \frac{\pi}{3}$

$|z_1| = 2$

$|z_2| = 2$

$\therefore$ base angles $\angle OAB = \angle ABO$

$= \frac{\pi}{2}$

since angle sum of triangle is π.

$\therefore \triangle OAB$ equilateral

Alternatively $|z_1| = |z_2| = 2$

show $|z_2 - z_1| = 2$.

$|AB| = |z_2 - z_1|$

$= 2\left|cis\frac{5\pi}{12} - cis\frac{3\pi}{4}\right|$

$= 2\sqrt{\left(\cos\frac{\pi}{12} - \cos\frac{3\pi}{4}\right)^2 + \left(\sin\frac{5\pi}{12} - \sin\frac{3\pi}{4}\right)^2}$

$= 2\sqrt{\cos^2\frac{5\pi}{12} + \cos^2\frac{3\pi}{4} - 2\cos\frac{5\pi}{12}\cos\frac{3\pi}{4} + \sin^2\frac{5\pi}{12} + \sin^2\frac{3\pi}{4} - 2\sin\frac{5\pi}{12}\sin\frac{3\pi}{4}}$

$= 2\sqrt{1 + 1 - 2\left(\cos\frac{5\pi}{12}\cos\frac{3\pi}{4} + \sin\frac{5\pi}{12}\sin\frac{3\pi}{4}\right.}$

$= 2\sqrt{2\left(1 - \cos\left(-\frac{\pi}{3}\right)\right)}$ since $\cos(-x) = \cos x$

$= 2\sqrt{2\left(1 - \frac{1}{2}\right)}$

$= 2$

$\therefore \triangle OAB$ is equilateral

(ii) Since ΔOAB is an equilateral triangle

$\therefore \angle AOB = \angle OBA = \angle BAO = \frac{\pi}{3}$

$\therefore z_2 - z_1$ is obtained by rotating z_2 by $\frac{\pi}{3}$ radians

(iii) $z_2 - z_1 = z_2 \operatorname{cis}\left(-\frac{\pi}{3}\right)$

$= 2\operatorname{cis}\left(\frac{5\pi}{12}\right)\operatorname{cis}\left(-\frac{\pi}{3}\right)$

$= 2\operatorname{cis}\frac{\pi}{12}$

b)

$z^2 = -8 - 6i$

$(x+iy)^2 = -8-6i$

$x^2 + 2xyi - y^2 = -8 - 6i$

$x^2 - y^2 = -8$

$2xy = -6$

$y = -\frac{3}{x}$

$x^2 - \frac{9}{x^2} = -8$

$x^4 - 9 = -8x^2$

$x^4 + 8x^2 - 9 = 0$

$(x^2-1)(x^2+9) = 0$

$x = \pm 1 \quad \therefore y = \mp 3$

$z = (1-3i)$ or $z = (-1+3i)$

C' (i) $I_n = \int \tan^n x \, dx$

$= \int \tan^{n-2} x \tan^2 x \, dx$

$= \int \tan^{n-2} x (\sec^2 x - 1) \, dx$

$= \int \tan^{n-2} x \sec^2 x - I_{n-2}$

$= \dfrac{\tan^{n-1} x}{n-1} - I_{n-2}$

Reverse chain rule
if unsure let do by substitution

$A = \tan x$

$dA = \sec^2 x \, dx$

$\therefore \int A^{n-2} \, dA = \dfrac{A^{n-1}}{n-1}$

$= \dfrac{\tan^{n-1} x}{n-1}$

(ii) $\int_0^{\frac{\pi}{4}} \tan^3 x \, dx$

$I_1 = \int_0^{\frac{\pi}{4}} \tan x \, dx$

$= \left[\ln(\cos x) \right]_0^{\frac{\pi}{4}}$

$= -\ln\left(\frac{1}{\sqrt{2}}\right)$

$I_3 = \dfrac{1}{3-1} \tan^2 x - I_{3-2}$

$= \frac{1}{2} \tan^2 x - I_1$

$= \frac{1}{2} \left[\tan^2 \frac{\pi}{4} - \tan^2 0 \right] + \ln\left(\frac{1}{\sqrt{2}}\right)$

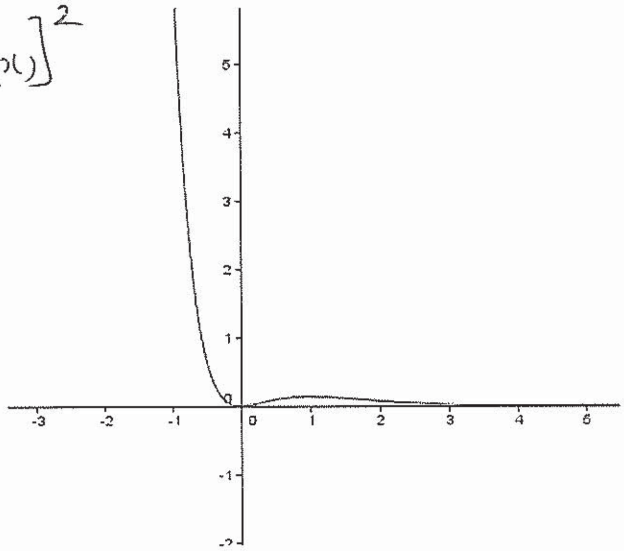
$= \frac{1}{2}(1) + \ln \frac{1}{\sqrt{2}}$

$= \frac{1}{2} - \ln 2^{\frac{1}{2}}$

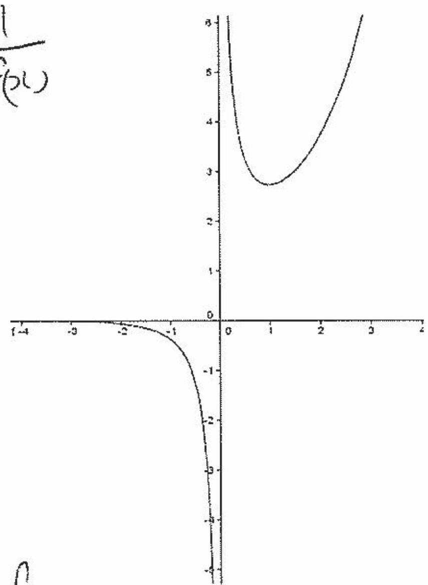
$= \frac{1}{2}(1 - \ln 2)$

Question 13

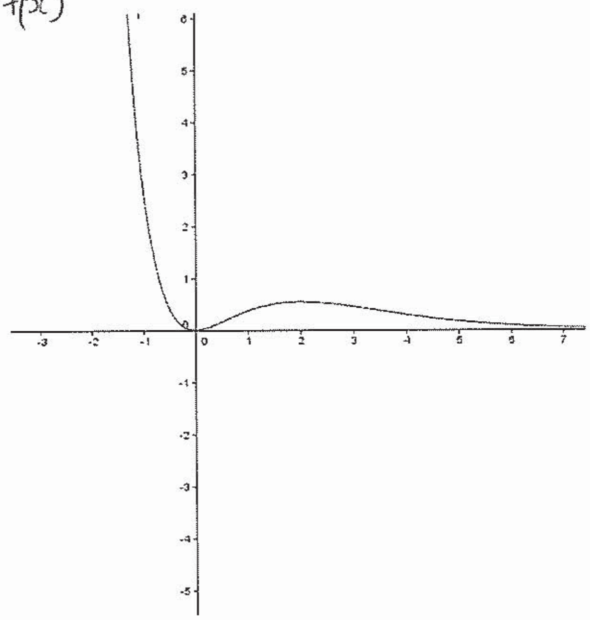
$y = [f(x)]^2$



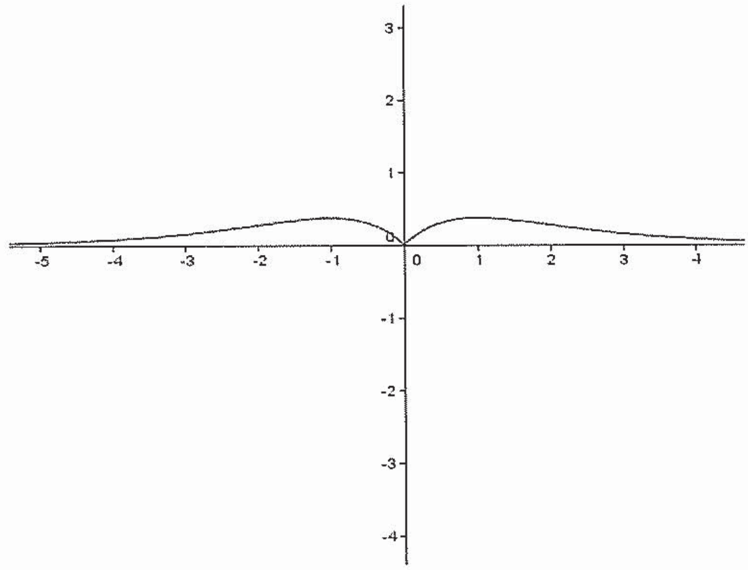
$y = \frac{1}{f(x)}$



$y = x f(x)$



$y = f(|x|)$



b) $9x^2 + 25y^2 = 225$

$\frac{x^2}{25} + \frac{y^2}{9} = 1$

(i) $b^2 = a^2(1 - e^2)$

$9 = 25(1 - e^2)$

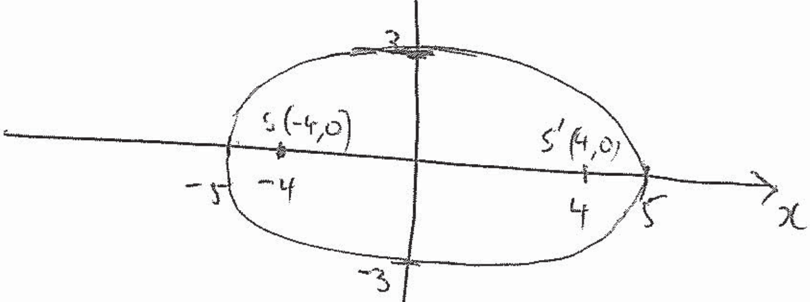
$\frac{9}{25} = 1 - e^2$

$e^2 = \frac{16}{25}$

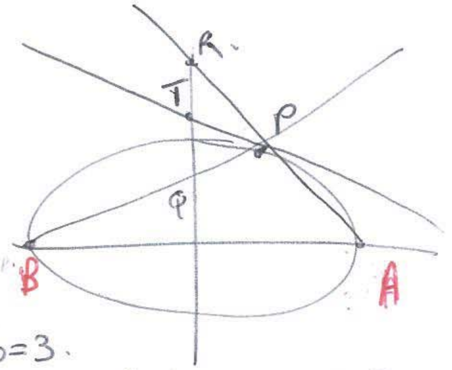
$e = \frac{4}{5}$

(ii) $S(ae, 0) \quad S'(-ae, 0)$

$\therefore S(4, 0) \quad S'(-4, 0)$



(iii)



$a=5 \quad b=3.$

equation of tangent at P

$$\frac{x\cos\theta}{5}+\frac{y\sin\theta}{3}=1$$

coordinates of $T\left(0, \frac{3}{\sin\theta}\right)$

AP equation of chord AP:

$$y-0=\frac{0-3\sin\theta}{5-5\cos\theta}(x-5)$$

$$5y=-\frac{3\sin\theta}{1-\cos\theta}(x-5)$$

$R. \left(0, \frac{3\sin\theta}{1-\cos\theta}\right)$

BP similarly equation of chord BP:

$$y-0=\frac{0-3\sin\theta}{-5-5\cos\theta}(x+5)$$

$Q. \left(0, \frac{3\sin\theta}{1+\cos\theta}\right)$

midpoint of Q and R

$$\left(\frac{0+0}{2}, \frac{1}{2}\left(\frac{3\sin\theta}{1-\cos\theta}+\frac{3\sin\theta}{1+\cos\theta}\right)\right)$$

$$=\left[0, \frac{3}{2}\sin\left(\frac{1+\cos\theta+1-\cos\theta}{1-\cos^2\theta}\right)\right]$$

$$=\left(0, \frac{3}{2}\sin\theta \times \frac{2}{\sin^2\theta}\right)$$

$$T=\left(0, \frac{3}{\sin\theta}\right)$$

= coordinates of T.

$\therefore$ T is the midpoint of Q and R

Question 14

a(i) $A\left(t, \frac{1}{t}\right)$ $B\left(-t, -\frac{1}{t}\right)$

$\underline{\quad\quad}$ $1 : 2.$

$$P(x,y) = \left(\frac{2t - t}{3}, \frac{2\left(\frac{1}{t}\right) - 1\left(\frac{1}{t}\right)}{3}\right)$$

$$= \left(\frac{t}{3}, \frac{1}{3t}\right)$$

$x = \frac{t}{3}$

$t = 3x$

$y = \frac{1}{3t} = \frac{1}{3(3x)}$

$= \frac{1}{9x}$

$\therefore 9xy = 1.$

(ii) $y = \frac{1}{x}$

$y' = -\frac{1}{x^2}$ at $x = t$

Gradient $= -\frac{1}{t^2}$

equation of tangent

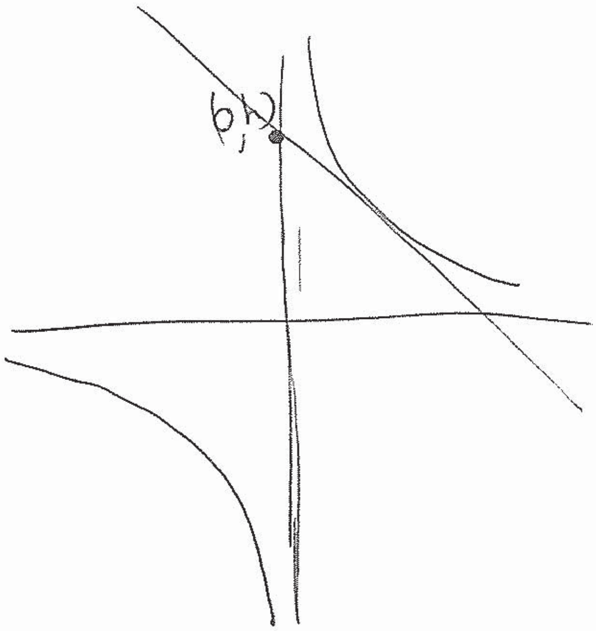
$y - \frac{1}{t} = -\frac{1}{t^2}(x - t)$

$t^2 y - t = -(x - t)$

$t^2 y - t + x - t = 0$

$t^2 y - 2t + x = 0.$

(iii)



$t^2y - 2t + x = 0$ from (ii)

since it passes through $(0, h)$

$\therefore t^2h - 2t + 0 = 0$

$t(th - 2) = 0$

$\therefore t = \frac{2}{h}$ or $t = 0$.

but $t \neq 0$.

$\therefore$ there is only one tangent from the point $R(0, h)$

b) $\int \frac{dx}{a^2\cos^2 x + b^2\sin^2 x}$

$= \int \frac{\sec^2 x\, dx}{a^2 + b^2\tan^2 x}$ Let $A = \tan x$
$dA = \sec^2 x\, dx$

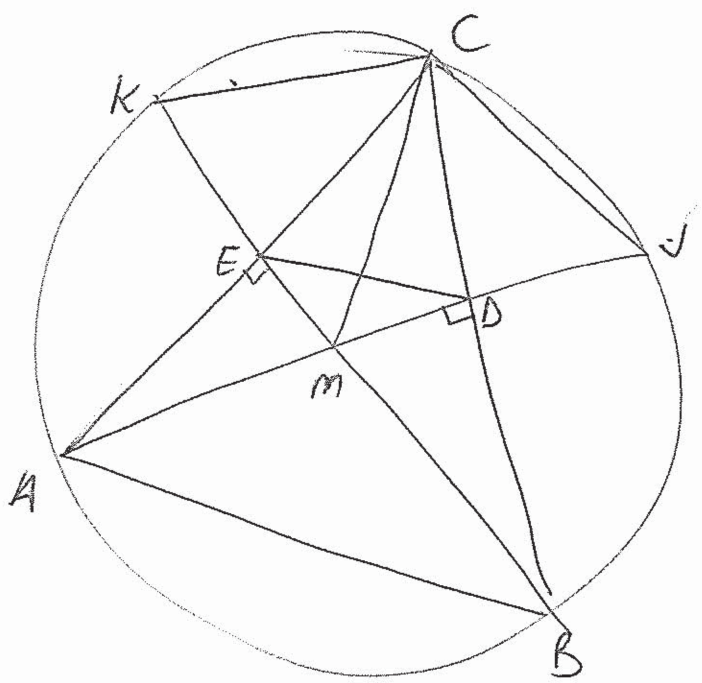
$= \int \frac{dA}{a^2 + b^2A^2}$

$= \int \frac{dA}{b^2\left(\frac{a^2}{b^2} + A^2\right)}$

$= \frac{1}{b^2}\int \frac{dA}{\left(\frac{a^2}{b^2}\right) + A^2}$

$= \frac{1}{b^2} \times \frac{b}{a}\tan^{-1}\left(\frac{A}{\frac{a}{b}}\right) + C = \frac{1}{ab}\tan^{-1}\left(\frac{b}{a}\tan x\right) + C$

C(i)



(ii) $\angle CEM + \angle CDM = 180^\circ$

$\therefore$ CEMD is a cyclic quadrilateral since opposite sides are supplementary.

Also AB subtends 90° at D and E.

$\therefore$ ABDE is cyclic AB being the diameter.

(iii) Let $\angle KCA = \alpha$.

since $\angle KCA = \angle KBA = \alpha$ (angles subtended by the same arc)

Similarly $\angle KBA = \angle EDA = \alpha$. (angles subtended by the same arc EM on CEMD)

$\therefore \angle KCA = \angle ACM = \alpha$

Hence AC bisects $\angle KCM$.

(iii) Join KA and BJ

In quadrilaterals KCMA and CJBM diagonals intersect at 90° and bisects one pair of angles (by part ii)

$\therefore$ both quadrilaterals are kites, having adjacent sides equal.

$\therefore$ KC = MC

MC = CJ

$\therefore$ KC = CJ.

Hence proved.

Question 15

(a) $F = ma = -m(1+v)$

(i) $\therefore a = -(1+v)$

(ii) a) $v\frac{dv}{dx} = -(1+v)$

$$\frac{dx}{dv} = -\frac{v}{1+v}$$

$$= -\frac{1+v-1}{1+v}$$

$$= -\left(1-\frac{1}{1+v}\right)$$

$$\frac{dx}{dv} = -1 + \frac{1}{1+v}$$

$$x = \int -1\,dv + \int \frac{1}{1+v}\,dv$$

① — $x = -v + \ln(1+v) + C$ $\quad x=0,\ t=0\ v=u.$

② — $0 = -u + \ln(1+u) + C$

①−② $\boxed{x = -v + u + \ln\left(\frac{(1+v)}{1+u}\right)}$

$= u$.

(b) $a = -(1+v)$

$$\frac{dv}{dt} = -(1+v)$$

$$\Rightarrow \frac{dt}{dv} = -\frac{1}{1+v}.$$

① — $t = -\ln(1+v) + c$ $\quad$ when $t=0\ v=u.$

② — $0 = -\ln(1+u) + C$

①−② $t = -\ln\left(\frac{1+v}{1+u}\right).$

or $t = \ln\left(\frac{1+u}{1+v}\right)$

$$e^t = \frac{1+u}{1+v}.$$

or $1+v = (1+u)e^{-t}$ $\quad \boxed{v = (1+u)e^{-t} - 1}$

c) $\frac{dx}{dt} = (1+u)e^{-t} - 1$

$x = \frac{1+u\,e^{-t}}{-1} - t + c$ $t=0,\ x=0$

$0 = -(1+u) + c$

$\therefore\ c = 1+u$

$\boxed{x = -(1+u)e^{-t} - t + (1+u)}$

(iii) $x + v + t = u.$

$\underbrace{-(1+u)e^{-t} - t + (1+u)}_{x} + \underbrace{(1+u)e^{-t} - 1}_{v} + t$

$= u.$ Hence proved.

(iv) $\frac{dv}{dt} = -(1+v)$

$\frac{dt}{dv} = -\frac{1}{1+v}$

$t = -\left[\ln(1+v)\right]_{u}^{0}$

$\boxed{t = \ln(1+u) \text{ seconds}}$

$x = -(1+u)e^{-\ln(1+u)} - \ln(1+u) + 1+u.$

$\boxed{= u - \ln(1+u)}$ metres

b) $P\,z = z^4 - z^3 + 6z^2 - z + 15$

(i) $z = 1-2i$ is a factor

$\therefore \bar{z} = 1+2i$ is also a factor

$[z-(1-2i)][z-(1+2i)]$

$= [(z-1)+2i][(z-1)-2i]$

$= (z-1)^2 + 4$

$= z^2 - 2z + 5$

$$
\begin{array}{r l}
 & z^2 + z + 3 \\
z^2 - 2z + 5 \; \big) & z^4 - z^3 + 6z^2 - z + 15 \\
 & \underline{z^4 - 2z^3 + 5z^2} \\
 & \quad z^3 + z^2 - z \\
 & \quad \underline{z^3 - 2z^2 + 5z} \\
 & \qquad 3z^2 - 6z + 15 \\
 & \qquad \underline{3z^2 - 6z + 15} \\
 & \qquad\qquad 0
\end{array}
$$

$P(z) = (z^2 - 2z + 5)(z^2 + z + 3)$

(ii) Solve for $P(z) = 0$

$z^2 - 2z + 5 = 0 \quad , \quad z^2 + z + 3 = 0$

$z = 1 \pm 2i \qquad\qquad z = \dfrac{-1 \pm \sqrt{11}i}{2}$

Question 16

a (i) $\frac{1}{2p+1} + \frac{1}{2p+2} > \frac{1}{p+1}$

now $2p+1 < 2p+2$, $p > 0$

$\therefore \frac{1}{2p+1} > \frac{1}{2p+2}$ $\left(\text{if } a<b \Rightarrow \frac{1}{a} > \frac{1}{b}\right)$

$\therefore \frac{1}{2p+1} + \frac{1}{2p+2} > \frac{1}{2p+2} + \frac{1}{2p+2}$

$= \frac{2}{2(p+1)}$

$= \frac{1}{p+1}$

$\therefore \frac{1}{2p+1} + \frac{1}{2p+2} > \frac{1}{p+1}$.

(ii) $\psi(m) = \frac{1}{m+1} + \frac{1}{m+2} + \dots + \frac{1}{2m} \geq \frac{37}{60}$.

Step 1 Test for $m=3$.

$\psi(3) = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} = \frac{37}{60}$ = R.H.S.

Also for $m=4$

$\psi(4) = \frac{1}{4} + \frac{1}{5} + \frac{1}{6} + \frac{1}{7} = \frac{37}{60} + \frac{1}{7}$.

$> \frac{37}{60}$

so the results are true for $m=3$ and $m=4$

Step 2 Let the result be true for $m=k$ $k \geq 3$.

ie $\psi(k) = \frac{1}{k+1} + \frac{1}{k+2} + \dots + \frac{1}{2k} \geq \frac{37}{60}$.

show the result is true for $m = k+1$

ie $\frac{1}{k+2} + \frac{1}{k+3} + \ldots + \frac{1}{2k+2} \geq \frac{37}{60}$

L.H.S.

$$\frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k} + \frac{1}{2k+1} + \frac{1}{2k+2}$$

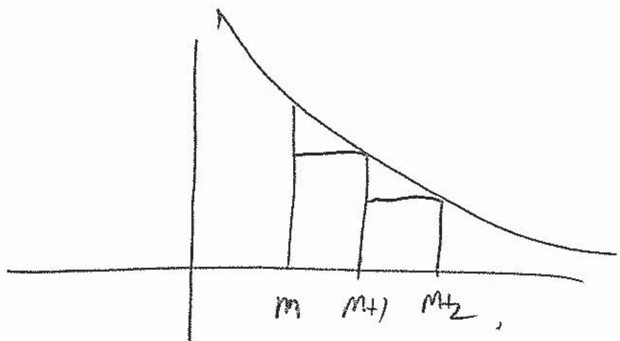
$$= \underbrace{\frac{1}{k+1} + \frac{1}{k+2} + \frac{1}{k+3} + \cdots + \frac{1}{2k}} + \frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1}$$

$$\geq \frac{37}{60} + \left(\frac{1}{2k+1} + \frac{1}{2k+2} - \frac{1}{k+1}\right)$$

$\geq \frac{37}{60}$ as $\frac{1}{2k+1} + \frac{1}{2k+2} > \frac{1}{k+1}$ using part a.

$\therefore$ ~~true~~ proved by mathematical induction.

(iii)



low rect $< \int_m^{m+1} \frac{1}{t}\,dt <$ upper rectangle

Area of lower rectangle $= 1\left(\frac{1}{m+1}\right)$

$$\therefore \int_m^{m+1} \frac{1}{t}\,dt > \frac{1}{m+1}.$$

(v) $$\int_m^{2m} \frac{1}{t}\,dt = \int_m^{m+1} \frac{1}{t}\,dt + \int_{m+1}^{m+2} \frac{1}{t}\,dt + \dots \int_{2m-1}^{2m} \frac{1}{t}\,dt$$

$$> \frac{1}{m+1} + \frac{1}{m+2} + \dots \frac{1}{2m} \quad (\text{using (iv)})$$

$$> \frac{37}{60} \quad (\text{using part ii})$$

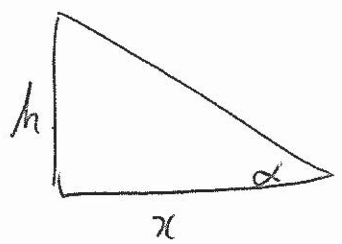
Also $$\int_m^{2m} \frac{1}{t}\,dt = \Big[\ln t\Big]_m^{2m}$$

$$= \ln 2m - \ln m$$

$$= \ln 2$$

$$\therefore \ln 2 > \frac{37}{60}.$$

b)



(i) Area $= \frac{1}{2} h \times x$

$$= \frac{1}{2} x \tan\alpha \times x$$

$$= \frac{1}{2} x^2 \tan\alpha$$

$$x^2 + y^2 = r^2$$
$$x^2 = r^2 - y^2$$

$$\therefore A = \frac{1}{2}(r^2 - y^2)\tan\alpha$$

(ii) $\delta V = \frac{1}{2}(r^2-y^2)\tan\alpha\ \delta y$

$$V = \lim_{\delta y \to 0} \sum_{-r}^{r} \frac{1}{2}(r^2-y^2)\tan\alpha\ \delta y$$

$$= \frac{1}{2}\tan\alpha \int_{-r}^{r} (r^2-y^2)\,dy$$

$$= \frac{2}{2}\tan\alpha \int_{0}^{r} (r^2-y^2)\,dy \quad \text{(since } r^2-y^2 \text{ is an even function)}$$

$$= \tan\alpha \left[r^2 y - \frac{y^3}{3}\right]_0^r$$

$$= \tan\alpha \left[r^3 - \frac{r^3}{3}\right]$$

$$= \frac{2}{3} r^3 \tan\alpha \text{ cubic units}$$