Sydney Girls High School
2015

TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred.
- Board-approved calculators may be used.
- A table of standard integrals is provided at the back of this paper.
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations.
- All answers should be given in simplest exact form unless otherwise specified.

Total marks – 100

Section I Pages 3 – 7

10 Marks

- Attempt Questions 1 – 10
- Answer on the Multiple Choice answer sheet provided
- Allow about 15 minutes for this section

Section II Pages 8 – 17

90 Marks

- Attempt Questions 11 – 16.
- Answer on the blank paper provided.
- Begin a new page for each question.
- Allow about 2 hours and 45 minutes for this section.

Name: …………………………………………………………. Teacher: …………………………………………………….	**THIS IS A TRIAL PAPER ONLY** It does not necessarily reflect the format or the content of the 2015 HSC Examination Paper in this subject.

Section I

10 marks

Attempt Questions 1–10

Use the multiple-choice answer sheet for Questions 1–10.

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(1) An object rotates at 40 rpm and is moving at 30 m/s. The radius of the motion is

(A) 1.33 m

(B) 6.37 m

(C) 7.16 m

(D) 20 m

(2) Let $z = 3 - i$. What is the value of $\overline{iz}$?

(A) $-1-3i$

(B) $-1+3i$

(C) $1-3i$

(D) $1+3i$

(3) Which of the following is an expression for $\int \frac{dx}{\sqrt{7-6x-x^2}}$?

(A) $\frac{1}{4}\sin^{-1}\left(\frac{x-3}{4}\right)+c$

(B) $\frac{1}{4}\sin^{-1}\left(\frac{x+3}{4}\right)+c$

(C) $\sin^{-1}\left(\frac{x-3}{4}\right)+c$

(D) $\sin^{-1}\left(\frac{x+3}{4}\right)+c$

(4) How many ways can 5 boys and 3 girls be arranged around a circular table such that no two girls sit next to each other?

(A) 144

(B) 432

(C) 720

(D) 1440

(5) What is the solution to the equation $\tan^{-1}(4x)-\tan^{-1}(3x)=\tan^{-1}\left(\frac{1}{7}\right)$?

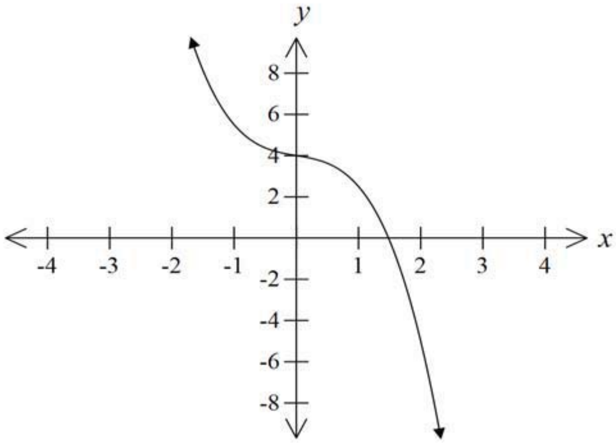
(A) $x=\frac{1}{7}$ or $x=\frac{2}{7}$

(B) $x=\frac{1}{3}$ or $x=\frac{2}{3}$

(C) $x=\frac{1}{3}$ or $x=\frac{1}{4}$

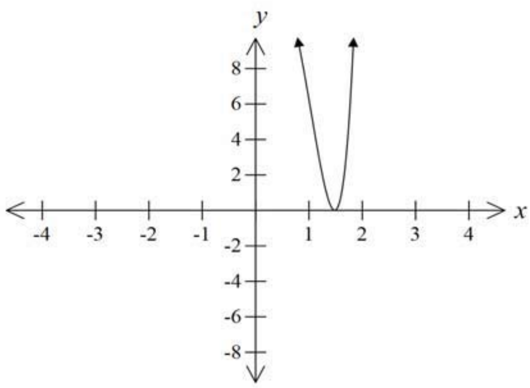
(D) $x=3$ or $x=4$

(6) The diagram below shows the graph of the function $y = f(x)$.



Which diagram represents the graph of $y^2 = f(x)$?

(A)



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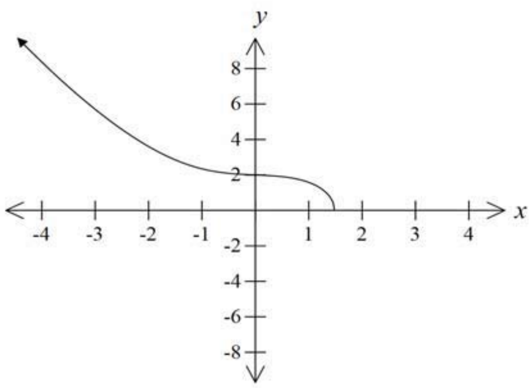
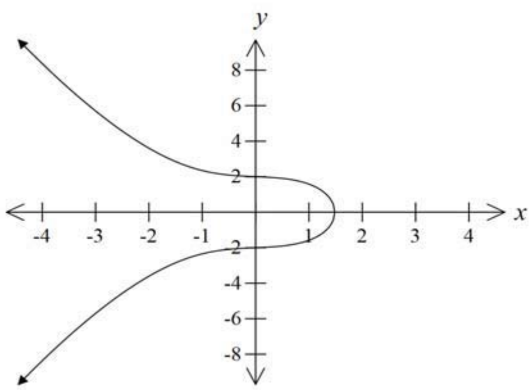
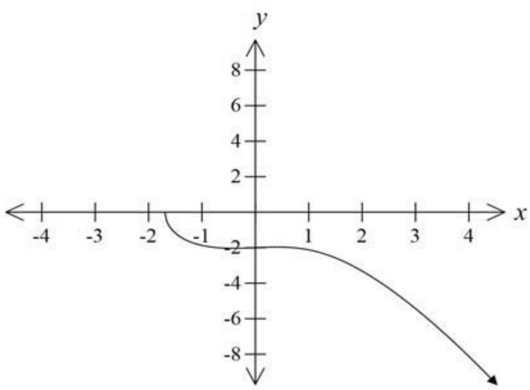


(C)



(D)



(7) Use the substitution $t = \tan\frac{x}{2}$ to find $\int -\sec x \; dx$.

(A) $\ln|(t-1)(t+1)| + c$

(B) $\ln|(1-t)(t+1)| + c$

(C) $\ln\left|\frac{1+t}{t-1}\right| + c$

(D) $\ln\left|\frac{t-1}{t+1}\right| + c$

(8) What is the eccentricity of the hyperbola $4x^2 - 25y^2 = 9$?

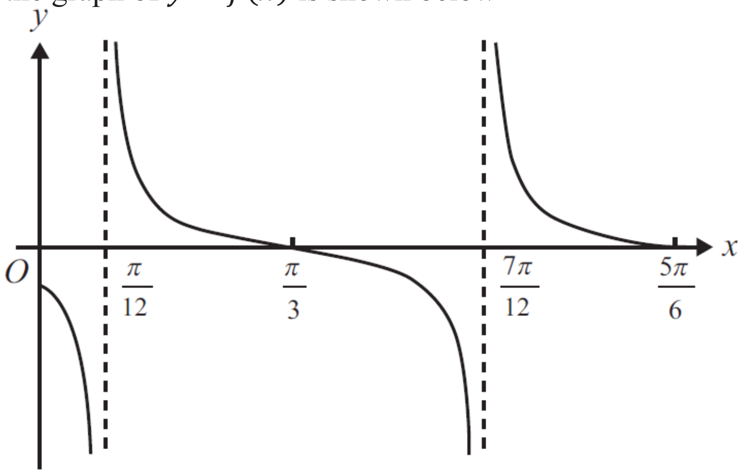
(A) $\frac{\sqrt{21}}{5}$

(B) $\frac{\sqrt{29}}{5}$

(C) $\frac{\sqrt{21}}{2}$

(D) $\frac{\sqrt{29}}{2}$

(9) Part of the graph of $y = f(x)$ is shown below



$y = f(x)$ could be

(A) $y = -\tan\left(2x - \frac{\pi}{6}\right)$

(B) $y = -\tan\left(2x - \frac{\pi}{3}\right)$

(C) $y = \cot\left(2x - \frac{\pi}{12}\right)$

(D) $y = \cot\left(2x - \frac{\pi}{6}\right)$

(10) The polynomial equation $x^3 - 3x^2 - x + 2 = 0$ has roots α, β and γ. Which one of the following polynomial equations has roots $2\alpha + \beta + \gamma$, $\alpha + 2\beta + \gamma$ and $\alpha + \beta + 2\gamma$?

(A) $x^3 - 6x^2 + 44x - 49 = 0$

(B) $x^3 - 12x^2 + 44x - 49 = 0$

(C) $x^3 + 3x^2 + 36x + 5 = 0$

(D) $x^3 + 6x^2 + 36x + 5 = 0$

Section II

90 marks

Attempt Questions 11–16

Start each question on a NEW sheet of paper.

Question 11 (15 marks)

Use a NEW sheet of paper.

(a) If $z = (1 - i)^{-1}$

(i) Express $\bar{z}$ in modulus-argument form. [2]

(ii) If $(\bar{z})^{13} = a + ib$ where *a* and *b* are real numbers, find the values of *a* and *b*. [2]

(b) Find

(i) [1]

$$\int x^3 e^{x^4+7} dx$$

(ii) [2]

$$\int \sec^3 x \tan x \ dx$$

(c) Find the Cartesian equation of the locus of a point P which represents the complex number *z* where $|z - 2i| = |z|$ [2]

(d) Sketch the region in the complex plane where $Re[(2 - 3i)z] < 12$ [2]

(e)

(i) Express $\frac{x^2+x+2}{(x^2+1)(x+1)}$ in the form $\frac{Ax+B}{x^2+1} + \frac{C}{x+1}$,
where A, B and C are constants. [2]

(ii) Hence find [2]

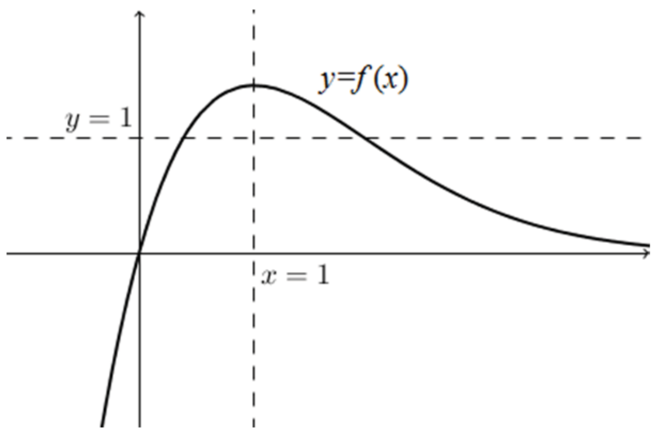
$$\int \frac{x^2 + x + 2}{(x^2 + 1)(x + 1)} dx$$

End of Question 11

Question 12 (15 marks)

Use a NEW sheet of paper.

(a)



Using four separate graphs sketch:

(i) $y = f'(x)$ [2]

(ii) $|y| = f(x)$ [2]

(iii) $y = \frac{1}{f(x)}$ [2]

(iv) $y = 3^{f(x)}$ [2]

(b) Evaluate [3]

$$\int_4^7 \frac{dx}{x^2 - 8x + 19}$$

(c) Let $f(x) = \frac{x^3+1}{x}$.

(i) Show that [1]

$$\lim_{x \to \pm\infty} [f(x) - x^2] = 0$$

(ii) Part (i) shows that the graph of $y = f(x)$ is asymptotic to the parabola $y = x^2$. Use this fact to help sketch the graph $y = f(x)$. [3]

End of Question 12

Question 13 (15 marks)

Use a NEW sheet of paper.

(a) If ω is the root of $z^5 - 1 = 0$ with the smallest positive argument, find the real quadratic equation with roots $\omega + \omega^4$ and $\omega^2 + \omega^3$. [3]

(b) Given the polynomial $P(x) = x^3 + x^2 + mx + n$ where m and n are real numbers:

(i) If $(1 - 2i)$ is a zero of $P(x)$ factorise $P(x)$ into complex linear factors. [2]

(ii) Find the values of m and n. [2]

(c)

(i) An ellipse has major and minor axes of lengths 12 and 8 respectively. Write a possible equation of this ellipse. [1]

(ii) A solid has the elliptical base from part (i). Sections of the solid, perpendicular to its base and parallel to the minor axis, are semi-circles. Find the volume of the solid. [3]

(d)

(i) Let $P(x)$ be a degree 4 polynomial with a zero of multiplicity 3. Show that $P'(x)$ has a zero of multiplicity 2. [2]

(ii) Hence find all the zeros of $P(x) = 8x^4 - 25x^3 + 27x^2 - 11x + 1$, given that it has a zero of multiplicity 3. [2]

End of Question 13

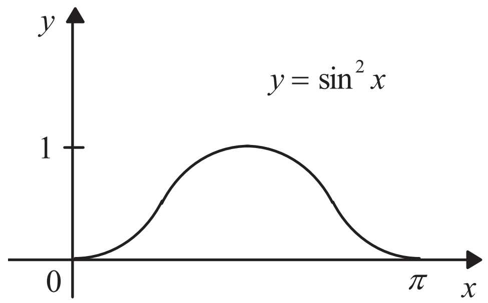
Question 14 (15 marks)

Use a NEW sheet of paper.

(a)

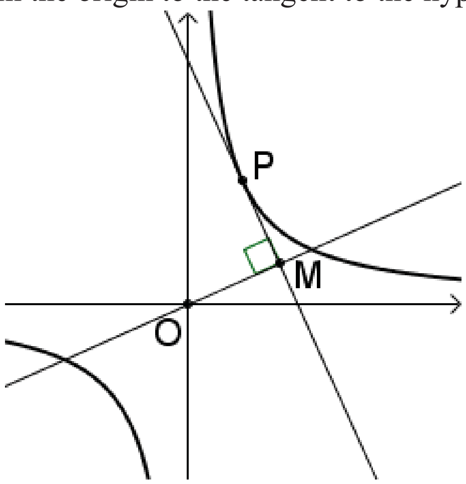
(i) Given that $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx$, show that $\int_0^\pi x\cos 2x\,dx = 0$. [2]

(ii)



The area bounded by the curve $y = \sin^2 x$ and the x-axis between $x = 0$ and $x = \pi$ is rotated through one revolution about the y-axis. By taking the limiting sum of the volumes of cylindrical shells find the volume of this solid. [2]

(b) $P\left(t, \frac{1}{t}\right)$ is a variable point on the rectangular hyperbola $xy = 1$. M is the foot of the perpendicular from the origin to the tangent to the hyperbola at P.

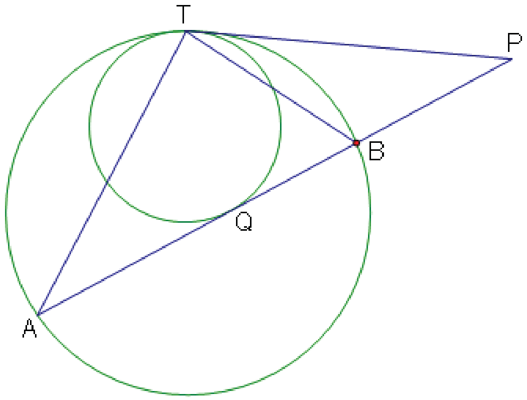


(i) Show that the tangent to the hyperbola at P has equation $x + t^2 y = 2t$. [2]

(ii) Find the equation of OM. [1]

(iii) Show that the equation of the locus of M as P varies is $x^4 + 2x^2y^2 - 4xy + y^4 = 0$ and indicate any restrictions on the values of x and y. [3]

(c) PT is a common tangent to the circles which touch at T. PA is a tangent to the smaller circle at Q.



(i) Prove that ΔBTP is similar to ΔTAP. [2]

(ii) Hence show that $PT^2 = PA \times PB$. [1]

(iii) If $PT = t$, $QA = a$ and $QB = b$ prove that $t = \dfrac{ab}{a-b}$. [2]

End of Question 14

Question 15 (15 marks)

Use a NEW sheet of paper.

(a) Evaluate [3]

$$\int_1^e x^7 \log_e x \, dx$$

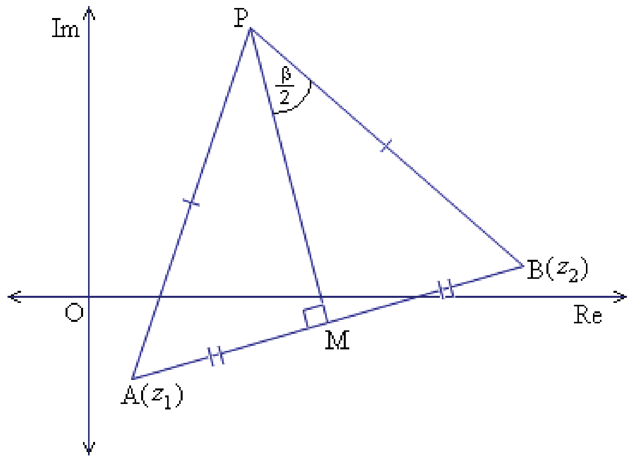
(b)

(i) On the same diagram sketch the graphs of the ellipses $E_1: \frac{x^2}{4} + \frac{y^2}{3} = 1$ and $E_2: \frac{x^2}{16} + \frac{y^2}{12} = 1$, showing clearly the intercepts on the axes. Find the coordinates of the foci and the equations of the directrices of the ellipse E_1. [2]

(ii) $P\left(2\cos p, \sqrt{3}\sin p\right)$, where $0 < p < \frac{\pi}{2}$, is a point on the ellipse E_1. Use differentiation to show that the tangent to the ellipse E_1 at P has equation $\frac{x\cos p}{2} + \frac{y\sin p}{\sqrt{3}} = 1$. [2]

(iii) The tangent to the ellipse E_1 at P meets the ellipse E_2 at the points $Q\left(4\cos q, 2\sqrt{3}\sin q\right)$ and $R\left(4\cos r, 2\sqrt{3}\sin r\right)$, where $-\pi < q < \pi$ and $-\pi < r < \pi$. Show that q and r differ by $\frac{2\pi}{3}$. [2]

(c) The diagram shows an isosceles triangle PAB. PM is the bisector of $\angle APB$, where $\angle APB = \beta$. PM bisects AB. A and B represent the complex numbers z_1 and z_2 respectively.



(i) Find the complex number represented by

(α) $\overrightarrow{AM}$ [1]

(β) $\overrightarrow{MP}$ [2]

(ii) Hence show that P represents the complex number

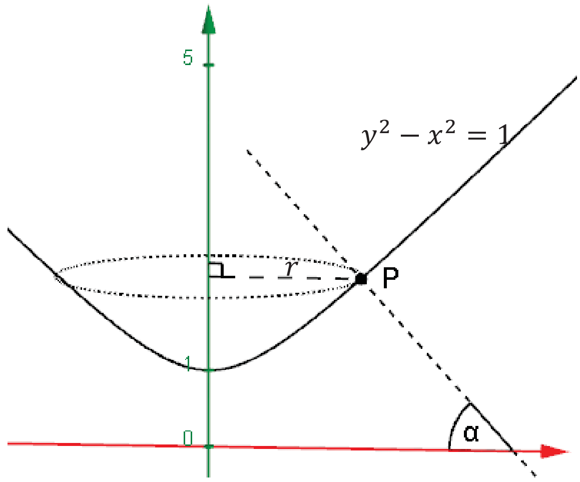
$$\frac{1}{2}\left(1 - i\cot\frac{\beta}{2}\right)z_1 + \frac{1}{2}\left(1 + i\cot\frac{\beta}{2}\right)z_2$$ [3]

End of Question 15

Question 16 (15 marks)

Use a NEW sheet of paper.

(a) A bowl is formed by rotating the hyperbola $y^2 - x^2 = 1$ for $1 \le y \le 5$ through 180° about the y-axis. Sometime later, a particle P of mass m moves around the inner surface of the bowl in a horizontal circle with constant angular velocity ω.



(i) Show that if the radius of the circle in which P moves is r, then the normal to the surface at P makes an angle α with the horizontal as shown in the diagram where $\tan\alpha = \frac{\sqrt{1+r^2}}{r}$. [2]

(ii) Draw a diagram showing the forces on P. [1]

(iii) Find the expressions for the radius r of the circle of motion and the magnitude of the reaction force between the surface and the particle in terms of m, g and ω. [3]

(iv) Find the values of ω for which the described motion of P is possible. [1]

(b) Let

$$I_n = \int_1^e (1 - \ln x)^n dx \quad \text{where } n = 0,1,2,\ldots$$

(i) Show [2]

$$I_n = -1 + nI_{n-1} \text{ where } n = 1,2,3,\ldots$$

(ii) Hence evaluate [2]

$$\int_1^e (1 - \ln x)^3 dx$$

(iii) Show that [2]

$$\frac{I_n}{n!} = e - \sum_{r=0}^{n} \frac{1}{r!} \text{ where } n = 1,2,3,\ldots$$

(iv) Show that $0 \leq I_n \leq e - 1$. [1]

(v) Deduce that [1]

$$\lim_{n \to \infty} \sum_{r=0}^{n} \frac{1}{r!} = e$$

End of Question 16

End of Exam

Sydney Girls High School

Mathematics Faculty

Multiple Choice Answer Sheet
2015 Trial HSC Mathematics Extension 2

Select the alternative A, B, C or D that best answers the question. Fill in the response oval completely.

Sample $2 + 4 = ?$ (A) 2 (B) 6 (C) 8 (D) 9

A ○ B ● C ○ D ○

If you think you have made a mistake, put a cross through the incorrect answer and fill in the new answer.

A ● B ☒ C ○ D ○

If you change your mind and have crossed out what you consider to be the correct answer, then indicate this by writing the word *correct* and drawing an arrow as follows:

correct

A ☒ B ☒ C ○ D ○

Student Number: Answers

Completely fill the response oval representing the most correct answer.

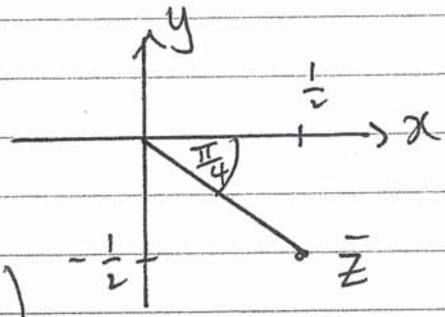
1. A ○ B ○ C ● D ○
2. A ○ B ○ C ● D ○
3. A ○ B ○ C ○ D ●
4. A ○ B ○ C ○ D ●
5. A ○ B ○ C ● D ○
6. A ○ B ○ C ● D ○
7. A ○ B ○ C ○ D ●
8. A ○ B ● C ○ D ○
9. A ○ B ○ C ○ D ●
10. A ○ B ● C ○ D ○

QUESTION 11

(a) (i) $z = \frac{1}{1-i} \times \frac{1+i}{1+i} = \frac{1+i}{1-i^2} = \frac{1}{2} + \frac{1}{2}i$

$\bar{z} = \frac{1}{2} - \frac{1}{2}i$

$|\bar{z}| = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{-1}{2}\right)^2} = \frac{1}{\sqrt{2}}$



$\arg(\bar{z}) = -\frac{\pi}{4}$ $\therefore \bar{z} = \frac{1}{\sqrt{2}} \text{cis}\left(-\frac{\pi}{4}\right)$

A common error was an incorrect calculation of the argument. A diagram is usually recommended.

(ii) $(\bar{z})^{13} = \left(\frac{1}{\sqrt{2}} \text{cis}\left(-\frac{\pi}{4}\right)\right)^{13}$

$= \left(\frac{1}{\sqrt{2}}\right)^{13} \text{cis}\left(-\frac{13\pi}{4}\right)$

$= \frac{1}{64\sqrt{2}} \text{cis}\left(\frac{3\pi}{4}\right)$

$= \frac{1}{64\sqrt{2}}\left(-\frac{1}{\sqrt{2}} + \frac{1}{\sqrt{2}}i\right)$

$\therefore a = \frac{-1}{128}, \quad b = \frac{1}{128}$

Some students didn't apply De Moivre's theorem correctly. Also, a reminder to answer the question being asked - some students did not state the values for a and b as asked.

(b) (i) $\int x^3 e^{x^4+7}\,dx = \frac{1}{4}e^{x^4+7} + C$

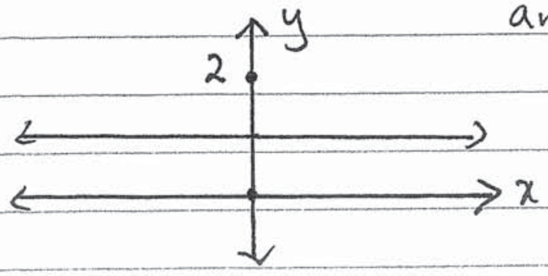
An easy question made complicated by some.

(ii) let $u = \sec x$ $\quad du = \sec x \tan x\,dx$

$I = \int u^2\,du = \frac{u^3}{3} + C = \frac{\sec^3 x}{3} + C$

(c) $|z - 2i| = |z| \rightarrow$ perpendicular bisector of $(0,0)$ and $(0,2)$.

Some used algebraic approach successfully.



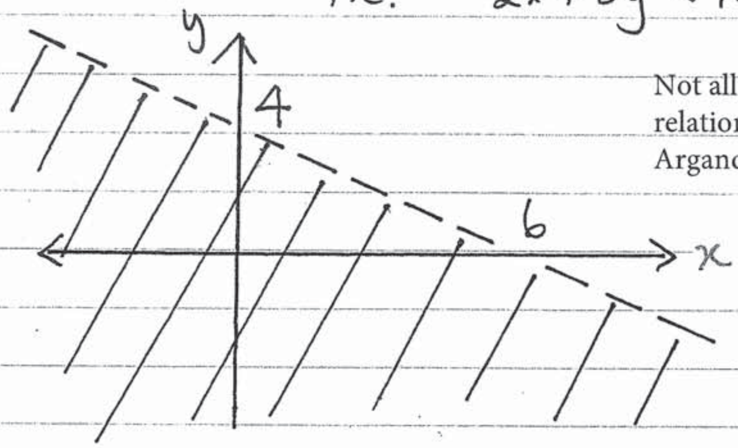
$\therefore$ The locus is the line $y = 1$.

Qn 11 (continued)

(d) let $z = x + iy$ $\quad \text{Re}\left[(2-3i)(x+iy)\right] < 12$

$$\text{Re}\left[2x+3y+i(2y-3x)\right] < 12$$

i.e. $2x + 3y < 12$



Not all students realised that the final relationship is just a region on the Argand diagram.

(e) (i) $\dfrac{x^2+x+2}{(x^2+1)(x+1)} \equiv \dfrac{Ax+B}{x^2+1} + \dfrac{C}{x+1}$

$$x^2+x+2 \equiv (Ax+B)(x+1) + C(x^2+1)$$

coeff. of x^2 $\quad 1 = A + C$ ①

coeff. of x $\quad 1 = A + B$ ②

constant $\quad 2 = B + C$ ③

① − ② $\quad C - B = 0$ i.e. $B = C$

subst. into ③ $\quad B + B = 2 \quad \therefore B = 1$ and $C = 1$

from ① $\quad A + 1 = 1 \quad \therefore A = 0$

$$\therefore \frac{x^2+x+2}{(x^2+1)(x+1)} = \frac{0x+1}{x^2+1} + \frac{1}{x+1} = \frac{1}{x^2+1} + \frac{1}{x+1}$$

A reminder to ANSWER THE QUESTION. :)

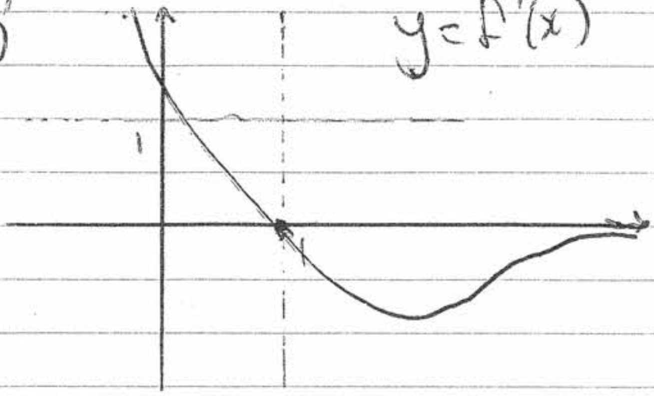
(ii) $\displaystyle\int \frac{x^2+x+2}{(x^2+1)(x+1)}\,dx = \int\left(\frac{1}{x^2+1} + \frac{1}{x+1}\right)$

$$= \tan^{-1}x + \ln(x+1) + C$$

2015 THSC ext 2

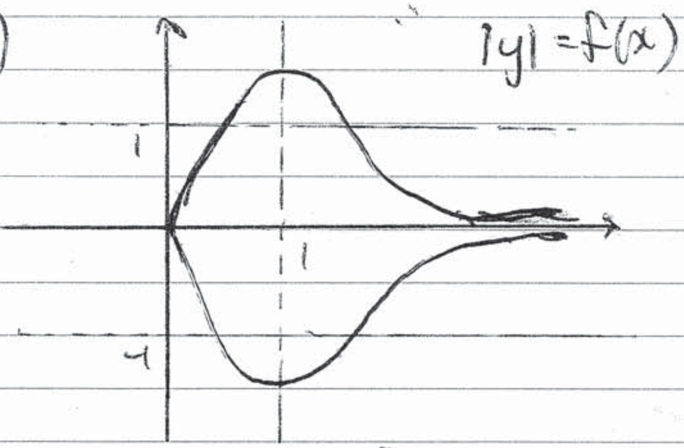
(a)

(i)

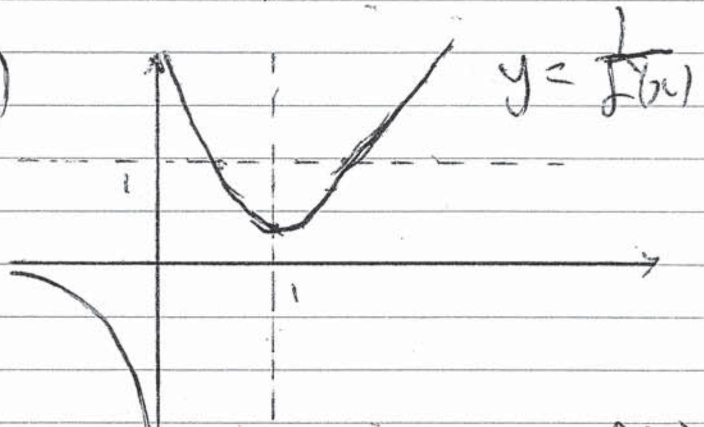


(Many student had problems with the first two graphs.)

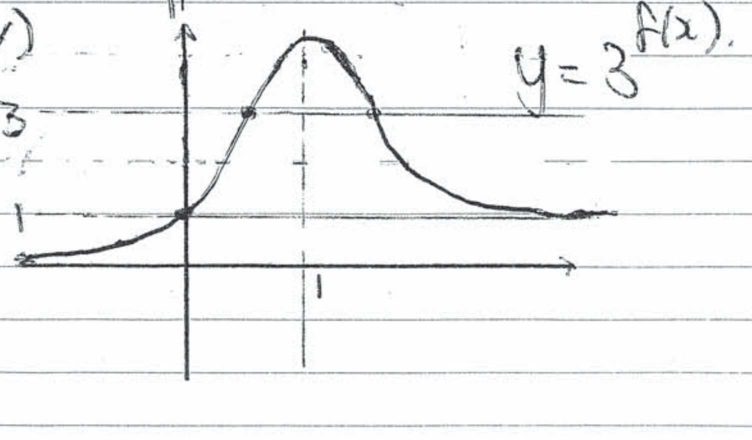
(ii)



(iii)



(iv)



(b) $\int_4^7 \frac{dx}{x^2-8x+19}$

$= \int_4^7 \frac{dx}{(x-4)^2-16+19}$

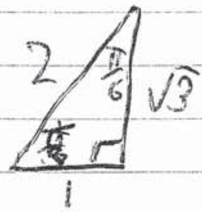
$= \int_4^7 \frac{dx}{\sqrt{3}^2+(x-4)^2}$

$= \frac{1}{\sqrt{3}}\left[\tan^{-1}\left(\frac{x-4}{\sqrt{3}}\right)\right]_4^7$

$= \frac{1}{\sqrt{3}}\left(\tan^{-1}\left(\frac{3}{\sqrt{3}}\right) - \tan^{-1} 0\right).$

$= \frac{\pi}{3\sqrt{3}}$

Most common error was to have 3 instead of $\sqrt{3}$. e.g. $\tan^{-1}\left(\frac{x-4}{3}\right)$.



(c) $f(x) = \frac{x^3+1}{x}$

(i) $\lim_{x\to\pm\infty}\left[\frac{x^3+1}{x} - x^2\right] = \lim_{x\to\pm\infty}\left[\frac{1}{x}\right]$

$= 0.$

(ii) $f'(x) = \frac{x(3x^2)-(x^3+1)}{x^2}$

$= \frac{2x^3-1}{x^2}$

$f'(x) = 0.$

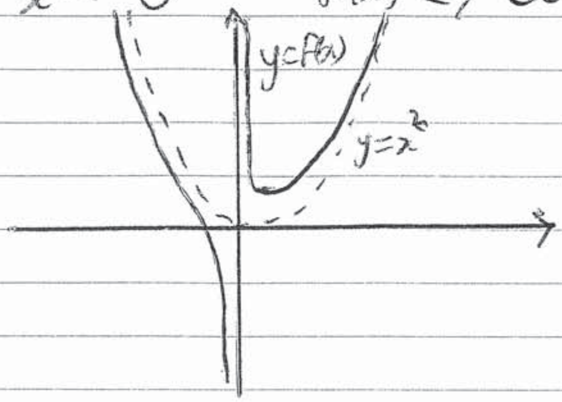
$2x^3 - 1 = 0$

$x = \frac{1}{\sqrt[3]{2}}$

x	0.7	$\frac{1}{\sqrt[3]{2}}$	0.9
$f'(x)$	-0.6	0	0.6
	\	—	/

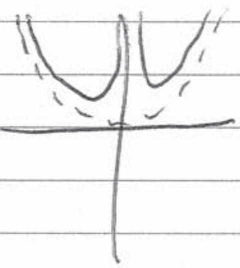
$f\left(\frac{1}{\sqrt[3]{2}}\right) = 1.89$

As $x \to 0^-$ $f(x) \to -\infty$
As $x \to 0^+$ $f(x) \to \infty$.



Common for student to have both piece inside the parabola.

e.g.



Some students ignored part (i) completely when graphing.

13 (a) $1 + w + w^2 + w^3 + w^4 = 0$

$\alpha + \beta = w + w^4 + w^2 + w^3$

$= -1$

$\alpha\beta = w^3 + w^4 + w^6 + w^7$

$= w^3 + w^4 + w + w^2$

$= -1$

Students finding w in mod-arg form had less success.

$\therefore\ x^2 - -1x + -1 = 0$

$x^2 + x - 1 = 0$

(b) (i) $1 - 2i + 1 + 2i + \alpha = -1$

$2 + \alpha = -1$

$\alpha = -3$

$\therefore\ P(x) = (x - 1 + 2i)(x - 1 - 2i)(x + 3)$

(ii) $(1-2i)(1+2i) \times -3 = -n$

$-15 = -n$

$n = 15$

~~$2(-3)^3 - 4\times(-3)^2 + m\times -3 + 15 = 0$~~

$(-3)^3 + (-3)^2 + m \times -3 + 15 = 0$

$-3m - 3 = 0$

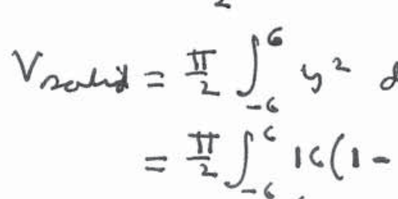
$m = -1$

(c) (i) $\dfrac{x^2}{6^2} + \dfrac{y^2}{4^2} = 1$

$\dfrac{x^2}{36} + \dfrac{y^2}{16} = 1$

Many students forgot to halve 12 and 8.

(ii)



$V_{slice} = \dfrac{\pi}{2} y^2 \delta x$

$V_{solid} = \dfrac{\pi}{2} \int_{-6}^{6} y^2\, dx$

$= \dfrac{\pi}{2} \int_{-6}^{6} 16\left(1 - \dfrac{x^2}{36}\right) dx$

This is not a semi-circle

$= 16\pi \int_0^6 \left(1 - \dfrac{x^2}{36}\right) dx$

$= 16\pi \left[x - \dfrac{x^3}{108}\right]_0^6 = 16\pi\left(6 - \dfrac{6^3}{108}\right) = 64\pi\,u^3$

(d) (i) $P(x) = (x-a)^3 Q(x)$

$P'(x) = (x-a)^3 Q'(x) + Q(x) \times 3(x-a)^2$

$= (x-a)^2 \{(x-a) Q'(x) + 3Q(x)\}$

Q.E.D.

(ii) $P'(x) = 32x^3 - 75x^2 + 54x - 1$

$P''(x) = 96x^2 - 150x + 54$

$= 6(16x^2 - 25x + 9)$

$= 6(16x - 9)(x - 1)$

$P(1) = 8 - 25 + 27 - 11 + 1$

$= 0$

$\therefore$ 1 is the triple root

$1^3 \times \alpha = \dfrac{1}{8}$

$\alpha = \dfrac{1}{8}$

$\therefore$ Zeros are $1, 1, 1, \dfrac{1}{8}$

14 (a) (i) $\int_0^{\pi} x \cos 2x \, dx$

$= \int_0^{\pi} (\pi - x) \cos (2\pi - 2x) \, dx$

$= \int_0^{\pi} (\pi - x) \cos 2x \, dx$

$= \int_0^{\pi} \pi \cos 2x \, dx - \int_0^{\pi} x \cos 2x \, dx$

$\therefore 2 \int_0^{\pi} x \cos 2x \, dx = \pi \int_0^{\pi} \cos 2x \, dx$

$= \pi \left[\frac{\sin 2x}{2} \right]_0^{\pi}$

$= \pi (0 - 0)$

$= 0$

$\therefore \int_0^{\pi} x \cos 2x \, dx = 0$

(ii) $V_{shell} = \pi \{ (x + \delta x)^2 - x^2 \} y$

$= 2\pi x y \, \delta x$

$V_{solid} = 2\pi \int_0^{\pi} x \sin^2 x \, dx$

$= 2\pi \int_0^{\pi} x \left(\frac{1 - \cos 2x}{2} \right) dx$

$= \pi \int_0^{\pi} (x - x \cos 2x) \, dx$

$= \pi \left[\frac{x^2}{2} \right]_0^{\pi}$ Using the result in (i)

$= \frac{\pi^3}{2}$

(b) (i) $\frac{dy}{dx} = \frac{dy}{dt} \times \frac{dt}{dx}$

$= -\frac{1}{t^2} \times 1$

$= -\frac{1}{t^2}$

$y - \frac{1}{t} = -\frac{1}{t^2} (x - t)$

$t^2 y - t = -x + t$

$\therefore x + t^2 y = 2t$

(ii) $y = t^2 x$ m_{OM} is perpendicular to the tangent at P

(iii) $t^2 = \frac{y}{x}$

$t = \pm \sqrt{\frac{y}{x}}$

$x + \frac{y}{x} \times y = 2x \pm \sqrt{\frac{y}{x}}$

$\left(x + \frac{y^2}{x} \right)^2 = \frac{4y}{x}$

$x^2 + 2y^2 + \frac{y^4}{x^2} = \frac{4y}{x}$

$x^4 + 2x^2 y^2 + y^4 = 4xy$

restriction: $\frac{y}{x} > 0$

The point M cannot lie on the hyperbola

(c) (i) In ΔBTP and ΔTAP

$\hat{P}$ is common

$P\hat{T}B = T\hat{A}P$ ($\angle$s in external segment)

$\therefore \Delta BTP \,|||\, \Delta TAP$ (equiangular)

(ii) $\frac{TP}{TA} = \frac{PB}{TP}$ (corresponding sides in similar Δs)

$\therefore TP^2 = TA \cdot PB$

Quoting the theorem is not showing.

(iii) $t^2 = (t + a)(t - b)$

$= t^2 - bt + at - ab$

~~ab~~ $= at - bt$

$= t(a - b)$

$\therefore t = \frac{ab}{a - b}$

$t = PG$ (tangents from an external pt.)

Q15.

(a) $I = \int_1^e x^7 \log_e x \, dx.$

$u = \log_e x \qquad v = \frac{x^8}{8}$

$u' = \frac{1}{x} \qquad v' = x^7$

$I = \frac{x^8}{8} \ln(x) \Big|_1^e - \frac{1}{8}\int_1^e x^7 \, dx$

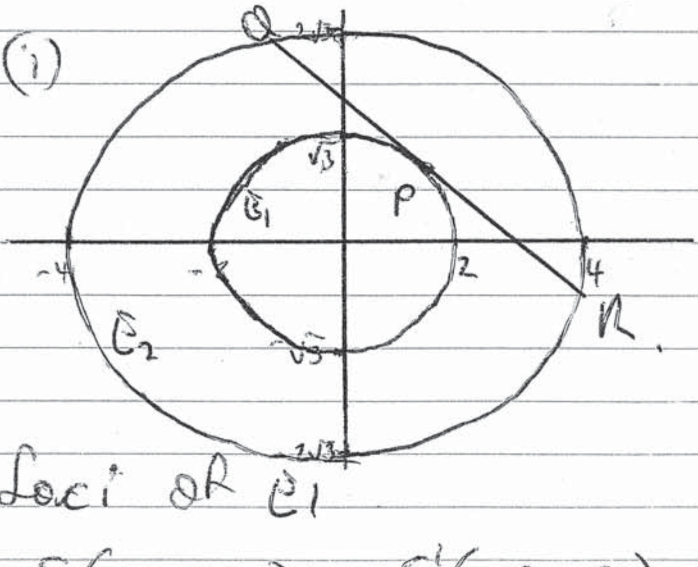
$= \frac{e^8}{8} - \frac{1}{8}\left[\frac{x^8}{8}\right]_1^e$

$= \frac{e^8}{8} - \frac{x^8}{64} + \frac{1}{64}$

$= \frac{7e^8}{64} + \frac{1}{64}$

Well done by most.

(b) (i)



$E_1: \frac{x^2}{4} + \frac{y^2}{3} = 1$

$E_2: \frac{x^2}{16} + \frac{y^2}{12} = 1$

For E_1 $3 = 4(1 - e^2)$

$1 - \frac{3}{4} = e^2$

$\frac{1}{4} = e^2$

$e = \frac{1}{2}.$

Foci of E_1

$S(ae, 0) \quad S'(-ae, 0).$

$S(1, 0) \quad S'(-1, 0)$

directrices

$x = \pm 4$

(b)(ii) $P(2\cos p, \sqrt{3}\sin p)$ $\quad E_1: \frac{x^2}{4} + \frac{y^2}{3} = 1.$

$$\frac{x}{2} + \frac{2y}{3}\frac{dy}{dx} = 0.$$

$$\frac{dy}{dx} = -\frac{x}{2} \times \frac{3}{2y}.$$

$$\frac{dy}{dx} = -\frac{3x}{4y}$$

At P,

$$m_T = -\frac{\sqrt{3}\cos p}{2\sin p}$$

Tangent at P,

$$y - \sqrt{3}\sin p = -\frac{\sqrt{3}\cos p}{2\sin p}(x - 2\cos p).$$

$$2y\sin p - 2\sqrt{3}\sin^2 p = -\sqrt{3}x\cos p + 2\sqrt{3}\cos^2 p$$

$$\sqrt{3}x\cos p + 2y\sin p = 2\sqrt{3}.$$

$$\frac{x\cos p}{2} + \frac{y\sin p}{\sqrt{3}} = 1.$$

(iii) Tangent to E_1 at P meets E_2 at some point $(4\cos t, 2\sqrt{3}\sin t)$.

So

$$\frac{4\cos t\cos p}{2} + \frac{2\sqrt{3}\sin t}{\sqrt{3}} = 1.$$

$$2(\cos t\cos p + \sin t\sin p) = 1$$

$$\cos(t - p) = \frac{1}{2}.$$

Since $0 < p < \frac{\pi}{2}$ and $-\pi < t < \pi$ $\quad t - p = \pm\frac{\pi}{3}.$

Hence Q and R have parametres

$$q = \frac{\pi}{3} + p, \quad r = -\frac{\pi}{3} + p.$$

$$\therefore |q - r| = \frac{2\pi}{3}.$$

Some arguments for this were poor and lost a mark.

(c)(i) $\overrightarrow{AB}$ represents $(z_2 - z_1)$

$\therefore \overrightarrow{AM}$ represents $\frac{1}{2}(z_2 - z_1)$.

(ii) $\tan\frac{\beta}{2} = \frac{|\overrightarrow{AM}|}{|\overrightarrow{PM}|}$

$|\overrightarrow{PM}| = |\overrightarrow{AM}| \cot\frac{\beta}{2}$.

Also $\left| \frac{\frac{1}{2}(z_2 - z_1)}{|\overrightarrow{AM}|} \right| = 1$

Find a unit vector in the right direction then multiply by the required modulus

So $\overrightarrow{PM}$ is represented by

$$\frac{\frac{1}{2}(z_2 - z_1)}{|\overrightarrow{AM}|} i \times |\overrightarrow{AM}| \cot\frac{\beta}{2}.$$

$$= \frac{i}{2}(z_2 - z_1) \cot\frac{\beta}{2}.$$

(iii) P represents $z_1 + \overrightarrow{AM} + \overrightarrow{PM}$

Must add 3 vectors to get P. Some student only add $\overrightarrow{AM}$ and $\overrightarrow{PM}$.

$$= z_1 + \frac{1}{2}(z_2 - z_1) + \frac{i}{2}(z_2 - z_1)\cot\frac{\beta}{2}.$$

$$= \frac{1}{2}z_1 + \frac{1}{2}z_2 + \frac{i}{2}z_2\cot\frac{\beta}{2} - \frac{i}{2}z_1\cot\frac{\beta}{2}$$

$$= \frac{1}{2}\left(1 - i\cot\frac{\beta}{2}\right)z_1 + \frac{1}{2}\left(1 + i\cot\frac{\beta}{2}\right)z_2.$$

Question 16

(a) (i) at P, $x = r$ and $y^2 - r^2 = 1$

$$y = \sqrt{1+r^2}$$

$$2y\frac{dy}{dx} - 2x = 0$$

$$\frac{dy}{dx} = \frac{x}{y}, \quad \text{at } P \quad m_T = \frac{r}{\sqrt{1+r^2}}$$

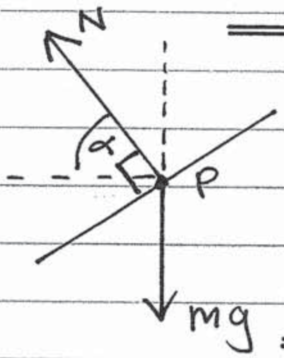
$$m_N = -\frac{\sqrt{1+r^2}}{r}$$

Many students did not make the connection to the gradient of the normal. Also, the angle between the line and the x-axis was overlooked.

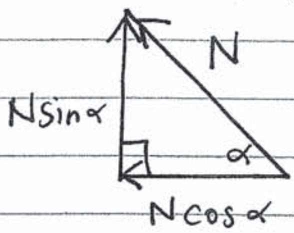
$$\tan(180 - \alpha) = -\frac{\sqrt{1+r^2}}{r}$$

$$-\tan\alpha = -\frac{\sqrt{1+r^2}}{r} \quad \therefore \tan\alpha = \frac{\sqrt{1+r^2}}{r}$$

(ii)



(iii) Resolving forces on P



Horizontally $\quad mr\omega^2 = N\cos\alpha \quad (1)$

Vertically $\quad N\sin\alpha - mg = 0$

$$\therefore mg = N\sin\alpha \quad (2)$$

$$(2) \div (1) \quad \tan\alpha = \frac{g}{r\omega^2}$$

Using (i) $\quad \frac{\sqrt{1+r^2}}{r} = \frac{g}{r\omega^2}$

$$1 + r^2 = \left(\frac{g}{\omega^2}\right)^2$$

$$r^2 = \frac{g^2}{\omega^4} - 1$$

$$\therefore r = \sqrt{\frac{g^2}{\omega^4} - 1}$$

Some students incorrectly resolved forces horizontally and vertically. Also, the question sought an expression for r and N in terms of m, g and w only. Solutions that included r in the expression for N were incomplete.

16 (a) (iii) continued

$① ^2 + ② ^2 \quad N^2(\sin^2\alpha + \cos^2\alpha) = m^2g^2 + m^2r^2\omega^4$

$\therefore \quad N^2 = m^2g^2 + m^2\omega^4\left(\frac{g^2}{\omega^4} - 1\right)$

$= m^2g^2 + m^2g^2 - m^2\omega^4$

i.e. $N = \sqrt{2m^2g^2 - m^2\omega^4}$

A small number of students answered this correctly. Most students failed to consider the restriction placed on w due to the range.

(iv) $y = \sqrt{1+r^2}$ and $r^2 = \frac{g^2}{\omega^4} - 1$

$\therefore \quad y = \sqrt{\frac{g^2}{\omega^4}} = \frac{g}{\omega^2}$

For movement to occur $r > 0 \quad \therefore y = \sqrt{1+r^2} > 1$

Since $1 \leq y \leq 5$ and $y > 1$ and $y = \frac{g}{\omega^2}$

$1 < \frac{g}{\omega^2} \leq 5$

$1 > \frac{\omega^2}{g} \geq \frac{1}{5}$ i.e. $\sqrt{\frac{g}{5}} \leq \omega < \sqrt{g}$

$\therefore \quad 1.4 \text{ rad/s} \leq \omega < 3.13 \text{ rad/s}$

Question 16 (cont.)

(b)(i) $I_n = \int_1^e (1-\ln x)^n \, dx$ where $n = 0, 1, 2 \ldots$

Mostly well done though substitution into the definite integral would show more clearly the required result.

let $u = (1-\ln x)^n$ $\quad v' = 1$

$u' = -\frac{n}{x}(1-\ln x)^{n-1}$ $\quad v = x$

$$\therefore I_n = \left[x(1-\ln x)^n\right]_1^e - \int -n(1-\ln x)^{n-1} dx$$

$$= e(1-\ln e)^n - 1(1-\ln 1)^n + n\int (1-\ln x)^{n-1} dx$$

$$\therefore I_n = -1 + nI_{n-1}$$

(ii) $I_3 = \int_1^e (1-\ln x)^3 dx$

$$= -1 + 3I_2$$

$$= -1 + 3(-1 + 2I_1)$$

$$= -4 + 6(-1 + I_0)$$

$$= -10 + 6\int_1^e dx$$

$$= -10 + 6\left[x\right]_1^e$$

$$= -10 + 6(e-1)$$

$$\therefore I_3 = 6e - 16$$

Some students made careless calculation errors. These questions require the ability to manage multiple substitutions with accuracy.

(iii) $\frac{I_n}{n!} = \frac{-1 + nI_{n-1}}{n!} = \frac{-1}{n!} + \frac{I_{n-1}}{(n-1)!}$

$$= -\frac{1}{n!} + \frac{-1 + (n-1)I_{n-2}}{(n-1)!}$$

$$= -\frac{1}{n!} - \frac{1}{(n-1)!} + \frac{I_{n-2}}{(n-2)!}$$

$$= \frac{-1}{n!} - \frac{1}{(n-1)!} - \frac{1}{(n-2)!} + \frac{I_{n-3}}{(n-3)!}$$

$$= -\frac{1}{n!} - \frac{1}{(n-1)!} - \frac{1}{(n-2)!} \cdots - \frac{1}{1!} + \frac{I_0}{0!}$$

This question was not done particularly well by many. Some students glossed over certain elements of the proof and need to consider any suggestions provided on their individual paper.

Qn 16 (b)(iii) Continued

$$\frac{I_n}{n!} = -\left(\frac{1}{1!}+\frac{1}{2!}+\dots+\frac{1}{(n-1)!}+\frac{1}{n!}\right)+\int_1^e 1\,dx$$

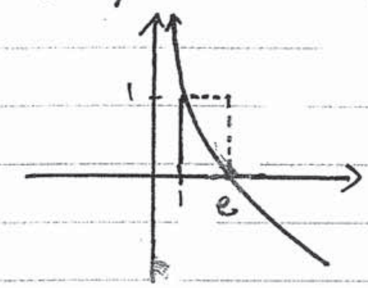
$$= -\sum_{r=1}^{n}\frac{1}{r!} + \left[x\right]_1^e$$

$$= -\sum_{r=1}^{n}\frac{1}{r!} + e - 1 \qquad \text{Note: } 1 = \frac{1}{0!}$$

$$= -\left(\sum_{r=1}^{n}\frac{1}{r!}+\frac{1}{0!}\right)+e$$

i.e. $\frac{I_n}{n!} = e - \sum_{r=0}^{n}\frac{1}{r!}$

(iv) Consider graph of $y = 1 - \ln x$ between $x = 1$ and $x = e$.



Note that the y-values for this domain are $0 \le y \le 1$.
The y-values for $y = (1-\ln x)^n$ will also be in the range $0 \le y \le 1$ where $n = 0, 1, 2 \dots$
Consider the area under the curve $y = (1-\ln x)^n$ for $1 \le x \le e$ where the area will always be smaller than the rectangle shown, and always above the x-axis.

A variety of approaches could be taken. However, the question required explanation for the zero part of the inequality and not just (e-1).

$$0 \le \int_1^e (1-\ln x)^n\,dx \le (e-1)\times 1$$

i.e. $0 \le I_n \le e-1$

(v) Using (iv) $\frac{0}{n!} \le \frac{I_n}{n!} \le \frac{e-1}{n!}$

as $n \to \infty$, $\frac{e-1}{n!} \to 0$ $\quad \therefore\ 0 \le \lim_{n\to\infty}\frac{I_n}{n!} \le 0$

i.e. $\lim_{n\to\infty}\frac{I_n}{n!} = 0$ $\quad \therefore \lim_{n\to\infty}\left(e - \sum_{r=0}^{n}\frac{1}{r!}\right) = 0$

Many students picked up the marks for this question but could have provided a solution with greater clarity.

$$e - \lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!} = 0$$

i.e. $\lim_{n\to\infty}\sum_{r=0}^{n}\frac{1}{r!} = e$