Caringbah High School

2016

Trial HSC Examination

Mathematics Extension 1

General Instructions
- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen Black pen is preferred
- Board-approved calculators may be used
- A data sheet is provided at the back of this paper
- In Questions 11–15, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I 10 marks
- Attempt Questions 1–10
- Allow about 15 minutes for this section

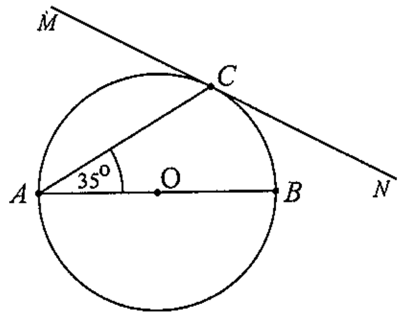
Section II 60 marks
- Attempt Questions 11–14
- Allow about 1 hour and 45 minutes for this section

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1–10.

---

1 In the diagram, AB is a diameter of the circle and MN is tangent to the circle at C. $\angle CAB = 35^o$. What is the size of $\angle MCA$?



A) 35^0 B) 45^0

C) 55^0 D) 65^0

2 Find $f^{-1}(x)$ given $f(x) = \dfrac{3x-3}{x-2}$

A) $f^{-1}(x) = \dfrac{3y-3}{x-2}$ B) $f^{-1}(x) = \dfrac{2x-3}{x-3}$

C) $f^{-1}(x) = \dfrac{x-2}{3x-3}$ D) $f^{-1}(x) = \dfrac{3-3x}{3-x}$

3 The acute angle between $2x + y - 3 = 0$ and $y = \dfrac{1}{3}x + 1$ is

A) 54^0 B) 82^0

C) 79^0 D) 45^0

4 The remainder when $P(x) = 2x^3 - 6x^2 + 4x + 3$ is divided by $2x - 1$ is

A) 3

B) -9

C) $3\frac{3}{4}$

D) $\frac{-3}{4}$

5 The exact value of $\cos 15^0$ is

A) $\frac{\sqrt{6}+\sqrt{2}}{4}$

B) $\frac{\sqrt{3}}{4}$

C) $\frac{1}{4}$

D) $\frac{\sqrt{6}-\sqrt{2}}{4}$

6 $\int 2\cos^2 x\,dx =$

A) $-\sin x \cos x + x + c$

B) $\frac{1}{2}\sin 2x + x + c$

C) $\frac{2}{3}\cos^3 x + c$

D) $\frac{-2}{\sqrt{1-x^2}} + c$

7 The velocity of a particle at a position x is given by $\dot{x} = 2e^{\frac{-x}{2}}$ m/s. The particles acceleration when its displacement is -2 metres is

A) $-e$ m/s^2

B) $\frac{-4}{e^2}$ m/s^2

C) $-2e^2$m/s^2

D) e^2 m/s^2

8 The value of $\sin^{-1}(\frac{1}{2}) + \cos^{-1}(\frac{-\sqrt{3}}{2})$ is

A) $\pi + \frac{\pi}{3}$

B) 0

C) $\frac{7\pi}{6}$

D) π

9 If $\log_a x = p$ and $\log_a y = q$, find the value of $\log_a x^2 y$ in terms of p and q.

A) $p^2 q$

B) $2p + q$

C) $p^2 + q$

D) $q - 2p$

10 When $y = e^{x+2}$ is rotated about the y axis between $x = 0$ and $x = 2$, its volume is given by

A) $\pi \int_{e^2}^{e^4} (\ln y - 2)^2 dy$

B) $\pi \int_{e^2}^{e^4} e^{2x+4} dx$

C) $\pi \int_{0}^{2} e^{x+2}\, dx$

D) $\pi \int_{0}^{2} (\ln y - 2)\, dy$

Section II

60 marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a) Find the gradient of the tangent to the curve $y = \cos^3 x$ at $x = \frac{\pi}{6}$ **2**

b) Evaluate $\lim_{x \to 0} \frac{\sin 3x}{5x}$ **1**

c) Consider the function $f(x) = 3\sin^{-1}(\frac{x}{2})$

i) Find the value of $f(2)$ **1**

ii) State the domain and range of this function **2**

iii) Draw the graph of $y = f(x)$ **2**

iv) Find $f'(x)$ **1**

d) A particle moves in a straight line so its position x from a fixed point 0 at time t is given by $x = 3\sin 2t + 4\cos 2t$.

i) If the motion is expressed in the form $x = r\sin(2t + \alpha)$ find the value of the constants r and α. (α to the nearest degree) **2**

ii) Show the motion is simple harmonic. **2**

iii) What is the period of the oscillation? **1**

iv) Determine the maximum displacement from the centre of the motion. **1**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) Write a primitive for $(5-2x)^4$ **2**

b) Find $\int \tan x \, dx$ **2**

c) If α, β, γ are the roots of the equation $x^3 - 4x + 1 = 0$ evaluate $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **2**

d) A bottle of lemonade is taken out of a fridge at $4^0 C$ into a room where the air temperature is $25^0 C$. The rate at which the lemonade warms follows Newton's law, that is $\frac{dT}{dt} = k(T-25)$ where $k < 0$, time t is measured in minutes, and the temperature T is in degrees celsius.

i) Show that $T = 25 + Ae^{kt}$ is a solution to $\frac{dT}{dt} = k(T-25)$ and find the value of A **2**

ii) The temperature of the lemonade reaches $15^0 C$ in 45 minutes. Find the value of k to four decimal places. **2**

iii) Find the temperature of the lemonade 90 minutes after being removed from the fridge, to nearest degree. **1**

e) Prove by the method of mathematical induction that $\sum_{r=1}^{n} 5^{r-1} = \frac{5^n - 1}{4}$ **4**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

a) Find the first derivative of $y = \log_e(\frac{1}{\sqrt{\cos x}})$ **3**

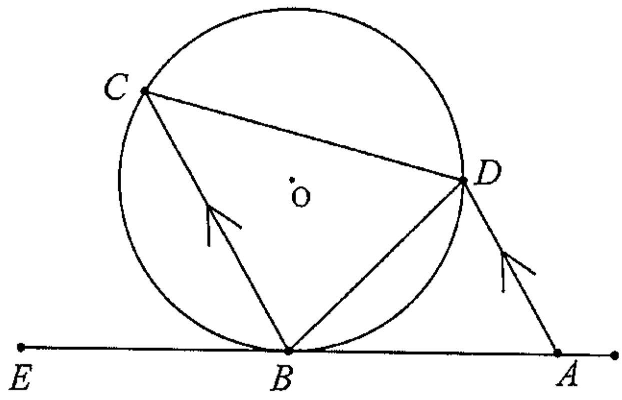
b) A capsule is in the shape of a cylinder with a hemisphere on each end. The radius of the cylindrical section is r cm, and the volume of the capsule is 16 cm^3.

i) If the height of the cylinder is $4\,cm$ show that $r^3 + 3r^2 = \frac{12}{\pi}$ **2**

ii) Show that one solution of the equation $r^3 + 3r^2 = \frac{12}{\pi}$ lies between $r = 0$ and $r = 1$ **1**

iii) The equation $r^3 + 3r^2 = \frac{12}{\pi}$ has one root close to $r = 0.9$. Use one application of Newton's method of approximation to give a better approximation to three decimal places. **2**

c) AE is tangent at B and $AD \parallel BC$. Prove that $\Delta BCD \parallel\!| \Delta DBA$ **2**



d) Find the indefinite integral of $\int \frac{1}{\sqrt{1-4x^2}}dx$ **2**

e) The polynomial $3x^3 - 17x^2 - 8x + 12 = 0$ has roots α, β, γ. Given that the product of two of the roots is 4, solve the equation for α, β, γ **3**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

a) Evaluate $\int_0^{\sqrt{3}} \frac{x}{\sqrt{1+x^2}}\,dx$ using the substitution $u = 1 + x^2$ **3**

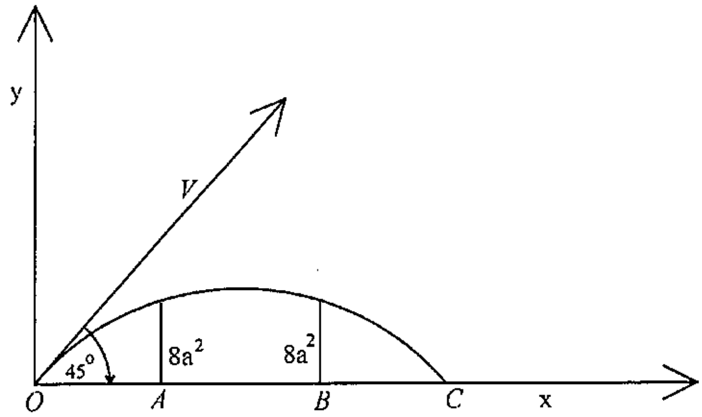
b) i) Sketch the graph of $y = \cos x$ and $y = \sin x$ on the same diagram for $0 \le x \le \frac{\pi}{2}$ **1**

ii) Show that if $0 < x < \frac{\pi}{4}$, then $\sin 2x > 2\sin^2 x$ **2**

Question 14 continues on page 10

Question 14 (continued)

c) A projectile fired with velocity V and at an angle of 45^0 to the horizontal, just clears the top of two vertical posts of height $8a^2$ units. The posts at A and B are $12a^2$ units apart. Also OA=BC = b units . There is no air resistance and the acceleration due to gravity is g.



If the projectile is at a point $P(x, y)$ at time t, expressions for x and y in terms of t are $x = \frac{Vt}{\sqrt{2}}$ and $y = \frac{-gt^2}{2} + \frac{Vt}{\sqrt{2}}$. **Do not prove these results.**

i) Show that the path of the projectile is given by $y = x - \frac{gx^2}{V^2}$ **2**

ii) Using the information in (ii) show that the range of the projectile is $\frac{V^2}{g}$ **2**

iii) If the first post is b units from the origin show that

(α) $\frac{V^2}{g} = 2b + 12a^2$ **1**

(β) $8a^2 = b - \frac{gb^2}{V^2}$ **1**

iv) Hence or otherwise prove that $V = 6a\sqrt{g}$. **3**

End of Exam

Carringbah High School YR 12 Ext 1 Trial H.S.C. 2016

SOLUTIONS

Multiple Choice

1, C 2, B 3, B 4, C 5, A 6, B 7, C 8, D 9, B 10, A

Question 11

(a) $y = \cos^3 x$

$y' = 3(\cos x)^2 \cdot -\sin x$

when $x = \pi/6$

$M_T = 3(\cos \frac{\pi}{6})^2 \cdot -\sin\frac{\pi}{6}$

$= 3\left(\frac{\sqrt{3}}{2}\right)^2 \cdot -\frac{1}{2}$

$= -\frac{9}{8}$

(b) $\lim_{x \to 0} \frac{\sin 3x}{5x}$

$= \frac{\sin 3x}{3x} \times \frac{3}{5}$

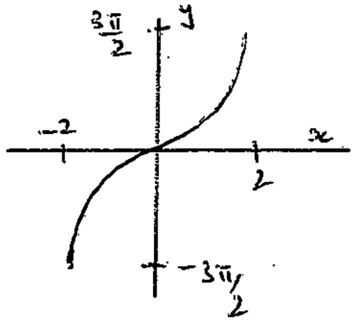
$= \frac{3}{5}$

(c) (i) $f(2) = \frac{3\pi}{2}$

(ii) $D: -2 \le x \le 2$

$R: -\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$

(iii)



(iv) $f(x) = 3\sin^{-1}(x/2)$

$f'(x) = 3 \cdot \frac{1}{\sqrt{1-(\frac{x}{2})^2}} \times \frac{1}{2}$

$= \frac{3}{2}\left[\frac{2}{\sqrt{4-x^2}}\right]$

$= \frac{3}{\sqrt{4-x^2}}$

(d) Question 11 (cont'd)

(i) $r = 5$, $\alpha = 53^\circ$

(ii) $x = 5\sin(2t + 53^\circ)$

$\dot{x} = 5\cos(2t + 53^\circ) \times 2$

$= 10\cos(2t + 53^\circ)$

$\ddot{x} = -20\sin(2t + 53^\circ)$

$\ddot{x} = -4x$.

(iii) $T = \frac{2\pi}{2}$

$= \pi$

(iv) 5

Question 12

(a) $\frac{(5-2x)^5}{-10} + C$

(b) $\int \tan x \, dx$

$= \int \frac{\sin x}{\cos x} dx$

$= -\int \frac{-\sin x}{\cos x}$

$= -\ln(\cos x) + C$

(c) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$

$= \frac{\beta\gamma + \alpha\gamma + \alpha\beta}{\alpha\beta\gamma}$

$= \frac{-4}{-1}$

$= +4$

(d) (i) $T = 25 + Ae^{k.t}$

$$\frac{dT}{dt} = K.Ae^{k.t}$$

$$= K.(T-25)$$

when $t = 0$, $T = 4$

$$4 = 25 + Ae^{0}$$

$$A = -21$$

$$\therefore T = 25 - 21e^{k.t}$$

(ii) $15 = 25 - 21e^{45K}$

$$\frac{10}{21} = e^{45K}$$

$$\ln\left(\frac{10}{21}\right) = 45K$$

$$K = -0.0165$$

(iii) $T = 25 - 21e^{-0.0165(90)}$

$T = 20°C$ (to nearest degree)

(e) $\sum_{r=1}^{n} 5^{r-1} = \frac{5^n - 1}{4}$

when $r = 1$: $5^{1-1} = \frac{5^1 - 1}{4}$

$$1 = 1$$

Assume true for $n = K$

$$S_K = \frac{5^K - 1}{4}$$

Prove true for $n = K+1$

$$S_{K+1} = S_K + T_{K+1}$$

$$= \frac{5^K - 1}{4} + 5^K$$

$$= \frac{5^K - 1 + 4.5^K}{4}$$

$$= \frac{5.5^K - 1}{4}$$

$$= \frac{5^{K+1} - 1}{4}$$ (plus Statement)

(a) $y = \log_e\left(\frac{1}{\sqrt{\cos x}}\right)$

$$y = \log_e\left[(\cos x)^{-\frac{1}{2}}\right]$$

$$y' = \frac{-\frac{1}{2}(\cos x)^{-\frac{3}{2}} \cdot -\sin x}{(\cos x)^{-\frac{1}{2}}}$$

$$y' = \frac{1}{2}\sin x(\cos x)^{-1}$$

$$= \frac{1}{2}\frac{\sin x}{\cos x}$$

$$= \frac{1}{2}\tan x$$

(b) (i) $V = \pi r^2 4 + \frac{4}{3}\pi r^3$

$$16 = 4\pi r^2 + \frac{4}{3}\pi r^3$$

$$48 = 12\pi r^2 + 4\pi r^3$$

$$48 = 4\pi(3r^2 + r^3)$$

$$\frac{12}{\pi} = r^3 + 3r^2$$

(ii) Let $f(r) = r^3 + 3r^2 - \frac{12}{\pi}$

$r = 0$: $f(0) = -\frac{12}{\pi}$

$r = 1$: $f(1) = 1 + 2 - \frac{12}{\pi}$

$= 0.18\ldots\ldots$

$f(0) < 0$ and $f(1) > 0$

(iii) $f(r) = r^3 + 3r^2 - \frac{12}{\pi}$

$f(0.9) \doteqdot -0.6607$

$f'(r) = 3r^2 + 6r$

$f'(0.9) = 7.83$

$$r_2 = r_1 - \frac{f(r_1)}{f'(r_1)}$$

$r_2 \doteqdot 0.984$

Question 13

(c)

$\angle ADB = \angle DBC$

(alt $\angle$'s // lines CB, DA)

$\angle DBA = \angle DCB$

(alt seg thm)

$\triangle BCD \parallel\!| \triangle DBA$

(equiangular)

(d) $\int \frac{1}{\sqrt{1-4x^2}}\,dx$

$= \int \frac{1}{\sqrt{4[(\frac{1}{2})^2 - x^2]}}\,dx$

$= \frac{1}{2}\int \frac{1}{\sqrt{(\frac{1}{2})^2 - x^2}}$

$= \frac{1}{2}\sin^{-1}\left(\frac{x}{1/2}\right) + c$

$= \frac{1}{2}\sin^{-1}2x + c$

(e) $\alpha+\beta+\gamma = 17/3$ — ①

$\alpha\beta+\alpha\gamma+\beta\gamma = -3/3$ — ②

$\alpha\beta\gamma = -4$ — ③

Let $\alpha\beta = 4 \Rightarrow \beta = \frac{4}{\alpha}$

$4\gamma = -4$

$\gamma = -1$

using ①

$\alpha+\beta-1 = 17/3$

$\alpha+\beta = \frac{20}{3}$

sub $\beta = 4/\alpha$

$\alpha + 4/\alpha = \frac{20}{3}$

$3\alpha^2 - 20\alpha + 12 = 0$

$(\alpha-6)(3\alpha-2) = 0$

$\alpha = 6,\ \alpha = 2/3$

$\therefore$ Roots are

$6,\ 2/3,\ -1$

Question 14

(a) $\int_0^{\sqrt{3}} \frac{x\,dx}{\sqrt{1+x^2}}$

If $u = 1+x^2$

$du = 2x.dx$

when $x = \sqrt{3}$

$u = 4$

when $x = 0$

$u = 1$

$\frac{1}{2}\int_0^{\sqrt{3}} \frac{2x}{\sqrt{1+x^2}}\,dx$

$\frac{1}{2}\int_1^4 \frac{du}{u^{1/2}}$

$\frac{1}{2}\int_1^4 u^{-1/2}\,du$

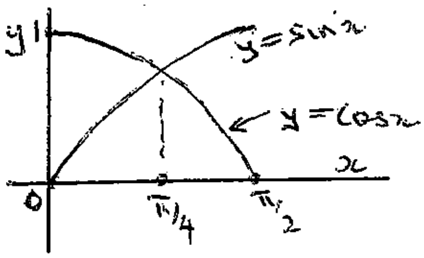
$\frac{1}{2}\left[\frac{u^{1/2}}{1/2}\right]_1^4$

$= \frac{1}{2}[4-2]$

$= 1$

(b)

(i)



for $0 < x < \pi/4$

(ii) $\cos x > \sin x$ and

since $\sin x > 0$

$\cos x \sin x > \sin^2 x$

$2\cos x \sin x > 2\sin^2 x$

$\sin 2x > 2\sin^2 x$

OR $\sin 2x > 2\sin^2 x$

$2\sin x \cos x > 2\sin^2 x$

$\cos x > \sin x$ (since $\sin x > 0$)

graph shows $\cos x$ is higher than $\sin x$ for

$0 < x < \pi/4$

(c) (i) $x = \dfrac{Vt}{\sqrt{2}}$, $y = -\dfrac{gt^2}{2} + \dfrac{Vt}{\sqrt{2}}$

$t = \dfrac{\sqrt{2}x}{V}$

$\therefore y = -\dfrac{g}{2}\left(\dfrac{\sqrt{2}x}{V}\right)^2 + \dfrac{V}{\sqrt{2}}\left(\dfrac{\sqrt{2}x}{V}\right)$

$y = -\dfrac{g}{2} \cdot \dfrac{2x^2}{V^2} + x$

$y = -\dfrac{gx^2}{V^2} + x$

(ii) when $y = 0$

$0 = -\dfrac{gx^2}{V^2} + x$

$0 = x\left(1 - \dfrac{gx}{V^2}\right) = 0$

start $x = 0$

$1 - \dfrac{gx}{V^2} = 0$

$\dfrac{gx}{V^2} = 1$

$x = \dfrac{V^2}{g}$

(iii) (α) $OC = \dfrac{V^2}{g} = OA + AB + BC$

$\dfrac{V^2}{g} = b + 12a^2 + b$

$\dfrac{V^2}{g} = 2b + 12a^2$

$y = x - \dfrac{gx^2}{V^2}$

(β) $x = b$, $y = 8a^2$

$8a^2 = b - \dfrac{gb^2}{V^2}$

Question 14

(iv) $\frac{V^2}{g} = 2b + 12a^2$ — ①

$8a^2 = b - \frac{gb^2}{V^2}$ — ②

from ①

$V^2 = g(2b + 12a^2)$

sub into ②

$8a^2 = b - \frac{gb^2}{g(2b+12a^2)}$

$8a^2 = \frac{2b^2 + 12a^2 b - b^2}{2b + 12a^2}$

$16a^2 b + 96a^4 = b^2 + 12a^2 b$

$b^2 - 4a^2 b - 96a^4 = 0$

$(b - 12a^2)(b + 8a^2) = 0$

$b = 12a^2 \quad b = -8a^2$

$\therefore b = 12a^2$

$\frac{V^2}{g} = 2(12a^2) + 12a^2$

$\frac{V^2}{g} = 36a^2$

$V = 6a\sqrt{g}$

OR (iv)

$\frac{V^2}{g} = 2b + 12a^2$ — ①

$8a^2 = b - \frac{gb^2}{V^2}$ — ②

from ① $b = \frac{V^2}{2g} - 6a^2$

sub into ②

$8a^2 = \frac{V^2}{2g} - 6a^2 - \frac{g}{V^2}\left(\frac{V^2}{2g} - 6a^2\right)^2$

$8a^2 = \left(\frac{V^2}{2g} - 6a^2\right)\left[1 - \frac{g}{V^2}\left(\frac{V^2}{2g} - 6a^2\right)\right]$

$8a^2 = \left(\frac{V^2}{2g} - 6a^2\right)\left[1 - \frac{1}{2} + \frac{6a^2 g}{V^2}\right]$

$8a^2 = \frac{V^2 - 12a^2 g}{2g}\left[\frac{1}{2} + \frac{6a^2 g}{V^2}\right]$

$8a^2 = \left(\frac{V^2 - 12a^2 g}{2g}\right)\left[\frac{V^2 + 12a^2 g}{2V^2}\right]$

$8a^2 = \frac{V^4 - 144a^4 g^2}{4gV^2}$

$32a^2 g V^2 = V^4 - 144a^4 g^2$

$V^4 - 32a^2 g V^2 - 144a^4 g^2 = 0$

$(V^2 - 36a^2 g)(V^2 + 4a^2 g) = 0$

$\therefore V^2 - 36a^2 g = 0$

$V^2 = 36a^2 g$

$V = 6a\sqrt{g}$.