GIRRAWEEN HIGH SCHOOL

2016

MATHEMATICS EXTENSION 2

YEAR 12 Trial

HIGHER SCHOOL CERTIFICATE

EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
  Black pen is preferred
- Board-approved calculators may be used
- A table of standard integrals is provided at the back of this paper
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II

90 marks

- Attempt Questions 11–16
- Allow about 2 hours and 45 minutes for this section

Section 1
10 marks
Attempt questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

1. If $z=1+2i$ and $w=3-i$, what is the value of $z-\overline{w}$?

A) $3i-2$ B) $4+3i$ C) $i-2$ D) $4+i$

2. What value of z satisfies $z^2=7-24i$?

A) $4-3i$ B) $-4-3i$ C) $3-4i$ D) $-3-4i$

3. The equation of the polynomial equation $x^3-3x^2+2=0$ has roots α, β and γ. What is the value of $\alpha^3+\beta^3+\gamma^3$?

A) 9 B) 13 C) 21 D) 25

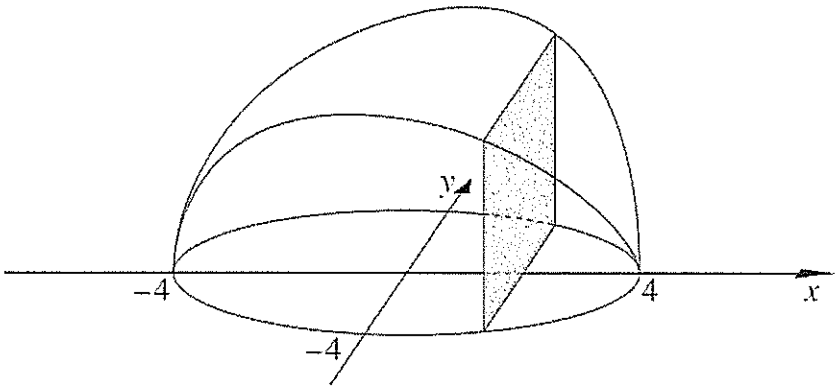
4. Which of the following is a correct expression for $\int x3^{x^2}\,dx$?

A) $\dfrac{3^{x^2+1}}{x^2+1}+C$ B) $\dfrac{3^{x^2}}{\ln 9}+C$ C) $\dfrac{3^{x^2}}{\ln 3}+C$ D) $3^{x^2}\ln 3+C$

5. What is the eccentricity of the ellipse $9x^2+16y^2=25$?

A) $\dfrac{7}{16}$ B) $\dfrac{\sqrt{7}}{4}$ C) $\dfrac{\sqrt{15}}{4}$ D) $\dfrac{5}{4}$

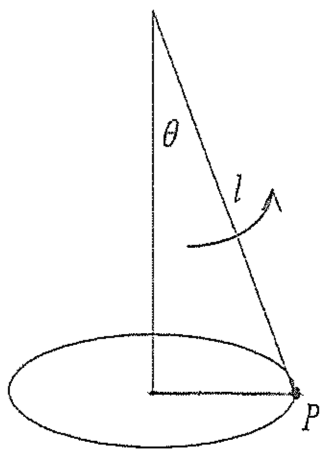
9. The base of a solid is the region bounded by the circle $x^2+y^2=16$. Vertical cross-sections are squares perpendicular to the x-axis as shown in the diagram.



Which integral represents the volume of the solid ?

A) $\int_{-4}^{4} 4x^2 dx$ B) $\int_{-4}^{4} 4\pi x^2 dx$ C) $\int_{-4}^{4} 4(16-x^2)dx$ D) $\int_{-4}^{4} 4\pi(16-x^2)dx$

10. A bob P of mass m kg is suspended from a fixed point A by a string of length l metres, and acceleration due to gravity g. P describes a horizontal circle with uniform angular velocity ω rad/sec.



Which of the following expressions represents the tension in the string ?

A) $ml\omega$ B) $ml\omega^2$ C) $mgl\omega$ D) $mgl\omega^2$

Question 12. (15 marks).

a) The polynomial $P(x) = x^4 + ax^2 + bx + 28$ has a double root at $x = 2$.

What are the values of a and b ? **2**

b) The polynomial $P(x) = x^4 - 4x^3 + 11x^2 - 14x + 10$ has roots $a + ib$ and $a + 2ib$

where a and b are real and $b \neq 0$.

i) By evaluating a and b, find all the roots of $P(x)$. **3**

ii) Hence, or otherwise, find the quadratic polynomials with real coefficients that are

factors of $P(x)$. **1**

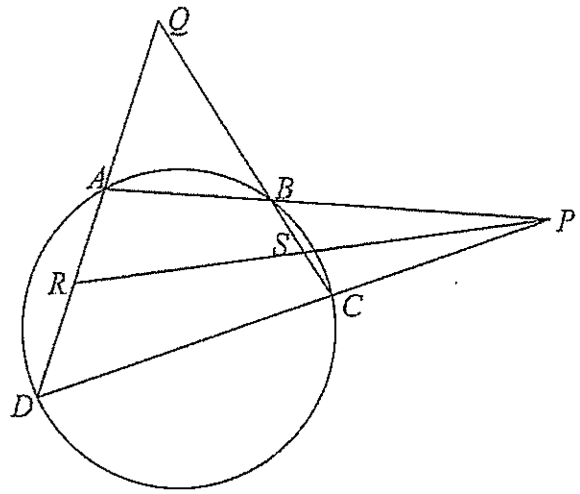
c) Let $I_n = \int_0^{\frac{\pi}{4}} \tan^n x dx$ for all integers $n \geq 0$.

i) Show that $I_n = \dfrac{1}{n-1} - I_{n-2}$ for integers $n \geq 2$. **3**

ii) Hence find $\int_0^{\frac{\pi}{4}} \tan^5 x dx$. **2**

Question 13. (15 marks)

a) The diagram shows a cyclic quadrilateral $ABCD$. Chords AB and DC produced meet at P and chords DA and CB produced meet at Q. PR is the internal bisector of $\angle APD$ meeting AD at R and BC at S.



Prove that ΔQRS is isosceles. **2**

b) i) Show that $4x^2 + 9y^2 + 16x + 18y - 11 = 0$ represents an ellipse. **2**

ii) Find the eccentricity and hence the coordinates of its foci and the equations of the directrices. **2**

Question 14.(15 marks)

a) Find the equation of the tangent to the curve $x^2 - xy + y^3 = 1$ at the point $P(1,1)$. **3**

b) Use the substitution $x = \tan\theta$ to evaluate $\int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}dx$. **3**

c) i) Use De Moivre's Theorem to prove that if $z = \cos\theta + i\sin\theta$,

$2\cos n\theta = z^n + \frac{1}{z^n}$. **2**

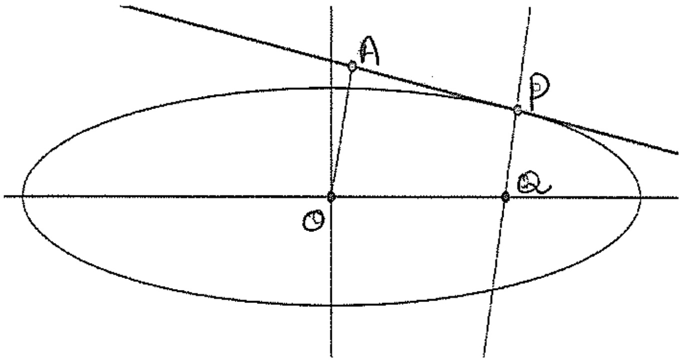
ii) Hence, or otherwise solve the equation $5x^4 - 11x^3 + 16x^2 - 11x + 5 = 0$ **3**

d) A mass of 5 kilogram attached to a light fishing line describes a circular path with radius 60 centimetres about a point P on a smooth table. It completes 2 revolutions per second.

i) Find the tension in the fishing line. **2**

ii) The line breaks under a tension of 900 Newtons. Find the maximum number of revolutions per second. **2**

c) An ellipse has the equation $x^2 + 16y^2 = 25$.

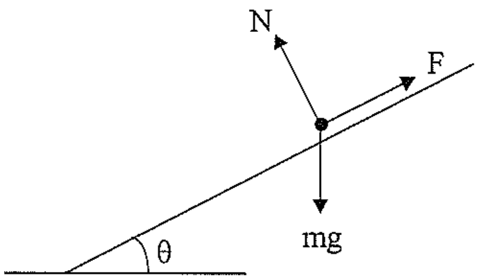


i) Find the gradient of the ellipse at the point $P(3,1)$. **1**

ii) Find the equation of the tangent and normal to the ellipse at P. **1**

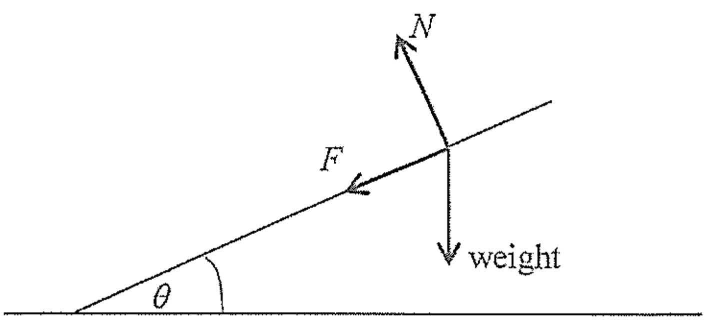
iii) The normal to the ellipse, at point P, meets the major axis at Q. A line from the centre, O to the tangent at P meets at right angles at point A.

Show that the value of $PQ \times OA$ is equal to the square of the semi-minor axis. **3** c)



A particle of mass m is lying on an inclined plane and does not move. The plane is at an angle θ to the horizontal. The particle is subject to a gravitational force mg, a normal reaction force N, and a frictional force F parallel to the plane, as shown in the diagram. Resolve the forces acting on the particle, and hence find an expression for $\frac{F}{N}$ in terms of θ. **2**

d)



A car of mass 2000 kg travels around a curve of radius 150m at a speed of 110 km/h.

The car experiences a lateral resistance force F of $0.22N$, where the normal force is N, as shown in the diagram. By resolving the forces vertically and horizontally, find the angle θ for the car to negotiate the curve. (Assume acceleration due to gravity is $10m/s^2$). **4**

END OF EXAMINATION

Year 12 Ext. 2 Solutions Trial 2016

Multiple choice Questions

(1) $z = 1+2i$, $w = 3-i$, $z - \bar{w} = ?$

$z - \bar{w} = 1+2i - (3+i) = 1+2i-3-i = -2+i$ (C)

(2) $z^2 = 7-24i$

Let $z = a+ib$

$\therefore z^2 = a^2 - b^2 + 2iab$

$\therefore a^2 - b^2 = 7$

$2ab = -24$

By inspection: $a = \pm 4$, $b = \mp 3$

$\therefore z = 4-3i$ (A)

(3) $x^3 - 3x^2 + 2 = 0$ has roots α, β, γ

$\alpha^3 + \beta^3 + \gamma^3 = 3(\alpha^2 + \beta^2 + \gamma^2) - 6$

$= 3\left[(\alpha+\beta+\gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)\right] - 6$

$= 3 \times 9 - 6 = 21$ (C)

(4) $\int x 3^{x^2} dx = \frac{1}{2}\int 2x 3^{x^2} dx = \frac{3^{x^2}}{2\ln 3} + c = \frac{3^{x^2}}{\ln 9} + c$ (B)

or

Let $u = x^2$ $\therefore \int x 3^{x^2} dx = \frac{1}{2}\int 2x 3^{x^2} dx$

$= \frac{1}{2}\int 3^u du = \frac{3^u}{2\ln 3} = \frac{3^{x^2}}{\ln 9} + c$

(5) $9x^2 + 16y^2 = 25 \Rightarrow \frac{x^2}{\frac{25}{9}} + \frac{y^2}{\frac{25}{16}} = 1$

$b^2 = a^2(1-e^2) \Rightarrow \frac{25}{16} = \frac{25}{9}(1-e^2) \Rightarrow 9 = 16 - 16e^2$

$\therefore 16e^2 = 7$

$e = \frac{\sqrt{7}}{4}$ (B)

Multiple choice Questions

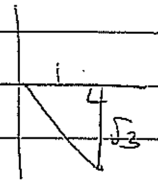
(1) C (2) A (3) C (4) B (5) B

(6) D (7) D (8) A (9) C (10) B

Question 11

$$Z = 1 - i\sqrt{3}$$

a) (i) $|z| = \sqrt{1^2 + (\sqrt{3})^2} = 2$



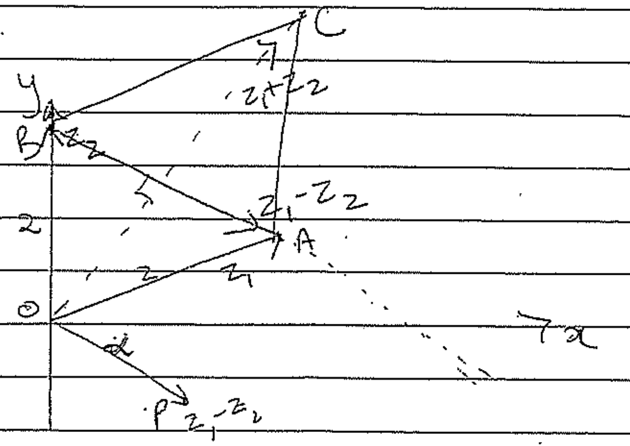
$\arg(z) = -\frac{\pi}{3}$ (2)

(ii) $Z_1 = 2\left[\cos\left(-\frac{\pi}{3}\right) + i\sin\left(-\frac{\pi}{3}\right)\right]$

$Z^6 = 2^6\left[\cos(-2\pi) + i\sin(-2\pi)\right] = 2^6\left[1 - 0\right] = 6$

(2)

b) (i)



(2)

(ii) Vectors OC and AB form a parallelogram
However, $OA = OB = 2$ $\therefore$ OACB is a rhombu

$$\angle AOC = \frac{1}{2}\left(\frac{\pi}{2} - \frac{\pi}{12}\right) = \frac{5\pi}{24}$$

$$\therefore \arg(Z_1 + Z_2) = \frac{5\pi}{24} + \frac{\pi}{12} = \frac{7\pi}{24}$$

e) (i) $\cot\theta + \operatorname{cosec}\theta = \cot(\theta/2)$

LHS $\cot\theta + \operatorname{cosec}\theta = \cancel{2\cos^2\theta/2}$

$= \frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta} = \frac{\cos\theta + 1}{\sin\theta}$

$= \frac{2\cos^2\theta/2}{2\sin\theta/2\cos\theta/2} = \cot\theta/2 = \text{RHS}$.

2

(ii) $\int(\cot\theta + \operatorname{cosec}\theta)\,d\theta = \int \cot\theta/2\,d\theta$.

$= \int \frac{\cos\theta/2}{\sin\theta/2}\,d\theta = 2\ln(\sin\theta/2) + c$

1

Question 12

a) $P(x) = x^4 + ax^2 + bx + 28$

$P'(x) = 4x^3 + 2ax + b$

Root at $x = 2 \Rightarrow$

$P(2) = 2^4 + 4a + 2b + 28 = 0$

$44 + 4a + 2b = 0$

$4a + 2b = -44$ — 1

$P'(2) = 0 \Rightarrow 32 + 4a + b = 0$

$\therefore 4a + b = -32$ — 2

1 − 2 $b = -12$

subs in 1 $\Rightarrow 4a - 24 = -44$

$\therefore 4a = -20$

$a = -5$

$a = -5$

$b = -12$

2

Question 12 (c) continued

(ii) $\int_0^{\frac{\pi}{4}} \tan^5 x\,dx = I_5 = \frac{1}{4} - I_3$

$I_3 = \frac{1}{2} - I_1$

$I_1 = \int_0^{\frac{\pi}{4}} \tan x\,dx = -\left[\ln(\cos x)\right]_0^{\pi/4}$

$= -\ln\frac{1}{\sqrt{2}} = \frac{1}{2}\ln 2.$

$\therefore I_3 = \frac{1}{2} - \frac{1}{2}\ln 2$

$\therefore I_5 = \frac{1}{4} - \left(\frac{1}{2} - \frac{1}{2}\ln 2\right) = \frac{1}{2}\ln 2 - \frac{1}{4}$ (2)

d) (i) Solving simultaneously $3 - x^2 = x + x^2$

$2x^2 + x - 3 = 0 \Rightarrow (2x+3)(x-1) = 0$

$\therefore x = -\frac{3}{2},\ x = 1$

Since P is in first quadrant, $x = 1$ (1)

(ii)

$\Delta x = 2\pi r h\,\delta x$

$= 2\pi(x+1)(y_1 - y_2)\,\delta x$

$\doteq 2\pi(x+1)\left[3 - x^2 - x - x^2\right]\delta x$

$= 2\pi(x+1)\left[3 - x - 2x^2\right]\delta x$

$\therefore V = \lim_{\delta x \to 0} 2\pi \sum_{x=-1}^{1} (x+1)(3 - x - 2x^2)\,\delta x$ (2)

(iii) $V = 2\pi\int_{-1}^{1} (3 + 2x - 3x^2 - 2x^3)\,dx = 8\pi\ u^3$ (1)

Question 13 continued

b) (ii) $b^2 = a^2(1-e^2)$

$4 = 9(1-e^2) \Rightarrow e^2 = \frac{5}{9}$

$\therefore e = \frac{\sqrt{5}}{3}$ as $(0<e<1)$

directrices: $x = -2 \pm \frac{9}{\sqrt{5}}$

$\frac{a}{e} = \frac{3}{\frac{\sqrt{5}}{3}} = \frac{9}{\sqrt{5}}$

focus:

$ae = 3 \times \frac{\sqrt{5}}{3} = \sqrt{5} \Rightarrow (-2+\sqrt{5}, -1)$ and $(-2-\sqrt{5}, -1)$

2

c) (i) $xy = c^2$

differentiating $\Rightarrow x \cdot \frac{dy}{dx} + y = 0$

$\therefore \frac{dy}{dx} = \frac{-y}{x}$

at P, $\frac{dy}{dx} = -\frac{\frac{c}{p}}{cp} = \frac{-1}{p^2}$

$\therefore$ equation of tangent: $y - \frac{c}{p} = \frac{-1}{p^2}(x-cp)$

$p^2y - cp = -x + cp \Rightarrow x + p^2y = 2cp$

1

(ii) Tangents at P and Q are

$x + p^2y = 2cp$ —— ①

$x + q^2y = 2cq$ —— ②

① − ② $y(p^2-q^2) = 2c(p-q)$

$\therefore y = \frac{2c}{p+q}$

sub. into ① $\Rightarrow x + p^2\left(\frac{2c}{p+q}\right) = 2cp$.

$x = 2cp - \frac{2cp^2}{p+q} = \frac{2cpq}{p+q}$

$\therefore T = \left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

Question 14

a) $x^2 - xy + y^3 = 1$ $\qquad P(1,1)$

differentiating, $2x - x\cdot\frac{dy}{dx} - y + 3y^2\frac{dy}{dx} = 0$

$$\frac{dy}{dx}(3y^2 - x) = y - 2x$$

$$\therefore \frac{dy}{dx} = \frac{y-2x}{3y^2-x}$$

at $P(1,1)$, $\frac{dy}{dx} = \frac{1-2}{3-1} = \frac{-1}{2} = -\frac{1}{2}$

$\therefore$ eq^n. of tangent:

$$y - 1 = -\tfrac{1}{2}(x-1)$$

$$2y - 2 = -x + 1 \Rightarrow x + 2y - 3 = 0$$

3

b) $x = \tan\theta \quad \therefore dx = \sec^2\theta\, d\theta$

Limits: $x = \sqrt{3} \Rightarrow \theta = \frac{\pi}{3}$

$x = 1 \Rightarrow \theta = \frac{\pi}{4}$

$$x^2\sqrt{1+x^2} = \tan^2\theta \times \sec\theta = \tan\theta \cdot \frac{\sin\theta}{\cos\theta} \cdot \sec\theta$$

$$\therefore = \tan\theta \cdot \sin\theta \sec^2\theta.$$

$$\therefore \int_1^{\sqrt{3}} \frac{1}{x^2\sqrt{1+x^2}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \operatorname{cosec}\theta \cot\theta\, d\theta$$

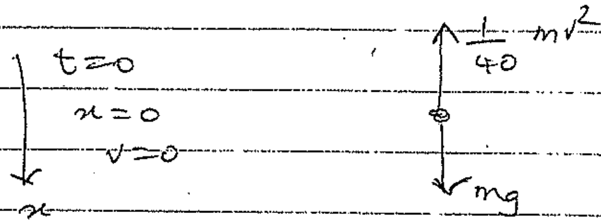
$$= -\left[\operatorname{cosec}\theta\right]_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \frac{-2}{\sqrt{3}} + \sqrt{2} = \frac{1}{3}(3\sqrt{2} - 2\sqrt{3})$$

3

Question 15

a) (i)



By Newton's second law, $m\ddot{x} = mg - \frac{1}{40}mv^2$

$$\therefore \ddot{x} = \frac{1}{40}(400 - v^2)$$

(1)

(ii)

$$\ddot{x} = \frac{1}{40}(400 - v^2)$$

$$\frac{dv}{dt} = \frac{1}{40}(400 - v^2)$$

$$\therefore \frac{dt}{dv} = \frac{40}{20^2 - v^2} = \frac{1}{20+v} + \frac{1}{20-v}$$

$$\therefore \int dt = \int \frac{1}{20+v}\,dv + \int \frac{1}{20-v}\,dv$$

$$\therefore t = \ln\left(\frac{20+v}{20-v}\right) + c$$

when $t=0$, $v=0 \Rightarrow c=0$.

$$\therefore t = \ln\left(\frac{20+v}{20-v}\right)$$

(2)

(iii)

$$\therefore e^t = \frac{20+v}{20-v}$$

$$\therefore (20-v)e^t = 20+v = 20e^t - ve^t$$

$$\therefore v(1+e^t) = 20(e^t - 1)$$

$$\therefore v = \frac{20(e^t-1)}{1+e^t} = 20\left(\frac{e^t+1-2}{e^t+1}\right)$$

$$= 20\left(1 - \frac{2}{e^t+1}\right)$$

(1)

b) (i) $h = AF = a(e-1)$

$\therefore PF = a(e^2-1) = h(e+1)$

e) (i) $x^2 + 16y^2 = 25$

differentiating, $2x + 32y \cdot \frac{dy}{dx} = 0$

$\therefore \frac{dy}{dx} = \frac{-2x}{32y} = \frac{-x}{16y}$

at $P(3,1)$, $\frac{dy}{dx} = -\frac{3}{16}$

(ii) Equation of tangent:

$y - 1 = \frac{-3}{16}(x-3) \Rightarrow 16y - 16 = -3x + 9$

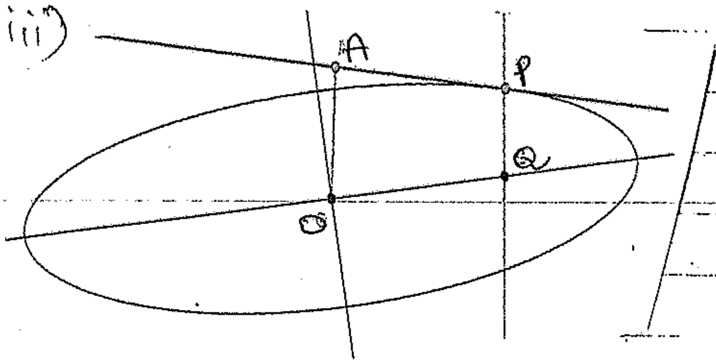
$\therefore 3x + 16y = 25$

Equation of normal:

$y - 1 = \frac{16}{3}(x-3) \Rightarrow 3y - 3 = 16x - 48$

$\therefore 16x - 3y = 45$

(iii)



$OA \perp AP$ and

eq^n AP $\Rightarrow 3x + 16y - 25 =$

$\therefore OA = \left| \frac{0 + 0 - 25}{\sqrt{9 + 256}} \right| = \frac{25}{\sqrt{2}}$

Coordinates of $Q \Rightarrow$ sub. in $y = 0$ in $16x - 3y = 45$

$\therefore x = \frac{45}{16}$

$\therefore Q = \left(\frac{45}{16}, 0\right)$

$\therefore PQ = \sqrt{\left(3 - \frac{45}{16}\right)^2 + 1} = \sqrt{\frac{9}{256} + 1} = \frac{\sqrt{265}}{16}$

$\therefore PQ \times OA = \frac{25}{\sqrt{265}} \times \frac{\sqrt{265}}{16} = \frac{25}{16}$

but in $x^2 + 16y^2 = 25 \Rightarrow \frac{x^2}{25} + \frac{y^2}{\frac{25}{16}} = 1 \Rightarrow b^2 = \frac{25}{16}$

$\therefore PQ \cdot \times OA = \frac{25}{16} =$ square of semi-minor axis

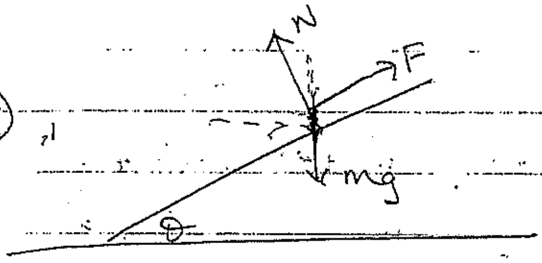
(iv) ① $x \sin\beta + 2 \times \cos\beta$

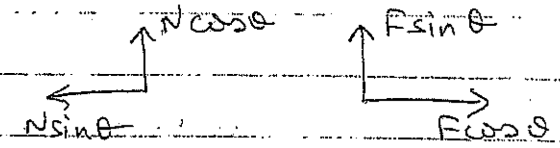
$\Rightarrow$

$$T_1(\sin\beta\cos\alpha + \cos\beta\sin\alpha) = Mg\sin\beta + Mr\omega^2\cos\beta$$

$$\therefore T_1 = M\left(\frac{r\omega^2\cos\beta + g\sin\beta}{\sin(\alpha+\beta)}\right)$$ ②

c)





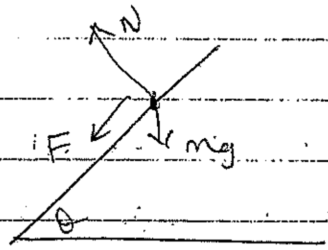
resolving forces (c)

normal to plane: $N = mg\cos\theta$

parallel to plane: $F = mg\sin\theta$

$$\therefore \frac{F}{N} = \frac{mg\sin\theta}{mg\cos\theta} = \tan\theta$$ ②

d)



Acceleration

$\xleftarrow{\quad}$ $\frac{v^2}{r}$

$110\ \text{km/h} = \frac{275}{9}\ \text{m/s}$

Resolve vertically $\Rightarrow$ $mg + F\sin\theta = N\cos\theta$

Resolve horizontally $\Rightarrow$ $F\cos\theta + N\sin\theta = \frac{mv^2}{r}$

$F = 0.22N$, $\Rightarrow$ $mg + 0.22N\sin\theta = N\cos\theta$.

$$N(\cos\theta - 0.22\sin\theta) = 20000$$

$$N(0.22\cos\theta + \sin\theta) = 12448.56$$

Dividing, $\frac{\cos\theta - 0.22\sin\theta}{0.22\cos\theta + \sin\theta} = 1.6066$

$\therefore \frac{1 - 0.22\tan\theta}{0.22 + \tan\theta} = 1.6066$ $\Rightarrow$ $\tan\theta = 0.3539$

$\therefore \theta = 19^\circ 29'$