Sydney Girls High School

2015

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 3 – 5

10 Marks

- Attempt Questions 1 – 10
- Answer on the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II Pages 6 – 9

60 Marks

- Attempt Questions 11 – 14
- Answer on the blank paper provided
- Begin a new page for each question
- Allow about 1 hour and 45 minutes for this section

Name: …………………………………………………………………………	**THIS IS A TRIAL PAPER ONLY** It does not necessarily reflect the format or the content of the 2015 HSC Examination Paper in this subject.
Teacher: ………………………………………………………………………	

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1. A committee of six is to be formed from seven women and nine men. Find the number of committees possible if exactly two members of the committee are to be men.

   (A) 1260

   (B) 2646

   (C) 36036

   (D) 60480

2. In the diagram below $\angle BAC = 34°$ and $\angle ADE = 85°$. What is the size of angle $\angle ACB$?

   (A) 51°

   (B) 56°

   (C) 61°

   (D) 60°

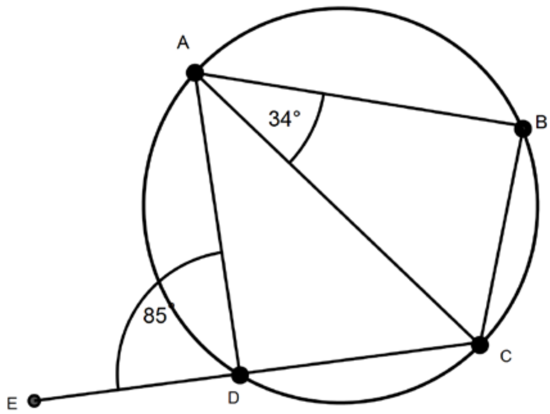


Diagram not to scale.

3. Use the substitution $u = e^x$ to determine which of the following is an expression for $\int \frac{e^x}{1+e^{2x}}\,dx$.

   (A) $\tan^{-1}\left(e^x\right)+C$

   (B) $\tan^{-1}\left(e^{2x}\right)+C$

   (C) $\frac{-1}{2\left(1+e^x\right)^2}+C$

   (D) $\frac{-e^x}{\left(1+e^x\right)^2}+C$

4. The radius of a spherical balloon is increasing at the rate of 2 cm/s. The rate at which the volume of the balloon is increasing when the radius is 10 cm is :

(A) $200\pi\ \text{cm}^3/\text{s}$

(B) $400\pi\ \text{cm}^3/\text{s}$

(C) $800\pi\ \text{cm}^3/\text{s}$

(D) $100\pi\ \text{cm}^3/\text{s}$

5. A stone is thrown vertically upwards with a speed of 21 m/s. How long is the stone in the air before it reaches its maximum height? (Assume acceleration due to gravity is 10 m/s^2.)

(A) 4.2 s

(B) 0.48 s

(C) 0.95 s

(D) 2.1 s

6. The polynomial equation $f(x) = x^3 + x - 1$ has a root near $x = 0.5$. Using this as the initial approximation, determine another approximation (correct to four decimal places) to the root using one application of Newton's method.

(A) $x = 0.7141$

(B) $x = 0.7142$

(C) $x = 0.7143$

(D) $x = 0.7144$

7. If $y = \sin^{-1}\left(\frac{a}{x}\right)$, then $\frac{dy}{dx} =$

(A) $\frac{-a}{\sqrt{x^2 - a^2}}$

(B) $\frac{a}{\sqrt{x^2 - a^2}}$

(C) $\frac{-a}{x\sqrt{x^2 - a^2}}$

(D) $\frac{a}{x\sqrt{x^2 - a^2}}$

8. What is the value of $\sum_{k=1}^{20} {}^{20}C_k$?

(A) 1 048 574

(B) 1 048 575

(C) 1 048 576

(D) 1 048 577

9. Given that a, b and c are the roots of the equation $x^3 - 3x^2 + mx + 24 = 0$, and that $-a$ and $-b$ are the roots of the equation $x^2 + nx - 6 = 0$, then the value of n is :

(A) 1

(B) −1

(C) 7

(D) −7

10. The sum $1^4 + 2^4 + 3^4 + 4^4 + ... + n^4$ is given by the expression $\dfrac{6n^5 + an^4 + bn^3 - n}{30}$.

The value of $a - b$ is :

(A) −25

(B) 25

(C) −5

(D) 5

Section II

Total 60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer all questions, starting each question on a new page.
In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

<u>Question 11</u> (15 marks)

Marks

(a) Solve $\dfrac{2}{x-5} \leq 3$. 3

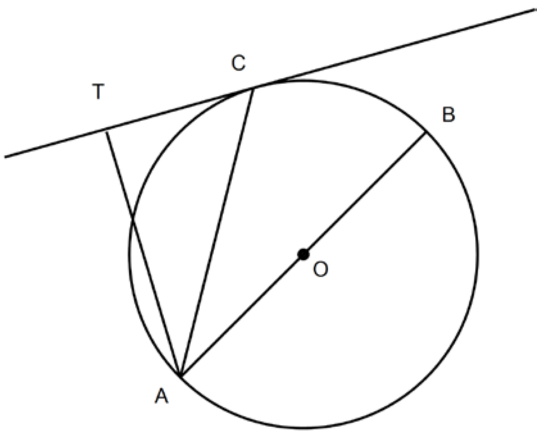
(b) Sketch the function $f\left(x\right)=\pi\cos^{-1}\left(\dfrac{x}{2}\right)$, clearly indicating the domain and range of the function. 3

(c) The velocity of a particle when x m from the origin is given by $v^2 = x^2e^{3x}+4$. Find the acceleration of the particle when $x=1$. 2

(d) Find the general solution to the equation $\sqrt{3}\tan\theta+1=0$. 2

(e) Find the value of $\displaystyle\lim_{x\to 0}\frac{5x+\sin 3x}{2x}$. 2

(f) In the diagram, AOB is the diameter of the circle with centre at O. TC is a tangent to the circle at the point C such that AC bisects $\angle TAB$. Copy the diagram onto your writing paper. Prove that AT is perpendicular to TC. 3



End of Question 11

Question 12 (15 marks) **Marks**

(a) By considering the derivative of $\ln(\tan x)$, find $\int \operatorname{cosec} 2x\, dx$. 3

(b) In the expansion of $\left(x+\frac{2}{x^2}\right)^{10}$, find the coefficient of x. 3

(c) $P(4p, 2p^2)$ and $Q(4q, 2q^2)$ are two variable points on the parabola $x^2 = 8y$. The tangents at P and Q intersect at the point T.

(i) Derive the equation of the tangent at P. 1

(ii) Hence, show that the point T has the coordinates $(2(p+q), 2pq)$. 2

(iii) Given that $p^2 + q^2 = 10$, determine the cartesian equation of the locus of T. 2

(d) Find $\int (\sin^2 x + 2\cos^2 x + 3\tan^2 x)\, dx$. 2

(e) Prove that $\tan^{-1}(x+1) + \cot^{-1} x = \tan^{-1}(-x^2 - x - 1)$ for $x > 0$. 2

End of Question 12

Question 13 **(15 marks)**

Marks

(a) The growth rate per month of the number N of birds on a property during a drought is -20% of the excess of the bird population over 1000.

(i) Express the information in the form of a differential equation and show that $N = 1000 + Ae^{-0.2t}$ (where t is the time in months) is a solution to this differential equation. **2**

(ii) Given that initially there are 8000 birds on the property, find the amount of time that will elapse before the population is reduced to half. **2**

(b) Ansett Airlines offer two options on all flights for their meal service – chicken or beef (vegetarians choose not to fly with Ansett). If 60% of the time Ansett passengers select the chicken dish, what is the probability that out of 7 randomly selected passengers at least 2 will select chicken for their meal? **2**

(c) An iPhone is thrown from the top of a building, 6 metres high, with an initial velocity of 8 m/s at an angle of $30°$ to the horizon.

(i) Using 10 m/s^2 for acceleration due to gravity, derive the horizontal and vertical equations of motion for the iPhone. **2**

(ii) Determine the greatest height of the iPhone above ground level. **2**

(iii) Find the velocity and direction of the iPhone's path after 1 second. **2**

(d) Prove the following statement is true by mathematical induction for all integers $n \geq 1$. **3**

$$\sum_{r=1}^{n} r(r!) = (n+1)! - 1$$

End of Question 13

Question 14 **(15 marks)** **Marks**

(a) A particle moves in a straight line with simple harmonic motion. At time t seconds, its displacement x metres from a fixed point O, is given by $x = 2 + 5\sin\left(3t + \frac{\pi}{4}\right)$.

(i) Show that $\ddot{x} = -9(x-2)$. 1

(ii) Determine the maximum speed of the particle and its displacement at this time. 2

(b) How many different arrangements of the word MAMMOTH can be made if only five letters are used? 2

(c) Use the substitution $u^2 = x+1$ to find the volume of the solid formed by rotating the area bounded by the curve $y = \frac{x-1}{\sqrt{x+1}}$, the x axis and the lines $x=3$ and $x=8$ about the x axis. Express your answer in exact form. 4

(d) Use the expansion of $(1+x)^n$ to prove that 3

$$\frac{n+(-1)^n}{n+1} = \frac{1}{2}{}^nC_1 - \frac{1}{3}{}^nC_2 + \cdots + \frac{(-1)^n}{n}{}^nC_{n-1}.$$

(e) Given that $f(x) = Ax^3 + Bx^2 + Cx + D$ is a function with a double zero at $x=1$, and with a minimum value of -4 when $x=-1$, find the values of A, B, C and D. 3

End of paper

2015 Trial HSC - Extension 1

SECTION I

1. $^9C_2 \times {^7C_4} = 1260$ A

2. $\angle ABC = 85$ (ext. $\angle$ of cyclic quad)
$\angle ACB + 34 + 85 = 180$ ($\angle$ sum of $\triangle ABC$)
$\therefore \angle ACB = 61^\circ$ C

3. let $u = e^x$ $du = e^x dx$
$$\int \frac{e^x}{1+(e^x)^2}\,dx = \int \frac{du}{1+u^2} = \tan^{-1}u + c = \tan^{-1}(e^x) + C$$
A

4. $$\frac{dV}{dt} = \frac{dV}{dr} \times \frac{dr}{dt}$$ $V = \frac{4}{3}\pi r^3$ $\frac{dV}{dr} = 4\pi r^2$
$$= 4\pi (10)^2 \times 2$$
$$= 800\pi \text{ cm}^3/\text{S}$$ C

5. when $t = 0$, $\dot{y} = 21$ m/S $t = ?$ when $\dot{y} = 0$
$\ddot{y} = -10$
$\dot{y} = -10t + c$ $\therefore c = 21$
$0 = -10t + 21$ $10t = 21$ $\therefore t = 2.1$ S D

6. $f(x) = x^3 + x - 1$ $f'(x) = 3x^2 + 1$
$$x_1 = x_0 - \frac{f(x_0)}{f'(x_0)} = 0.5 - \frac{(0.5^3 + 0.5 - 1)}{3(0.5)^2 + 1}$$
$$\doteqdot 0.7143$$ C

7. $$\frac{dy}{dx} = \frac{1}{\sqrt{1-\frac{a^2}{x^2}}} \times \frac{-a}{x^2} = \frac{-a}{x\sqrt{x^2-a^2}}$$ C

8. Consider $(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_n x^n$

let $x=1$

$$(1+1)^n = {}^nC_0 + {}^nC_1 + {}^nC_2 + \dots + {}^nC_n$$

let $n=20$

$$\therefore \sum_{k=1}^{20} {}^{20}C_k + {}^{20}C_0 = (2)^{20}$$

$$\therefore \sum_{k=1}^{20} {}^{20}C_k = 2^{20} - 1 = 1\,048\,575$$

(B)

9. $a+b+c = 3$ $\qquad abc = -24$

$-a-b = -n$ $\qquad ab = -6$

$a+b = n$ $\qquad \therefore c = 4$

$\therefore n = 3 - c = 3 - 4$

$\therefore n = -1$ (B)

10. $1^4 = \dfrac{6+a+b-1}{30}$ $\qquad a+b = 30-5 = 25$ (1)

$1^4 + 2^4 = \dfrac{6(2)^5 + a(2)^4 + b(2)^3 - 2}{30}$

$190 + 16a + 8b = 510$

$2a+b = \dfrac{320}{8} = 40$ (2)

$a = 15,\ b = 10$

$a-b = 5$ (D)

Question 11 (15 Marks).

a) $\frac{2}{x-5} \le 3 \quad (x \ne 5$

$$\frac{2(x-5)^2}{(x-5)} \le 3(x-5)^2$$

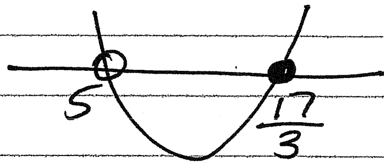
$$2(x-5) \le 3(x-5)^2$$

$$3(x-5)^2 - 2(x-5) \geq 0$$

$$(x-5)[3(x-5)-2] \geq 0$$

$$(x-5)(3x-15-2) \geq 0$$

$$(x-5)(3x-17) \geq 0$$



$x < 5$ or $x \geq \frac{17}{3}$ ✓✓✓

$(x \ne 5)$

Many students lost a mark as $x \ne 5$.

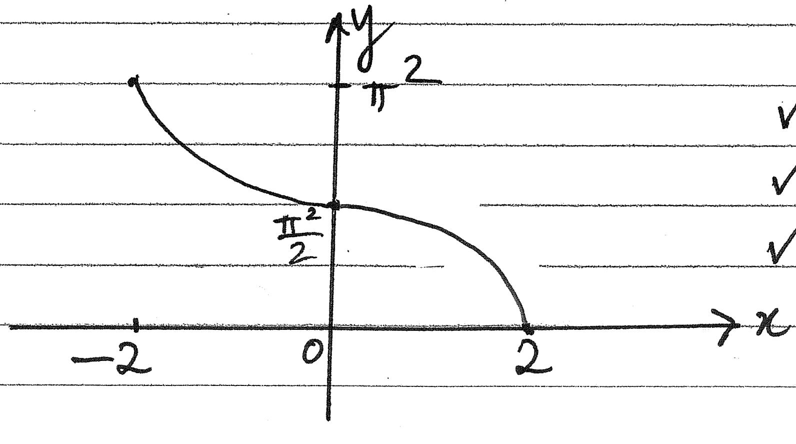
b) $f(x) = \pi \cos^{-1}\left(\frac{x}{2}\right)$

Domain: $-1 \le \frac{x}{2} \le 1$

$-2 \le x \le 2$

Range: $0 \le \frac{y}{\pi} \le \pi$

$0 \le y \le \pi^2$



✓D

✓R

✓shape.

c) $v^2 = x^2 e^{3x} + 4$

$$a = \frac{d}{dx}\left(\frac{1}{2}v^2\right)$$

$$= \frac{d}{dx}\,\frac{1}{2}\left(x^2 e^{3x} + 4\right)$$

$$= \frac{1}{2}\left[x^2 . 3e^{3x} + e^{3x} . 2x\right]$$

$$= \frac{1}{2}\left[3x^2 e^{3x} + 2x e^{3x}\right]$$

$$= \frac{1}{2} e^{3x}\left(3x^2 + 2x\right) \quad \checkmark$$

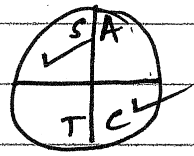
when $x = 1$

$$a = \frac{1}{2} \cdot e^3 \cdot (5)$$

$$a = \frac{5}{2} e^3 \ \ m/s^2 \quad \checkmark$$

$\left[a \doteq 50.21 \ m/s^2 \ (2 \text{ dec.pl.})\right]$

d) $\sqrt{3}\tan\theta = -1$



$$\tan\theta = -\frac{1}{\sqrt{3}}$$

$$\theta = \pi n + \tan^{-1}\left(-\frac{1}{\sqrt{3}}\right)$$

$$\theta = n\pi - \frac{\pi}{6}. \quad (n \text{ is an integer}).$$

$\checkmark \quad \checkmark$

OR $\theta = n\pi + \frac{5\pi}{6}$

e) $$\lim_{x \to 0} \frac{5x + \sin 3x}{2x}.$$

$$= \lim_{x \to 0} \frac{5\cancel{x}}{2\cancel{x}} + \lim_{x \to 0} \frac{\sin 3x}{2x}$$

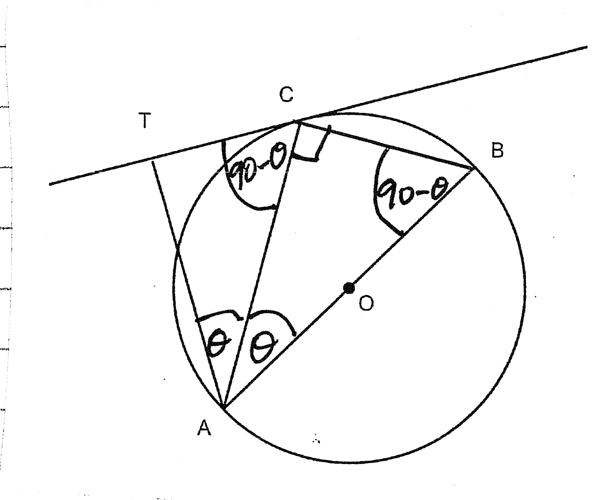
$$= \frac{5}{2} + \underbrace{\frac{3}{2} \lim_{x \to 0} \cancel{\frac{\sin 3x}{3x}}}_{1}.$$ ②

$$= \frac{8}{2}$$

$$= 4.$$

✱ Some confusion for students with this limit.

f)



✱ Many more ways to prove that $AT \perp TC$

✱ The proof had to be logical from start to finish.

Join B to C

$\angle ACB = 90°$ ($\angle$ in a semi circle).

Let $\angle TAC = \angle CAB = \theta$ (AC bisects $\angle TAB$)

In ΔACB: $\angle ABC = 90 - \theta$ ($\angle$ sum of ΔACB).

$\angle TCA = \angle ABC = 90 - \theta$ ($\angle$ in alternate segment).

In ΔATC: $\angle ATC + \angle TAC + \angle TCA = 180°$ ($\angle$ sum ΔATC).

$$\angle ATC + \theta + 90 - \theta = 180$$

$$\angle ATC + 90 = 180$$

$$\angle ATC = 90°.$$

$\therefore AT \perp TC$

③

Q12

a) $y = \ln(\tan x)$

$$y' = \frac{\sec^2 x}{\tan x} = \frac{\frac{1}{\cos^2 x}}{\frac{\sin x}{\cos x}} = \frac{1}{\cos x \sin x}$$

Now $$\frac{1}{\cos x \sin x} = \frac{2}{2\sin x \cos x} = \frac{2}{\sin 2x} = 2\,\mathrm{cosec}\, 2x$$

$$\frac{1}{2}\int 2\,\mathrm{cosec}\,2x\,dx = \frac{\ln[\tan x]}{2} + C$$

[Some students could not find $\frac{d[\ln(\tan x)]}{dx}$, hence couldn't get the final Ans.]

b) $\left(x + \frac{2}{x^2}\right)^{10}$

$$T_{k+1} = {}^nC_k \cdot a^{n-k} \cdot b^k$$
$$= {}^{10}C_k \cdot x^{10-k} \cdot (2x^{-2})^k$$
$$= {}^{10}C_k \cdot x^{10-k} \cdot 2^k \cdot x^{-2k}$$
$$= {}^{10}C_k \cdot 2^k \cdot x^{10-3k}$$

Equating the coefficients of x

$$\therefore 10 - 3k = 1$$
$$3k = 9$$
$$k = 3$$

The coefficient of x is

$${}^{10}C_3 \cdot 2^3 = 960$$

[Most students did well in this question.]

Q12

c) $P(4p, 2p^2)$ $Q(4q, 2q^2)$

i) $x^2 = 8y$

$y = \frac{x^2}{8}$ $\therefore y' = \frac{x}{4}$

At $P(4p, 2p^2)$ $\therefore$ $y' = \frac{4p}{4} = p$

Equation of the tangent at P.

$$y - 2p^2 = p(x - 4p)$$
$$y = px - 2p^2 \quad (1)$$

Similarly the equation of the tangent at Q

$$y = qx - 2q^2 \quad (2)$$

ii) Solve (1) and (2)

$$px - 2p^2 = qx - 2q^2$$
$$(p-q)x = 2(p+q)(p-q)$$
$$x = 2(p+q) \quad \text{Sub into (1)}$$

$$y = p[2(p+q)] - 2p^2$$
$$y = 2p^2 + 2pq - 2p^2$$
$$y = 2pq$$

Thus $T\,(2(p+q),\ 2pq)$

[Almost every one did well in parts (c/i) and (c/ii)]

$\boxed{Q12}$

c(iii) $x = 2(p+q)$ Given $p^2+q^2=10$

$y = 2pq$

$$x^2 = [2(p+q)]^2$$

$$x^2 = 4(p^2+q^2+2pq)$$

$$x^2 = 4(10+y)$$

$$x^2 = 4(y+10) \text{ : Locus of } T$$

d) $\int (\sin^2 x + 2\cos^2 x + 3\tan^2 x)\,dx$

$$I = \int \left[\sin^2 x + \cos^2 x + \cos^2 x + 3(\sec^2 x - 1)\right] dx$$

$$I = \int \left[3\sec^2 x + \tfrac{1}{2}(1+\cos 2x) - 2\right] dx$$

$$I = \int \left(3\sec^2 x + \tfrac{1}{2}\cos 2x - \tfrac{3}{2}\right) dx$$

$$= 3\tan x + \tfrac{1}{4}\sin 2x - \tfrac{3}{2}x + C$$

[A number of students forgot Trig identities, double-angle formula, hence could not integrate this question correctly.]

$\boxed{Q12}$

e) Let $A = \tan^{-1}(x+1)$, let $B = \cot^{-1} x$

$\tan A = x+1$, $\cot B = x$

$\tan B = \frac{1}{x}$

$$\tan^{-1}[-(x^2+x+1)] = -\tan^{-1}(x^2+x+1)$$

$$\tan(A+B) = \frac{\tan A + \tan B}{1 - \tan A \cdot \tan B}$$

$$= \frac{x+1+\frac{1}{x}}{1-(x+1)\times\frac{1}{x}}$$

$$= \frac{\frac{x^2+x+1}{x}}{\frac{x-x-1}{x}} = -(x^2+x+1)$$

Thus $A + B = \tan^{-1}[-(x^2+x+1)]$

OR $\tan^{-1}(x+1) + \cot^{-1} x = \tan^{-1}(-x^2-x-1)$

[Many students could not simplify $\tan^{-1}(x+1)$, $\cot^{-1} x$ and $\tan[-(x^2+x+1)]$ correctly]

Q13 Ext 1 2015

a) i) $\frac{dN}{dt} = k(N - 1000)$

$= -0.2(N - 1000)$

$N = 1000 + Ae^{-0.2t}$

$\frac{dN}{dt} = -0.2Ae^{-0.2t}$

$Ae^{-0.2t} = N - 1000$

$= -0.2(N - 1000)$

* The setting out for this question was not very good. For a show question you need to show all the steps.

ii) at $t = 0$, $N = 8000$

$8000 = 1000 + Ae^{0}$

$\therefore A = 7000$

$N = 1000 + 7000e^{-0.2t}$

$4000 = 1000 + 7000e^{-0.2t}$

$\frac{3}{7} = e^{-0.2t}$

$\ln\left(\frac{3}{7}\right) = -0.2t$

$t \doteq 4.23$ months

* This question was done very well.

b) P(at least 2 chicken) =

$1 - {}^7C_1(0.6)(0.4)^6 - {}^7C_0(0.6)^0(0.4)^7$

$= 1 - \frac{1344}{78125} - \frac{128}{78125}$

$\doteq 0.98$

* Many students couldn't do this question

c)

i)

$\ddot{x} = 0$

$\dot{x} = C$

$\dot{x} = V\cos\theta$

$x = Vt\cos\theta + C$

at $t = 0$, $x = 0$

$\therefore C = 0$

$x = Vt\cos\theta$

$= 8t\cos 30^\circ$

$= 8t \times \frac{\sqrt{3}}{2}$

$\boxed{x = 4\sqrt{3}t}$

* Many students didn't show all the steps of deriving these equations

$\ddot{y} = -10$

$\dot{y} = -10t + C$

$C = V\sin\theta$

$\dot{y} = -10t + V\sin\theta$

$y = -5t^2 + Vt\sin\theta + C$

at $t = 0$ $C = 6$

$\therefore y = 6$

$y = -5t^2 + Vt\sin\theta + 6$

$= -5t^2 + 8t \times \frac{1}{2} + 6$

$\boxed{y = -5t^2 + 4t + 6}$

ii) $\dot{y} = 0$

$-10t + 4 = 0$

$t = \frac{2}{5}$

$y = -5\left(\frac{4}{25}\right) + \frac{8}{5} + 6$

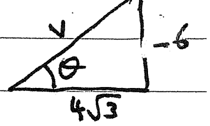
$y = 6.8$

* This question was done well

iii) $\dot{x} = 4\sqrt{3}$

$\dot{y} = -10(1) + 8\left(\frac{1}{2}\right)$

$= -6$



$v^2 = (4\sqrt{3})^2 + 36$

$v = 9.2$ m/s

$\tan\theta = \frac{6}{4\sqrt{3}}$ $\therefore \theta = 41^\circ$

The particle is travelling at 9.2 m/s downwards at $\theta = 41^\circ$ to the horizontal.

* many students didn't know how to find the direction

d)

$1 + 2(2!) + 3(3!) + \dots\ n(n!) = (n+1)! - 1$

when $n = 1$

$LHS = 1$

$RHS = 2! - 1$

$= 1$

$\therefore$ true for $n=1$

when

$n=k$

$1 + 2(2!) + \dots\dots\ k(k!) = (k+1)! - 1$ ✓

prove true for $n=k+1$

$1 + 2(2!) + \dots\ k(k!) + (k+1)(k+1)! = (k+2)! - 1$

$LHS = \underbrace{(k+1)!}\ - 1 + \underbrace{(k+1)(k+1)!}$

$= (k+1)!\,[1 + k + 1] - 1$ ✓

$= (k+1)!\,(k+2) - 1$

$= (k+2)! - 1$ ✓

$= RHS$

$\therefore$ true for $n \geq 1$

By Mathematical induction

* This question was done well. except some students didn't show the last step properly $\therefore$ lost mark.

Question 14 (15 Marks)

a) $x = 2 + 5\sin\left(3t + \frac{\pi}{4}\right) \Rightarrow x - 2 = 5\sin\left(3t + \frac{\pi}{4}\right)$

i) $\dot{x} = 0 + 5\cos\left(3t + \frac{\pi}{4}\right) \times 3$

$= 15\cos\left(3t + \frac{\pi}{4}\right)$

$\ddot{x} = -15\sin\left(3t + \frac{\pi}{4}\right) \times 3$

$= -45\sin\left(3t + \frac{\pi}{4}\right)$ ✓

$= -9\left[5\sin\left(3t + \frac{\pi}{4}\right)\right]$

$\therefore \ddot{x} = -9(x-2)$, as required.

ii) Maximum speed occurs at the centre of motion i.e. at $x = 2$

When $x = 2$: $5\sin\left(3t + \frac{\pi}{4}\right) = 0$

$3t + \frac{\pi}{4} = \pi$ $(t > 0)$

$3t = \frac{3\pi}{4}$

$t = \frac{\pi}{4}$ sec.

At $x = 2$: $\dot{x} = 15\cos\left(\frac{3\pi}{4} + \frac{\pi}{4}\right)$

$= 15\cos\pi$

$= -15$ m/sec.

$\therefore |\dot{x}|_{max} = 15$ m/sec at $x = 2$. ✓ ✓

b). different arrangements of the word MAMMOTH using only 5 letters.

Case 1: (1M) $\underline{M}$ _ _ _ _ $= 5!$ ways.

Case 2: (2M's) $\underbrace{\underline{M}\ \underline{M}}$ _ _ _ $= \frac{5!}{2!} \times {}^4C_3$

Case 3: (3M's) $\underbrace{\underline{M}\ \underline{M}\ \underline{M}}$ _ _ $= \frac{5!}{3!} \times {}^4C_2$

Total arrangements

$$= 5! + \frac{5!}{2!} \times {}^4C_3 + \frac{5!}{3!} \times {}^4C_2$$

$$= 120 + 60 \times 4 + 20 \times 6$$

$$= 120 + 240 + 120$$

$$= \underline{\underline{480}}.$$

☆ One mark awarded for considering the different cases of repetition.
One mark for the correct answer.

c) $V = \pi \int_3^8 y^2 \, dx$

$$= \pi \int_3^8 \frac{(x-1)^2}{(x+1)} \, dx$$

$u^2 = x+1$

$2u\,du = dx$.

$$= \pi \int_2^3 \frac{(u^2-2)^2}{u^{\cancel{2}}} \cdot 2\cancel{u}\,du$$

$\begin{cases} u^2 - 1 = x. \\ u^2 - 2 = x - 1 \end{cases}$

$$= 2\pi \int_2^3 \left(\frac{u^4}{u} - \frac{4u^2}{u} + \frac{4}{u} \right) du$$

when $x = 8$, $u^2 = 9$
$u = 3$

$x = 3$, $u^2 = 4$
$u = 2$.

$$= 2\pi \int_2^3 \left(u^3 - 4u + \frac{4}{u} \right) du$$

$$= 2\pi \left[\frac{u^4}{4} - 2u^2 + 4\ln u \right]_2^3$$

$$= 2\pi \left[\left(\frac{81}{4} - 18 + 4\ln 3 \right) - \left(4 - 8 + 4\ln 2 \right) \right]$$

$$= 2\pi \left[\frac{25}{4} + 4\ln 3 - 4\ln 2 \right].$$

$$= \frac{25\pi}{2} + 8\pi(\ln 3 - \ln 2) \text{ units}^3.$$

* Most students substituted incorrectly with the new variable, back into the integral.

d)

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \dots + {}^nC_{n-1} x^{n-1} + {}^nC_n x^n$$

Integrate both sides w.r.t. x:

$$\frac{(1+x)^{n+1}}{n+1} + C = {}^nC_0 x + {}^nC_1 \frac{x^2}{2} + {}^nC_2 \frac{x^3}{3} + \dots + {}^nC_{n-1} \frac{x^n}{n} + {}^nC_n \frac{x^{n+1}}{n+1}$$

Let $x=0$:

$$\frac{1}{n+1} + C = 0 \qquad \therefore \; C = -\frac{1}{n+1}$$

$$\therefore \frac{(1+x)^{n+1} - 1}{(n+1)} = {}^nC_0 x + \frac{1}{2}{}^nC_1 x^2 + \frac{1}{3}{}^nC_2 x^3 + \dots + \frac{1}{n}{}^nC_{n-1} x^n + \frac{1}{n+1}{}^nC_n x^{n+1}$$

Let $x = -1$: noting that ${}^nC_0 = {}^nC_n = 1$

$$\frac{-1}{n+1} = -1 + \frac{1}{2}{}^nC_1 - \frac{1}{3}{}^nC_2 + \dots + \frac{(-1)^n}{n}{}^nC_{n-1} + \frac{(-1)^{n+1}}{n+1}$$

$$1 - \frac{1}{n+1} - \frac{(-1)(-1)^n}{n+1} = \frac{1}{2}{}^nC_1 - \frac{1}{3}{}^nC_2 + \dots + \frac{(-1)^n}{n}{}^nC_{n-1}$$

$$\frac{n+1-1}{n+1} + \frac{(-1)^n}{n+1} = \frac{1}{2}{}^nC_1 - \frac{1}{3}{}^nC_2 + \dots + \frac{(-1)^n}{n}{}^nC_{n-1}$$

$$\therefore \quad \frac{n+(-1)^n}{n+1} = \frac{1}{2}{}^nC_1 - \frac{1}{3}{}^nC_2 + \dots + \frac{(-1)^n}{n}{}^nC_{n-1}.$$

* Most students forgot the constant of integration in the first step.

e) $f(x) = Ax^3 + Bx^2 + Cx + D$

$f'(x) = 3Ax^2 + 2Bx + C$

$f(1) = 0: \quad A + B + C + D = 0 \quad \cdots (1)$

$f'(1) = 0: \quad 3A + 2B + C = 0 \quad \cdots (2)$

$f'(-1) = 0: \quad 3A - 2B + C = 0 \quad \cdots (3)$

$f(-1) = -4: \quad -A + B - C + D = -4 \quad \cdots (4)$

$(1) + (4): \quad 2B + 2D = -4 \quad \cdots (5)$

$(2) - (3): \quad 4B = 0$

$\underline{\underline{B = 0}}$

From (5): $\quad 2D = -4$

$\underline{\underline{D = -2}}$

From (1): $\quad A + C - 2 = 0 \quad \cdots (6)$

From (2): $\quad 3A + C = 0 \quad \cdots (7)$

$(7) - (6): \quad 2A + 2 = 0$

$2A = -2$

$\underline{\underline{A = -1}}$

From (6): $\quad -1 + C - 2 = 0$

$\underline{\underline{C = 3}}$

$\therefore \quad A = -1, \; B = 0, \; C = 3, \; D = -2.$