NORMANHURST BOYS HIGH SCHOOL

MATHEMATICS EXTENSION 1

2016 HSC Course Assessment Task 4 (Trial Examination)
Friday August 12, 2016

General instructions

- Working time – 2 hours. (plus 5 minutes reading time)
- Write using blue or black pen. Where diagrams are to be sketched, these may be done in pencil.
- Board approved calculators may be used.
- Attempt **all** questions.
- At the conclusion of the examination, bundle the booklets used in the correct order within this paper and hand to examination supervisors.

SECTION I

- Mark your answers on the answer grid provided (on page 11)

SECTION II

- Commence each new question on a new booklet. Write on both sides of the paper.
- All necessary working should be shown in every question. Marks may be deducted for illegible or incomplete working.

BOSTES NUMBER: .... **# BOOKLETS USED:** .....

Class (please ✔)

◯ 12M1 – Mr Sekaran
◯ 12M2 – Mrs Bhamra
◯ 12M3 – Mr Lam
◯ 12M4 – Mr Wall
◯ 12M5 – Mrs Gan

Marker's use only.

QUESTION	1-10	11	12	13	14	Total	%
MARKS	/10	/15	/15	/15	/15	/70	

Section I

10 marks
Attempt Question 1 to 10
Mark your answers on the answer grid provided (labelled as page 11).

Questions **Marks**

1. What is the number of asymptotes on the graph of $y = \dfrac{1}{x^2 - 1}$? **1**

(A) 1 (B) 2 (C) 3 (D) 4

2. A is the point $(-1, 4)$ and B is $(9, -6)$. Which of the following are coordinates of a point that divides AB internally in the ratio $3 : 2$? **1**

(A) $(1, 3)$ (B) $(3, 1)$ (C) $(-2, 5)$ (D) $(5, -2)$

3. The equation of the normal to the parabola $x^2 = 4ay$ at the variable point $P\left(2ap, ap^2\right)$ is given by $x + py = 2ap + ap^3$. **1**

How many different values of p are there such that the normal passes through the focus of the parabola?

(A) 0 (B) 1 (C) 2 (D) 3

4. A curve has parametric equations $x = t - 3$ and $y = t^2 + 2$. **1**

What is the Cartesian equation of this curve?

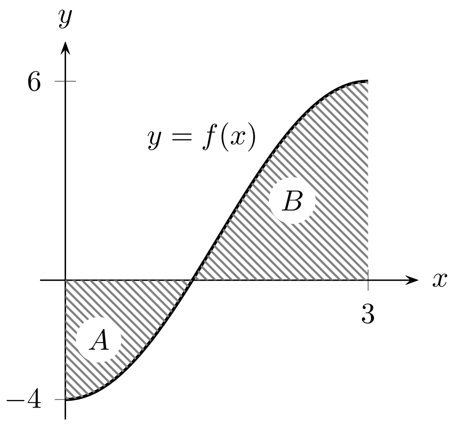
(A) $y = x^2 - x - 1$ (C) $y = x^2 - 6x + 11$

(B) $y = x^2 + x - 1$ (D) $y = x^2 + 6x + 11$

5. What is the value of $\tan^{-1}\left(\tan\frac{5\pi}{4}\right)$? **1**

(A) $\frac{\pi}{4}$ (B) $-\frac{\pi}{4}$ (C) $\frac{5\pi}{4}$ (D) $-\frac{5\pi}{4}$

6. A graph of the function $y = f(x)$ is shown. **1**



Area A is equal to 3 square units, and area B is equal to 7 square units.

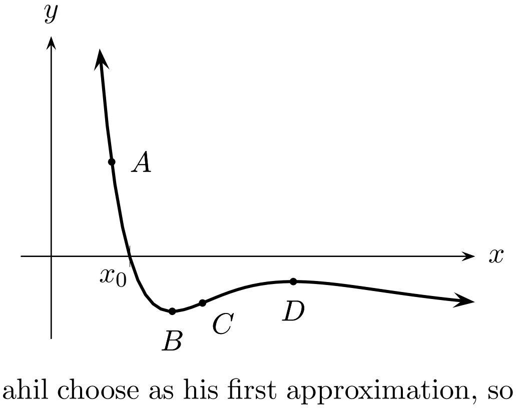
What is the value of $\int_{-4}^{6} f^{-1}(x)\,dx$?

(A) 4 (B) 10 (C) 14 (D) 20

7. What is the largest value of m such that the equation $\sin^{-1} x - mx = 0$ has three unique solutions? **1**

(A) $m = 0$ (B) $m = 1$ (C) $m = \frac{\pi}{2}$ (D) $m = \pi$

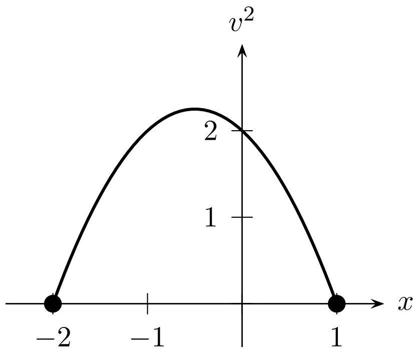
8. $f(x)$ is shown in the following diagram, and $f(x)$ has a root at $x = x_0$. Sahil is trying to find the first approximation to this root, from where a second approximation will be found by Newton's Method. **1**



Which point should Sahil choose as his first approximation, so that his subsequent application of Newton's Method will produce a better approximation?

(A) A (B) B (C) C (D) D

9. A graph of the square of the velocity against displacement is shown. **1**



What type of motion does this graph best describe?

(A) Parabolic

(B) Exponential

(C) Projectile

(D) Simple harmonic

10. Which of the following is *not* equal to $\dfrac{d}{dx}\left(\dfrac{1}{2}v^2\right)$? **1**

(A) $v\dfrac{dv}{dx}$

(B) $\dfrac{dv}{dt}$

(C) $\ddot{x}$

(D) $\dot{x}$

Examination continues overleaf. . .

Section II

60 marks
Attempt Questions 11 to 14
Allow approximately 1 hour and 45 minutes for this section.

Write your answers in the writing booklets supplied. Additional writing booklets are available. Your responses should include relevant mathematical reasoning and/or calculations.

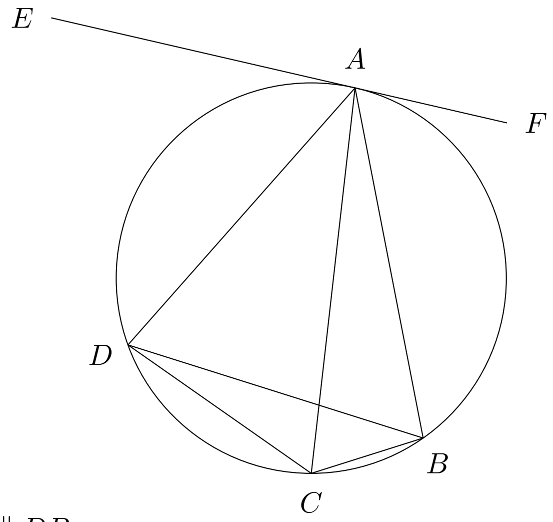
Question 11 (15 Marks) Commence a NEW booklet. **Marks**

(a) Find the acute angle between the lines $2x - y = 0$ and $x - 2y = 0$, giving the answer correct to the nearest degree. **2**

(b) Solve the inequality $\dfrac{x^2 - 4}{x} \geq 0$. **3**

(c) Find the roots of the equation $x^3 + 6x^2 - x - 30 = 0$ given one of the roots of the equation is the sum of the other roots. **3**

(d) $ABCD$ is a cyclic quadrilateral. EF is a tangent at A to the circle. CA bisects $\angle BCD$. **3**

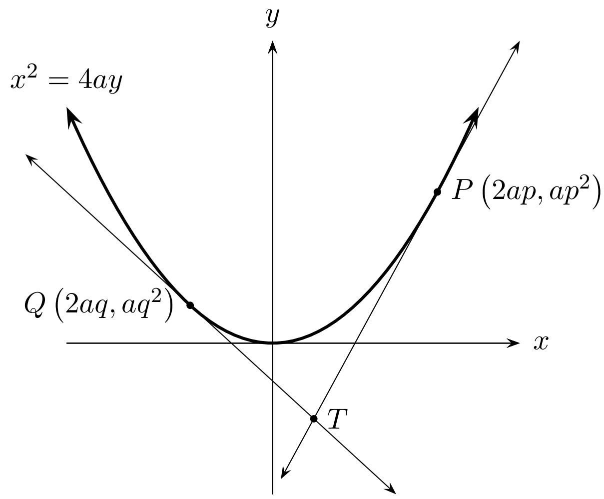


Show that $EF \parallel DB$.

Question 11 continued overleaf...

Question 11 continued from previous page...

(e) The diagram shows the parabola $x^2 = 4ay$. The points $P\left(2ap, ap^2\right)$ and $Q\left(2aq, aq^2\right)$ lie on the parabola. The tangents at P and Q intersect at the point T.



i. By referring to the Reference Sheet or otherwise, show that T has coordinates $\Big(a(p+q), apq\Big)$. **2**

ii. The points P and Q move on the parabola such that the gradient of the chord PQ is always equal to $\dfrac{a}{2}$. **2**

Show that the locus of T is parallel to the axis of the parabola.

Examination continues overleaf...

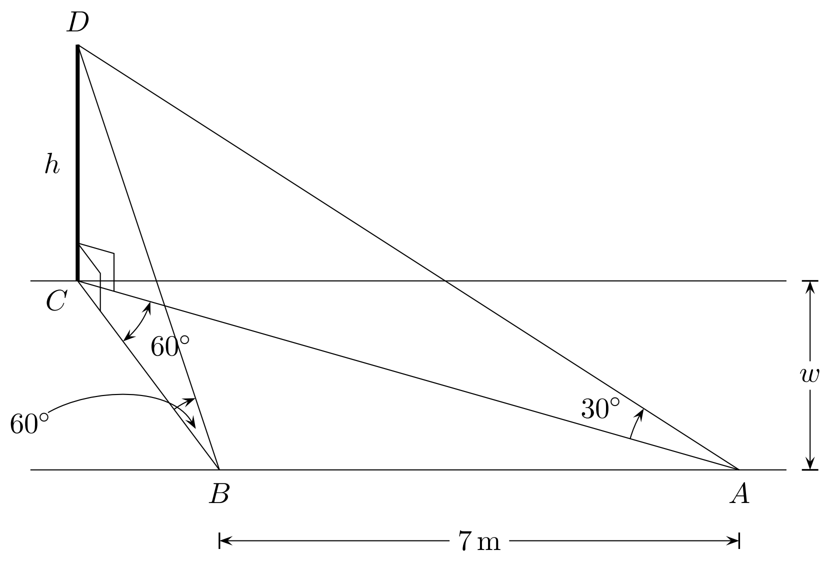
Question 12 (15 Marks) Commence a NEW booklet. **Marks**

(a) Prove by induction that: **3**

$$\frac{2\times 1}{2\times 3}+\frac{2^2\times 2}{3\times 4}+\frac{2^3\times 3}{4\times 5}+\cdots+\frac{2^n\times n}{(n+1)(n+2)}=\frac{2^{n+1}}{n+2}-1$$

for all integers $n \geq 1$.

(b) A footpath on horizontal ground has two parallel edges. CD is a vertical street sign pole of height h metres which stands with its base C on one edge of the footpath. A and B are two points on the other edge of the footpath such that $AB = 7\,\text{m}$ and $\angle ACB = 60°$. From A and B, the angles of elevation to the top of the pole at D are $30°$ and $60°$ respectively.



i. Show that the exact height of the flagpole is $h = \sqrt{21}$ metres. **3**

ii. By considering the area of $\triangle ABC$, find the exact width w of the footpath. **3**

(c) i. Find the derivative of $f(x) = \tan\left(x^2\right)$. **2**

ii. Hence or otherwise, evaluate $\displaystyle\int_{-a}^{a} x\sec^2\left(x^2\right)\,dx$. **2**

(d) Find the simplest expression for $\sin\Big(\cos^{-1}(x-1)\Big)$ in terms of x only. **2**

Examination continues overleaf...

Question 13 (15 Marks) Commence a NEW booklet. **Marks**

(a) Evaluate $\displaystyle\int \frac{1}{\sqrt{9-4x^2}}\,dx$. **2**

(b) Use the substitution $u = 3 + e^x$ to find the exact value of **3**

$$\int_0^{\ln 5} \frac{e^x}{\sqrt{3+e^x}}\,dx$$

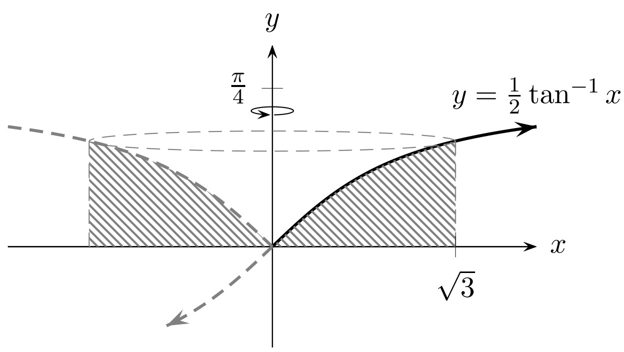
(c) i. Differentiate $y = \tan^{-1}(e^x)$. **2**

ii. Find the equation of the tangent to $y = \tan^{-1}(e^x)$ at $x = 0$. **2**

iii. Discuss the behaviour of the curve as $x \to \infty$. **1**

(d) i. Evaluate $\displaystyle\int \tan^2 2x\,dx$. **2**

ii. Find the exact volume generated when the area between curve $y = \dfrac{1}{2}\tan^{-1} x$, the x axis between $x = 0$ and $x = \sqrt{3}$ is rotated about the y axis. **3**

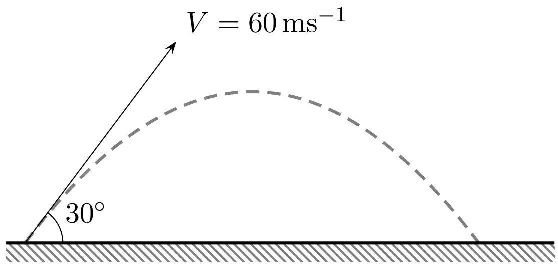


(Extension 2 students: Do NOT use volumes by slicing or cylindrical shells)

Examination continues overleaf...

Question 14 (15 Marks) Commence a NEW booklet. **Marks**

(a) A projectile is launched from the origin, across a level plain at $30°$ to the horizontal and at an initial speed of $60\,\text{ms}^{-1}$. Use $g = 10\,\text{ms}^{-2}$.



The displacement equations of motion are:

$$\begin{cases} x = 30\sqrt{3}t \\ y = 30t - 5t^2 \end{cases} \qquad \text{(Do NOT prove these)}$$

i. Find the maximum height of the particle. **2**

ii. Find the velocity of the particle one second after launch. **2**

(b) A particle is moving in simple harmonic motion with

$$\ddot{x} = -4x + 4$$

Initially, the particle is at the origin and is moving away from the origin with a speed of $2\sqrt{3}$ ms^{-1}.

i. Use integration to show that $v^2 = -4(x-3)(x+1)$. **2**

ii. Find the centre and amplitude of the motion. **2**

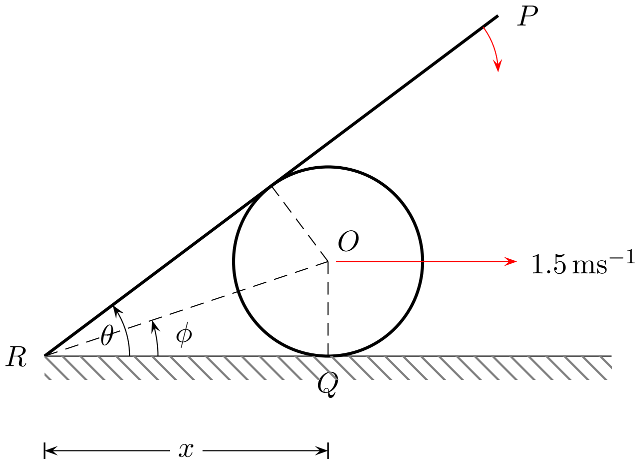
iii. Find the displacement-time equation for this particle. **3**

Question 14 continued overleaf. . .

Question 14 continued from previous page...

(c) A rod RP is leaning on a disc of radius 1 m as shown in the diagram. The centre of the disc is labelled O.

One end of the rod R, is fixed to the ground, whilst the disc is tangential to the ground at Q.



The disc rolls to the right with a constant speed of $1.5\,\text{ms}^{-1}$. As the disc rolls, the other end of the rod P commences its descent towards the ground.

Let $\angle ORQ = \phi$ and $\angle PRQ = \theta$ and $RQ = x$.

i. Write a relationship between $\angle ORQ$, OQ and RQ. **1**

ii. Hence or otherwise, find the rate of change of θ in radians per second, when $x = 6\,\text{m}$. **3**

End of paper.

Answer sheet for Section I

Mark answers to Section I by fully blackening the correct circle, e.g "●"

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2 – Ⓐ Ⓑ Ⓒ Ⓓ
3 – Ⓐ Ⓑ Ⓒ Ⓓ
4 – Ⓐ Ⓑ Ⓒ Ⓓ
5 – Ⓐ Ⓑ Ⓒ Ⓓ
6 – Ⓐ Ⓑ Ⓒ Ⓓ
7 – Ⓐ Ⓑ Ⓒ Ⓓ
8 – Ⓐ Ⓑ Ⓒ Ⓓ
9 – Ⓐ Ⓑ Ⓒ Ⓓ
10 – Ⓐ Ⓑ Ⓒ Ⓓ

2016 Mathematics Extension 1 HSC Course Trial Examination STUDENT SELF REFLECTION

1. In hindsight, did I do the best I can? Why or why not?

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2. Which topics did I need more help with, and what parts specifically?

- Q1-4, 8, 11 - Miscellaneous topics, Polynomials, Circle Geometry, Parametric Representation

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- Q5, 7, 12 - Induction, 3D Trigonometry, Inverse Trigonometry

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- Q6, 13 - Integration (of inverse trig/subsitution/volumes), Curve Sketching.

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- Q9-10, 14 - Further applications of calculus to the physical world.

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3. What other parts from the feedback session can I take away to refine my solutions for future reference?

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Sample Band 6 Responses

Section I

1. (C) **2.** (D) **3.** (B) **4.** (D) **5.** (A)
6. (C) **7.** (C) **8.** (A) **9.** (D) **10.** (D)

Section II

Question 11 (Wall)

(a) (2 marks)

$$2x - y = 0 \qquad x - 2y = 0$$
$$y = 2x \qquad y = \frac{1}{2}x$$
$$\therefore m_1 = 2 \quad m_2 = \frac{1}{2}$$

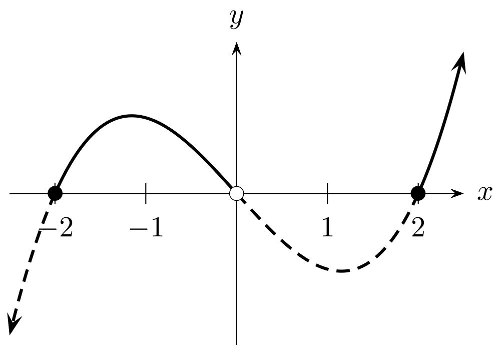
Apply angle between two lines formula,

$$\tan\theta = \left|\frac{m_1 - m_2}{1 + m_1 m_2}\right| = \left|\frac{2 - \frac{1}{2}}{1 + (2)\left(\frac{1}{2}\right)}\right|$$
$$= \left|\frac{\frac{3}{2}}{2}\right| = \frac{3}{4}$$
$$\therefore \theta = 36.869^\circ \cdots \approx 37^\circ$$

(b) (3 marks)

- ✓ [1] for multiplying by the square of the denominator.
- ✓ [1] for significant progress towards solution.
- ✓ [1] for removing $x = 0$ as a possible solution.

$$\underset{\times x^2}{\frac{x^2 - 4}{x}} \geq \underset{\times x^2}{0}$$
$$x(x^2 - 4) \geq 0$$
$$x(x - 2)(x + 2) \geq 0$$



$$\therefore -2 \leq x < 0 \text{ or } x \geq 2$$

(c) (3 marks)

- ✓ [1] for correct usage of elementary symmetric functions.
- ✓ [1] for significant progress towards solution.
- ✓ [1] for correct final solutions.

$$x^3 + 6x^2 - x - 30 = 0$$

Roots are α, β and $\alpha + \beta$

- Sum of roots:

$$\alpha + \beta + (\alpha + \beta) = -\frac{6}{1} = -6$$
$$\therefore 2(\alpha + \beta) = -6$$
$$\alpha + \beta = -3$$

- Product:

$$\alpha\beta(\alpha + \beta) = -\frac{d}{a} = 30$$
$$\alpha\beta(-3) = 30$$
$$\therefore \alpha\beta = -10$$

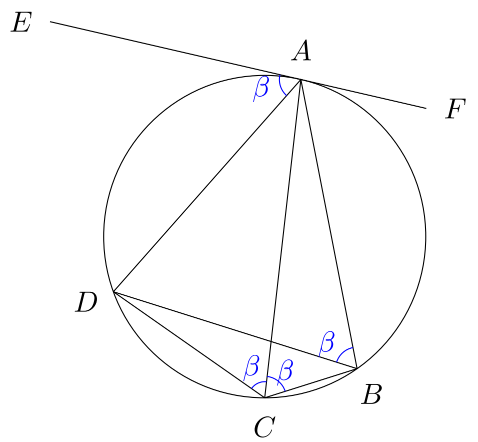
Form a quadratic with sum of roots -3 and product of roots -10:

$$x^2 - (\alpha + \beta)x + \alpha\beta = 0$$
$$x^2 + 3x - 10 = 0$$
$$(x + 5)(x - 2) = 0$$
$$\therefore x = 2, -5$$

Hence roots are $x = 2, -5, -3$.

(d) (3 marks)

- ✓ [1] for appropriate usage of ∠ in the alternate segment.
- ✓ [1] for appropriate usage of ∠ in the same segment.
- ✓ [1] for final justification.

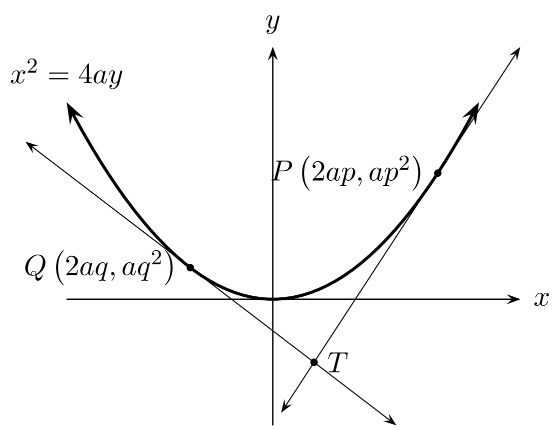


To prove $EF \parallel DB$: let $\angle BAF = \alpha$ and $\angle BCA = \angle ACD = \beta$.

- Let $\angle ACD = \beta$.
  Then $\angle ACB = \beta$
  (CA bisects $\angle BCD$, given)
- $\angle ABD = \angle ACD = \beta$
  (∠ in the same segment, on chord AD).
- $\angle FAB = \angle ACB = \beta$
  (∠ in the alternate segment)
- $\therefore EF \parallel DB$
  (alternate angles $\angle FAB = \angle ABD$)

(e) i. (2 marks)

- ✓ [1] for adequate steps to show $x = a(p+q)$.
- ✓ [1] for adequate steps to show $y = apq$.



Solving simultaneously for equation of tangent at P and Q

$$\begin{cases} y = px - ap^2 \\ y = qx - aq^2 \end{cases}$$

$$px - ap^2 = qx - aq^2$$
$$px - qx = ap^2 - aq^2$$
$$x\cancel{(p-q)} = a\cancel{(p-q)}(p+q)$$
$$\therefore x = a(p+q)$$

Subtitute into $y = px - ap^2$:

$$\begin{aligned} y &= p(a)(p+q) - ap^2 \\ &= ap^2 + apq - ap^2 \\ &= apq \end{aligned}$$

ii. (2 marks)

- ✓ [1] for finding relationship between $p+q$ and a.
- ✓ [1] for correct justification of locus.

$$m_{PQ} = \frac{a}{2}$$

Applying gradient formula with $m = \frac{a}{2}$ between $P(2ap, ap^2)$ and $Q(2aq, aq^2)$:

$$\frac{aq^2 - ap^2}{2aq - 2ap} = \frac{a}{2}$$
$$\frac{\cancel{a}\cancel{(q-p)}(q+p)}{2\cancel{a}\cancel{(q-p)}} = \frac{a}{2}$$
$$\therefore p + q = a$$

Substituting into x_T:

$$x_T = a(p+q) = a^2$$

As the x coordinate at T is a constant, the locus of T is a straight line at $x = a^2$.

Question 12 (Lam)

(a) (3 marks)

- ✓ [1] for testing base case.
- ✓ [1] for correct use of the inductive hypothesis in the testing.
- ✓ [1] for final answer.

Let $P(n)$ be the proposition

$$\frac{2\times 1}{2\times 3}+\frac{2^2\times 2}{3\times 4}+\frac{2^3\times 3}{4\times 5}+\cdots+\frac{2^n\times n}{(n+1)(n+2)}=\frac{2^{n+1}}{n+2}-1$$

for all integers $n\geq 1$.

- Base case: $P(1)$

Left: $\dfrac{2\times 1}{2\times 3}=\dfrac{2}{6}=\dfrac{1}{3}$

Right: $\dfrac{2^{1+1}}{1+2}-1=\dfrac{2^2}{3}-1=\dfrac{4}{3}-\dfrac{3}{3}=\dfrac{1}{3}$

Hence $P(1)$ is true.

- Inductive hypothesis: assume $P(k)$ is true for $k\in\mathbb{Z}^+$, i.e. $P(k)$ is

$$\frac{2\times 1}{2\times 3}+\frac{2^2\times 2}{3\times 4}+\frac{2^3\times 3}{4\times 5}+\cdots+\frac{2^k\times k}{(k+1)(k+2)}=\frac{2^{k+1}}{k+2}-1$$

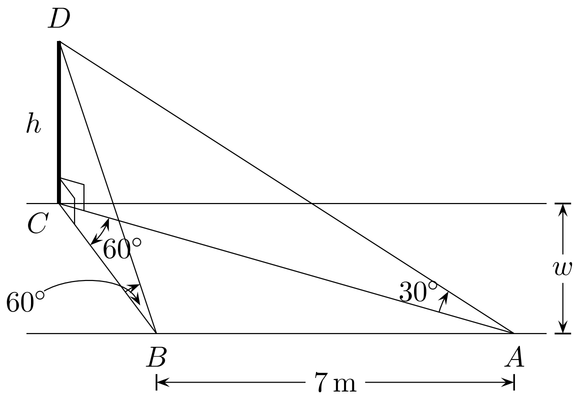
- Examine $P(k+1)$:

$$\begin{aligned}&\frac{2\times 1}{2\times 3}+\frac{2^2\times 2}{3\times 4}+\frac{2^3\times 3}{4\times 5}+\cdots+\frac{2^k\times k}{(k+1)(k+2)}+\frac{2^{k+1}(k+1)}{(k+2)(k+3)}\\
&=\frac{2^{k+1}}{k+2}\frac{\times(k+3)}{\times(k+3)}-1+\frac{2^{k+1}(k+1)}{(k+2)(k+3)}\\
&=\frac{2^{k+1}(k+3)+2^{k+1}(k+1)}{(k+2)(k+3)}-1\\
&=\frac{2^{k+1}(2k+4)}{(k+2)(k+3)}-1\\
&=\frac{2\times 2^{k+1}\cancel{(k+2)}}{\cancel{(k+2)}(k+3)}-1\\
&=\frac{2^{(k+1)+1}}{(k+1)+2}-1\end{aligned}$$

$\therefore P(k+1)$ is also true, and $P(n)$ is true by induction.

(b) i. (3 marks)

- ✓ [1] for both CB and CA in terms of h.
- ✓ [1] for correct usage of cosine rule in $\triangle ABC$.
- ✓ [1] for demonstrating $h=\sqrt{21}$.



- In $\triangle DCB$,

$$\frac{h}{CB}=\tan 60^\circ=\sqrt{3}$$
$$\therefore CB=\frac{h}{\sqrt{3}}$$

- In $\triangle DCA$,

$$\frac{h}{CA} = \tan 30^\circ = \frac{1}{\sqrt{3}}$$
$$\therefore CA = h\sqrt{3}$$

- Applying cosine rule in $\triangle ABC$:

$$AB^2 = BC^2 + AC^2 - 2(BC)(AC)\cos 60^\circ$$
$$7^2 = \frac{h^2}{3} + 3h^2 - 2h^2 \times \frac{1}{2}$$
$$\therefore 49 = \frac{7}{3}h^2$$
$$h^2 = 21$$
$$h = \sqrt{21}$$

ii. (3 marks)

✓ [1] for area by sine rule

✓ [1] for area by $\frac{1}{2} \times h_\perp \times$ base

✓ [1] for final value of w.

- Finding the area by base and perpendicular height, where the perpendicular height is denoted w:

$$A = \frac{1}{2} \times 7 \times w$$

- Finding the area by the sine rule:

$$CB = \frac{\sqrt{21}}{\sqrt{3}} = \sqrt{7}$$
$$CA = \sqrt{21}\sqrt{3} = 3\sqrt{7}$$
$$A = \frac{1}{2}ab\sin C$$
$$= \frac{1}{2}(BC)(AC)\sin 60^\circ$$
$$= \frac{1}{2}\left(\sqrt{7}\right)\left(3\sqrt{7}\right)\frac{\sqrt{3}}{2}$$

Equating areas,

$$\frac{7}{2}w = \frac{1}{2} \times \left(\sqrt{7}\right)\left(3\sqrt{7}\right)\frac{\sqrt{3}}{2}$$
$$\frac{7}{2}w = \frac{7}{4} \times 3\sqrt{3}$$
$$w = \frac{3\sqrt{3}}{2} \text{ m}$$

(c) i. (2 marks)

$$f(x) = \tan\left(x^2\right)$$

Applying chain rule,

$$f'(x) = \sec^2\left(x^2\right) \times 2x$$
$$= 2x\sec^2\left(x^2\right)$$

ii. (2 marks)

$$\int_{-a}^{a} x\sec^2\left(x^2\right)\,dx = 0$$

($y = x$ is odd, $y = \sec^2\left(x^2\right)$ is even, , i.e. odd $\times$ even = odd. Integrating an odd function over a balanced interval $-a < x < a$ produces zero.)

Alternatively,

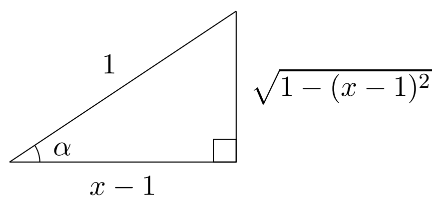
$$\int_{-a}^{a} x\sec^2\left(x^2\right)\,dx = \frac{1}{2}\left[\tan\left(x^2\right)\right]_{-a}^{a}$$
$$= \frac{1}{2}\left(\tan a^2 - \tan a^2\right)$$
$$= 0$$

(d) (2 marks)

✓ [1] for relationship between $\cos\alpha$ and x.

✓ [1] for simplest expression.

Let $\alpha = \cos^{-1}(x-1)$. Hence $\cos\alpha = x-1$. Draw a right $\angle$ triangle with adjacent side to α of length $x-1$ and unit hypotenuse.



$$\therefore \sin\left(\cos^{-1}(x-1)\right) = \sin\alpha$$
$$= \sqrt{1-(x-1)^2}$$
$$= \sqrt{(1-(x-1))(1+(x-1)}$$
$$= \sqrt{(2-x)x}$$

Question 13 (Lawson)

(a) (2 marks)

$$\int \frac{1}{\sqrt{9-4x^2}}\,dx = \int \frac{1}{\sqrt{4\left(\frac{9}{4}-x^2\right)}}\,dx$$
$$= \frac{1}{2}\sin^{-1}\left(\frac{2x}{3}\right) + C$$

(b) (3 marks)

- ✓ [1] for correct relationship between differentials.
- ✓ [1] for correct primitive.
- ✓ [1] for final answer.

$$\int_0^{\ln 5} \frac{e^x}{\sqrt{3+e^x}}\,dx$$

With substitution $u = 3 + e^x$:

$$\frac{du}{dx} = e^x$$
$$\therefore du = e^x\,dx$$
$$x = 0 \quad u = 3 + e^0 = 4$$
$$x = \ln 5 \quad u = 3 + e^{\ln 5} = 8$$
$$\int_0^{\ln 5} \frac{e^x}{\sqrt{3+e^x}}\,dx = \int_{u=4}^{u=8} \frac{du}{\sqrt{u}}$$
$$= \int_4^8 u^{-\frac{1}{2}}\,du$$
$$= \left[2u^{\frac{1}{2}}\right]_4^8$$
$$= 2\left(\sqrt{8}-\sqrt{4}\right)$$
$$= 2\left(2\sqrt{2}-2\right)$$
$$= 4\left(\sqrt{2}-1\right)$$

(c) i. (2 marks)

$$y = \tan^{-1}(e^x)$$
$$\frac{dy}{dx} = \frac{1}{1+(e^x)^2} \times e^x = \frac{e^x}{1+e^{2x}}$$

ii. (2 marks)

- ✓ [−1] for each newly introduced error.

At $x = 0$

$$y = \tan^{-1} e^0 = \tan^{-1} 1$$
$$= \frac{\pi}{4}$$

Finding the gradient at $x = 0$:

$$\frac{dy}{dx} = \frac{e^0}{1+e^0} = \frac{1}{2}$$

Applying point-gradient formula,

$$\frac{y-\frac{\pi}{4}}{x-0} = \frac{1}{2}$$
$$y - \frac{\pi}{4} = \frac{1}{2}x$$
$$\therefore y = \frac{1}{2}x + \frac{\pi}{4}$$

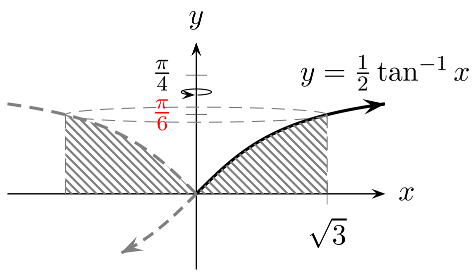
iii. (1 mark) As $x \to \infty$, $e^x \to \infty$. Hence $y = \tan^{-1}(e^x) \to \frac{\pi}{2}$.

(d) i. (2 marks)

$$\int \tan^2 2x\,dx = \int \left(\sec^2 2x - 1\right)\,dx$$
$$= \frac{1}{2}\tan 2x - x + C$$

ii. (3 marks)

- ✓ [1] for changing subject to x, and limits of integration along y axis.
- ✓ [1] for correct primitive
- ✓ [1] for final answer.



When $x = \sqrt{3}$, $y = \frac{1}{2}\tan^{-1}\sqrt{3} = \frac{\pi}{6}$. Changing subject from x to y:

$$y = \frac{1}{2}\tan^{-1} x$$
$$2y = \tan^{-1} x$$
$$\therefore x = \tan 2y$$

Volume by rotating along y axis:

$$\begin{aligned}V_1 &= \pi \int_a^b x^2\,dy\\ &= \pi \int_0^{\frac{\pi}{6}} \tan^2 2y\,dy\\ &= \pi \left[\frac{1}{2}\tan 2y - y\right]_0^{\frac{\pi}{6}}\\ &= \pi\left(\frac{1}{2}\tan\frac{\pi}{3} - \frac{\pi}{6}\right)\\ &= \pi\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)\end{aligned}$$

Finding the volume of the cylinder,

$$\begin{aligned}V_{\text{cylinder}} &= \pi r^2 h\\ &= \pi\left(\sqrt{3}\right)^2 \times \frac{\pi}{6}\\ &= \frac{\pi^2}{2}\end{aligned}$$

Volume generated (shaded):

$$\begin{aligned}V &= V_{\text{cylinder}} - V_1\\ &= \frac{\pi^2}{2} - \left(\pi\left(\frac{\sqrt{3}}{2} - \frac{\pi}{6}\right)\right)\\ &= \frac{4\pi^2}{6} - \frac{\pi\sqrt{3}}{2}\\ &= \frac{2\pi^2}{3} - \frac{\pi\sqrt{3}}{2}\end{aligned}$$

Question 14 (Gan)

(a) i. (2 marks)

✓ [1] for identifying parameters for maximum height.

✓ [1] for finding value of maximum height.

Maximum height occurs when $\dot{y} = 0$:

$$\begin{aligned}y &= 30t - 5t^2\\ \dot{y} &= 30 - 10t = 0\\ \therefore 10t &= 30\\ t &= 3\end{aligned}$$

Substitute into y:

$$\begin{aligned}y &= 30(3) - 5\left(3^2\right)\\ &= 90 - 45\\ &= 45\,\text{m}\end{aligned}$$

ii. (2 marks)

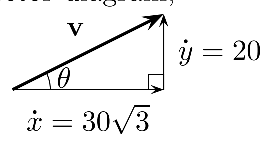
✓ [1] for correct vector diagram

✓ [1] for magnitude and direction of velocity.

When $t = 1$:

$$\begin{aligned}\dot{x} &= 30\sqrt{3}\\ \dot{y} &= 30 - 10t\Big|_{t=1}\\ &= 20\end{aligned}$$

Drawing vector diagram,



$$\begin{aligned}v^2 &= (20)^2 + \left(30\sqrt{3}\right)^2\\ &= 400 + 2\,700\\ &= 3\,100\\ &= 10\sqrt{31}\ \text{ms}^{-1}\end{aligned}$$

Finding the angle at $t = 1$

$$\begin{aligned}\tan\theta &= \frac{20}{30\sqrt{3}}\\ \theta &= 21^\circ 3'\end{aligned}$$

(b) i. (2 marks)

✓ [1] for integrating to obtain $\frac{1}{2}v^2$

✓ [1] for showing required result.

$$\ddot{x} = -4x + 4$$

Integrating,

$$\begin{aligned}\frac{1}{2}v^2 &= \int(-4x + 4)\,dx\\ &= -2x^2 + 4x + C_1\\ v^2 &= -4x^2 + 8x + C_2\end{aligned}$$

When $t = 0$, $v^2 = 4 \times 3 = 12$, $x = 0$:

$$\begin{aligned}12 &= -4(0) + 8(0) + C_2\\ \therefore C_2 &= 12\\ \therefore v^2 &= -4x^2 + 8x + 12\\ &= -4\left(x^2 - 2x - 3\right)\\ &= -4(x-3)(x+1)\end{aligned}$$

ii. (2 marks)

- Extremities of motion at $x = 3$ and $x = -1$.
- Centre of motion at $x = \frac{3-1}{2} = 1$
- Amplitude $a = 2$

iii. (3 marks)

✓ [1] for obtaining completed square form

✓ [1] for obtaining an inverse trigonometric primitive

✓ [1] for correct equation

Completing the square,

$$\begin{aligned} v^2 &= -4\left(x^2 - 2x + 1 - 4\right) \\ &= -4\left((x-1)^2 - 4\right) \\ &= 4\left(4 - (x-1)^2\right) \\ \therefore v &= \pm 2\sqrt{4-(x-1)^2} \end{aligned}$$

As v is non determinant due to initial conditions, assume positive root and perform separation of variables:

$$\begin{aligned} \frac{dx}{dt} &= 2\sqrt{4-(x-1)^2} \\ \frac{dx}{\sqrt{4-(x-1)^2}} &= 2\,dt \end{aligned}$$

Integrating,

$$\begin{aligned} \int \frac{dx}{\sqrt{4-(x-1)^2}} &= \int 2\,dt \\ \sin^{-1}\left(\frac{x-1}{2}\right) &= 2t + C_3 \end{aligned}$$

When $t = 0$, $x = 0$:

$$\begin{aligned} \sin^{-1}\left(-\frac{1}{2}\right) &= C_3 \\ \therefore C_3 &= -\frac{\pi}{6} \\ \frac{x-1}{2} &= \sin\left(2t - \frac{\pi}{6}\right) \\ x &= 1 + 2\sin\left(2t - \frac{\pi}{6}\right) \end{aligned}$$

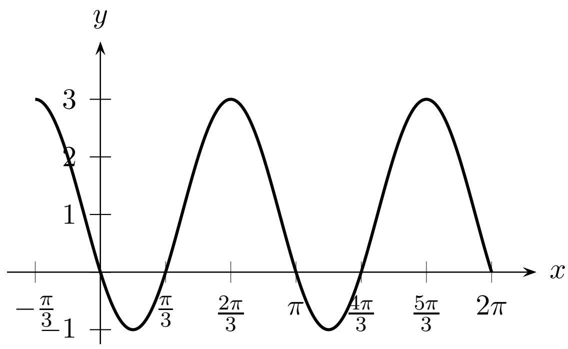
Alternatively, given the initial velocity was not defined in terms of direction, students may have obtained a different phase shift when performing the steps above and assumed a negative root. This results in $x = 2\cos\left(2t - \frac{2\pi}{3}\right) + 1$.

Alternatively, sketch a graph assuming a cosine as the sinusoid with centre of motion at $x = 1$, and use initial conditions $t = 0$, $x = 0$ to find the phase shift:

$$x = 2\cos(2t + \alpha) + 1$$

When $t = 0$, $x = 0$:

$$\begin{aligned} 0 &= 2\cos\alpha + 1 \\ \cos\alpha &= -\frac{1}{2} \\ \alpha &= \frac{2\pi}{3} \\ x &= 2\cos\left(2t + \frac{2\pi}{3}\right) + 1 \end{aligned}$$



(c) i. (1 mark)

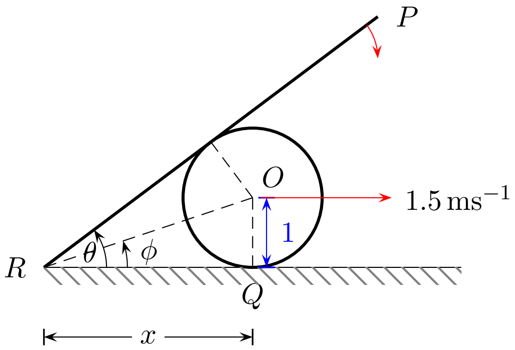
- $OQ = 1$
- $\tan\phi = \frac{1}{x}$

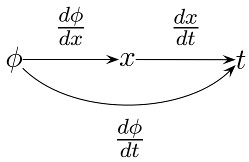
ii. (3 marks)

✓ [1] correct derivative $\frac{d\phi}{dx}$.

✓ [1] for finding $\frac{d\phi}{dx} = -\frac{1}{37}$.

✓ [1] for finding $\frac{d\theta}{dt}$.





Applying chain rule,

$$\frac{d\phi}{dt} = \frac{d\phi}{dx} \times \frac{dx}{dt}$$

Given $\tan\phi = \frac{1}{x}$, rearranging to find $\dfrac{d\phi}{dx}$:

$$\phi = \tan^{-1}\left(x^{-1}\right)$$

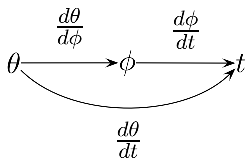
$$\frac{d\phi}{dx} = \frac{1}{1+\left(x^{-1}\right)^2} \times -x^{-2}$$

$$= \frac{1}{1+\frac{1}{x^2}} \times \frac{-1}{x^2} = -\frac{1}{x^2+1}\bigg|_{x=6}$$

$$= -\frac{1}{37}$$

Evaluating $\dfrac{d\phi}{dt}$:

$$\frac{d\phi}{dt} = -\frac{1}{37} \times \frac{3}{2} = -\frac{3}{2\times 37}$$



Using the $\theta - \phi$ chain,

$$\theta = 2\phi$$

$$\therefore \frac{d\theta}{d\phi} = 2$$

$$\frac{d\theta}{dt} = \frac{d\theta}{d\phi} \times \frac{d\theta}{dt}$$

$$= 2 \times -\frac{3}{2\times 37}$$

$$= -\frac{3}{37} \text{ rad/s}$$