NORTH SYDNEY GIRLS HIGH SCHOOL

2016 TRIAL HSC EXAMINATION

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black pen
- Board approved calculators may be used
- A reference sheet has been provided
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 2 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 7 – 15

90 Marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

NAME:_____________________ **TEACHER:**_____________________

STUDENT NUMBER:_____________________

Question	1-10	11	12	13	14	15	16	Total
Mark	/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Questions 1−10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following is a focus of $\dfrac{x^2}{11}-\dfrac{y^2}{25}=1$?

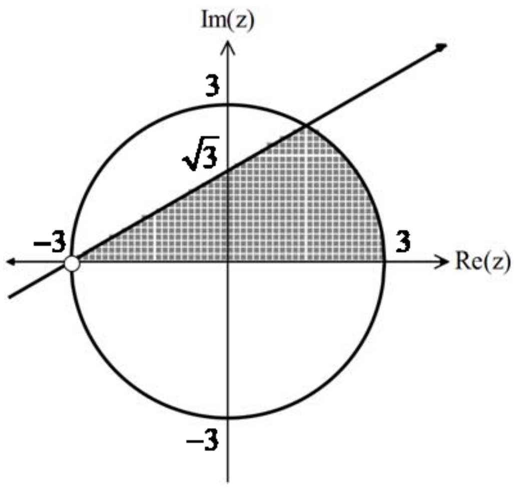
(A) $(0,5)$

(B) $(5,0)$

(C) $(0,6)$

(D) $(6,0)$

2 Which of the following describes the locus shown below?



(A) $|z|\le 3$ and $0\le \arg(z)\le \dfrac{\pi}{6}$

(B) $|z|\le 3$ and $0\le \arg(z)\le \dfrac{\pi}{3}$

(C) $|z|\le 3$ and $0\le \arg(z+3)\le \dfrac{\pi}{6}$

(D) $|z|\le 3$ and $0\le \arg(z+3)\le \dfrac{\pi}{3}$

3 In which quadrant is the complex number $(-3+3i)^3$ located on the Argand plane?

(A) the first quadrant

(B) the second quadrant

(C) the third quadrant

(D) the fourth quadrant

4 z_1 and z_2 are complex numbers such that $z_1 + z_2$ lies on the real axis. Which of the following statements can be inferred?

(A) $z_1 = \overline{z_2}$

(B) $\arg(z_1) = -\arg(z_2)$

(C) $|\mathrm{Im}\, z_1| = |\mathrm{Im}\, z_2|$

(D) $|z_1| = |z_2|$

5 Consider the equation $z^3 - 2z^2 + bz + c = 0$, where b and c are real numbers. If one of the roots of the equation is $2 - i$, what is the value of b?

(A) -3

(B) -19

(C) 3

(D) 19

6 Which of the following correctly describes some of the features of the graph of

$$y = \frac{x^2 - 4x + 3}{x^2 - x - 6}?$$

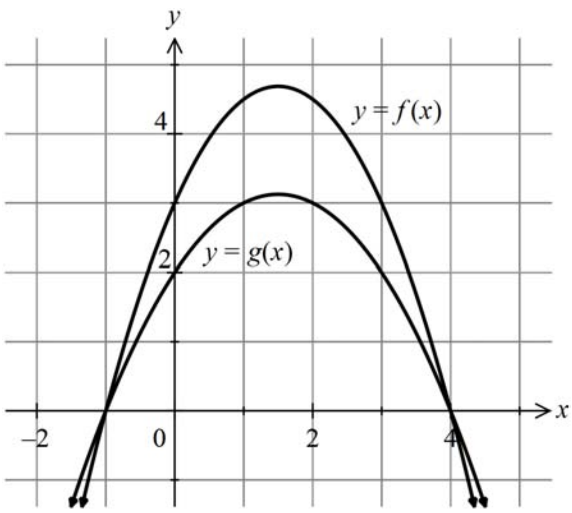
(A) x-intercepts at 3 and 1 and horizontal asymptote at $y = 1$

(B) Vertical asymptotes at $x = 3$ and $x = -2$ and horizontal asymptote at $y = 0$

(C) Vertical asymptotes at $x = 3$ and $x = -2$ and horizontal asymptote at $y = 1$

(D) Vertical asymptote at $x = -2$ and horizontal asymptote at $y = 1$

7 Shown below are the graphs of $y = f(x)$ and $y = g(x)$.



Given that $f(g(x)) = 3$, what are all the possible values of x?

(A) $x = 0,\ 3$

(B) $x = 1,\ 2$

(C) $x = -1,\ 4$

(D) $x = -1,\ 1,\ 2,\ 4$

8 Which of the following statements is NOT true?

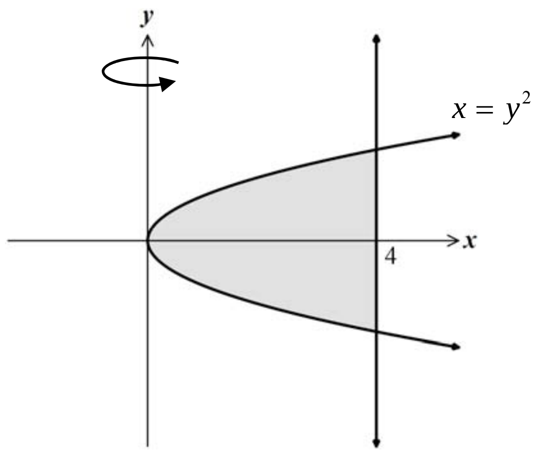
(A) $\int_{-1}^{1}\left(e^{x}-e^{-x}\right)dx=0$

(B) $\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin x\,dx=0$

(C) $\int_{0}^{\pi}\cos 2x\,dx=0$

(D) $\int_{-1}^{1}\tan^{-1}x\,dx=0$

9 The region enclosed by the curve $x=y^2$ and the line $x=4$ is rotated around the y-axis. Which of the following gives an expression for the volume of the resulting solid of revolution?



(A) $V=\pi\int_{-2}^{2}\left(4-y^2\right)^2 dy$

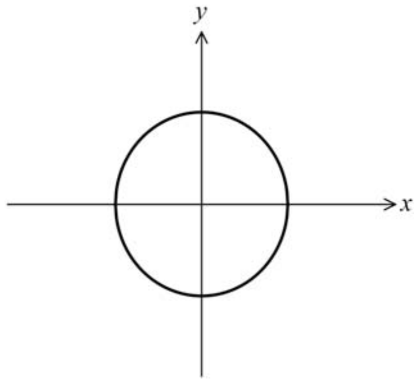
(B) $V=2\pi\int_{0}^{2}\left(16-y^4\right)dy$

(C) $V=\pi\int_{-2}^{2}\left(4-y^2\right)dy$

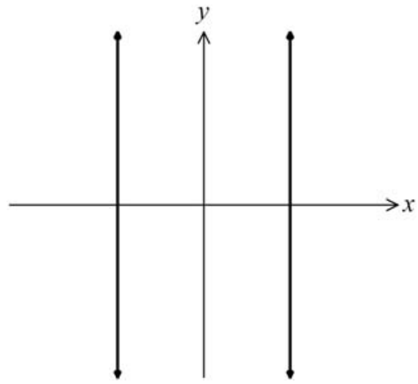
(D) $V=2\pi\int_{-2}^{2}\left(16-y^4\right)dy$

10 Consider the conic $\frac{x^2}{p}+\frac{y^2}{q}=1$ with an eccentricity e, where $p>0$. Which of the following describes the limiting shape of the conic for a given value of p, as $e \to 1^{+}$?

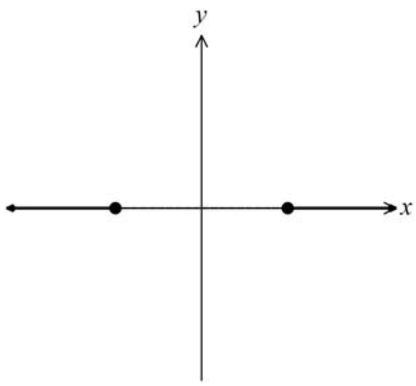
(A)



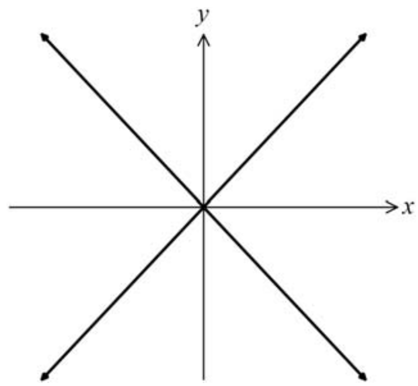
(B)



(C)



(D)



Section II

Total marks − 90
Attempt Questions 11−16
Allow about 2 hour 45 minutes for this section.
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 to 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) If $u = -\sqrt{3} + i$ and $v = 2 + 2i$:

(i) Find $\dfrac{u}{v}$ in the form $x + iy$. **2**

(ii) Write u and v in modulus-argument form. **2**

(iii) Write down the argument of uv. **1**

(b) Sketch the locus of z if $\dfrac{z+2i}{z-2i}$ is purely imaginary. **2**

(c) Consider the curve with the equation $x^2 - xy + y^2 - 21 = 0$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{y-2x}{2y-x}$. **1**

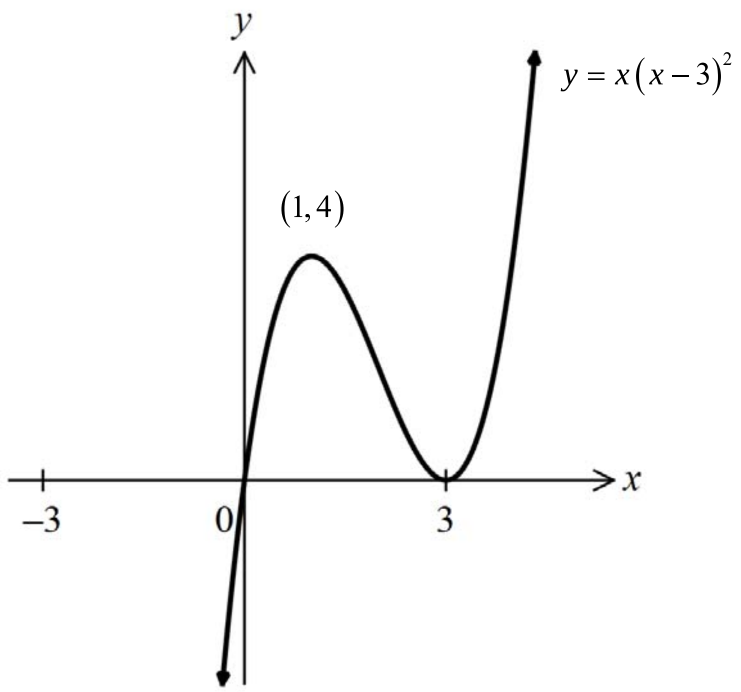
(ii) Find the points on the curve where the tangents are parallel to the line $y = 3x + 1$. **2**

(d) (i) Using partial fraction decomposition, show that $\displaystyle\int_0^1 \frac{10dt}{(1+3t)(3-t)} = \log_e 6$. **3**

(ii) Hence, by using the substitution $t = \tan\dfrac{\theta}{2}$, evaluate $\displaystyle\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta + 3\cos\theta}$. **2**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The graph of the curve $y = x(x-3)^2$ is shown below.



(i) Neatly sketch the graph of the curve $y^2 = x(x-3)^2$ on one-third of a page, clearly showing the coordinates of any turning points. **2**

(ii) Hence, on a new number plane, sketch the graph of $y^2 = |x|(|x|-3)^2$. **1**

(b) The equation $x^3 - 2x^2 + 3x - 1 = 0$ has roots α, β and γ.

(i) Find a polynomial equation whose roots are α^2, β^2 and γ^2. **3**

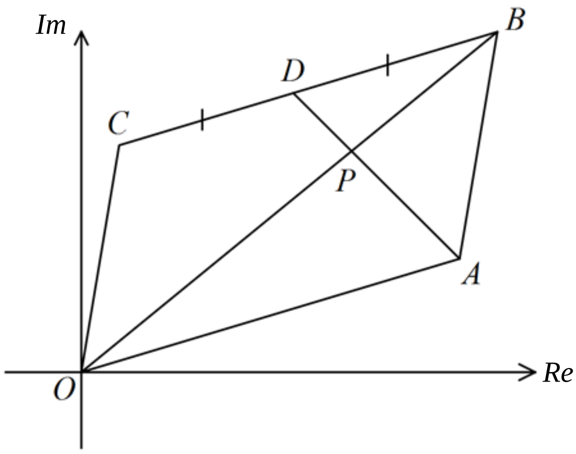
(ii) Hence or otherwise, find a polynomial equation whose roots are $\alpha^2\beta^2$, $\beta^2\gamma^2$ and $\gamma^2\alpha^2$. **2**

(c) Given that the polynomial equation $x^3 - ax^2 + b = 0$ ($a, b \neq 0$) has a non-zero multiple root, show that $4a^3 - 27b = 0$. **3**

Question 12 continues on Page 9

(d) *OABC* is a parallelogram and *D* is the midpoint of *BC*. *OB* and *AD* intersect at *P*.

Let $\overrightarrow{OA}$ represent the complex number z_1 and $\overrightarrow{OC}$ represent the complex number z_2.



(i) Given that $\overrightarrow{AP} = m \times \overrightarrow{AD}$, for some real number *m*, explain why **1**

$\overrightarrow{AP}$ is equal to $m\left(z_2 - \dfrac{z_1}{2}\right)$.

(ii) Given that $\overrightarrow{OP} = n \times \overrightarrow{OB}$, for some real number *n*, find another expression **1**

for $\overrightarrow{AP}$ in terms of n, z_1 and z_2.

(iii) Hence, deduce the values of *m* and *n*. **2**

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Let $P\left(cp, \frac{c}{p}\right)$ be a variable point on the hyperbola $xy = c^2$.

(i) Show that the equation of the tangent to the hyperbola at P is $x + p^2 y = 2cp$. **2**

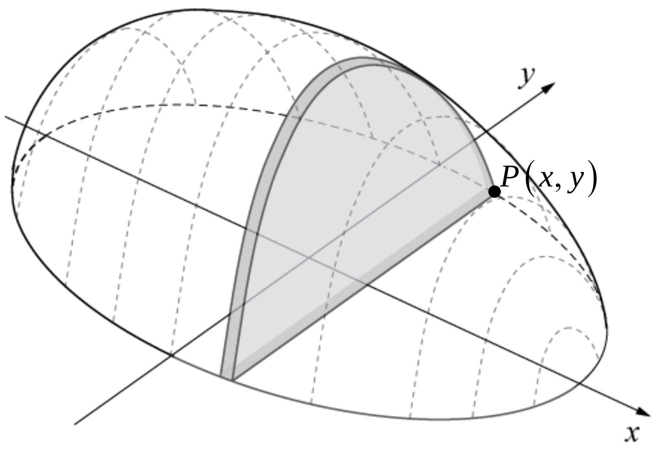
(ii) The tangent meets the x-axis at Q. Show that the equation of the line through Q perpendicular to the tangent at P is $p^2 x - y = 2cp^3$. **2**

(iii) The line through Q in part (ii) cuts the hyperbola at two points R and S. Find the Cartesian equation of the locus of M, the midpoint of RS, noting any restrictions that may apply. **3**

(b) A solid has as its base in the xy plane, the ellipse $\frac{x^2}{16} + \frac{y^2}{4} = 1$.

Cross-sections taken perpendicular to the x-axis and the base are inverted parabolas with latus rectums in the base.

The *latus rectum* is the focal chord of the parabola which is perpendicular to the axis of the parabola.

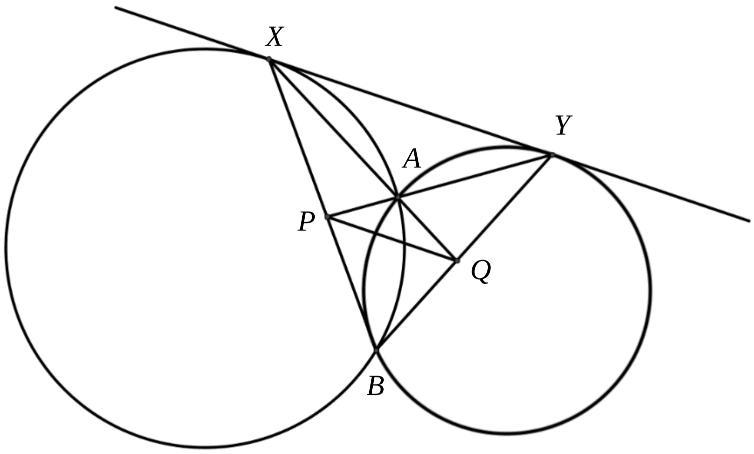


(i) Explain why the height of any cross-sectional slice is equal to $\frac{\sqrt{16 - x^2}}{4}$. **1**

(ii) By first using Simpson's rule to find an expression for the area of the cross-section, find the volume of the solid. **3**

Question 13 continues on Page 11

(c) Two circles meet at the points A and B. The line XY is a common tangent to the two circles as shown. XA produced meets BY at Q and YA produced meets BX at P.



(i) Prove that $PAQB$ is a cyclic quadrilateral. **3**

(ii) Hence, or otherwise, prove that $PQ \parallel XY$. **1**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Show that $\int \frac{dx}{\sqrt{x^2 - a^2}} = \ln\left(x + \sqrt{x^2 - a^2}\right) + C$, where C is a constant and $x > a > 0$. **2**

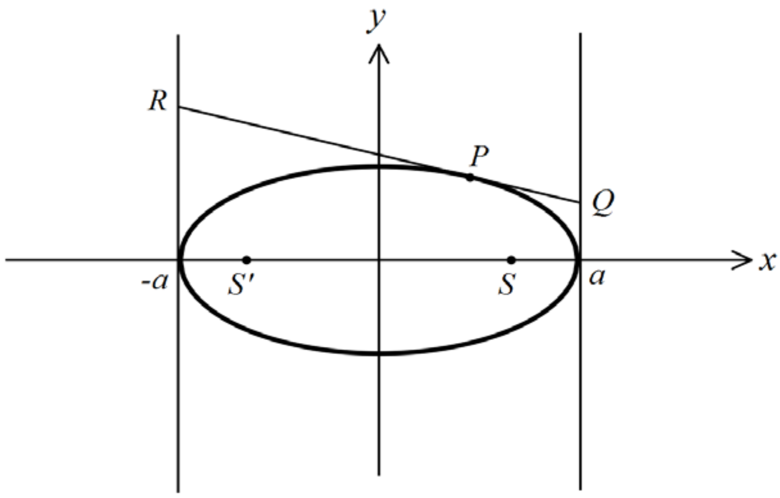
(b) Using the substitution $u = \tan x$, or otherwise, evaluate $\int_0^{\frac{\pi}{3}} \frac{\tan x \, dx}{7\sin^2 x + 5\cos^2 x}$. **3**

(c) (i) If a and b are real numbers, prove that $a^2 + b^2 \geq 2ab$. **1**

(ii) Show that $a^2 + b^2 + c^2 \geq ab + ac + bc$, where a, b and c are real. **1**

(iii) Hence or otherwise, show that the equation $x^3 - 3x^2 + 4x + k = 0$ cannot have three real roots for any real constant k. **2**

(d)



The point $P(x_1, y_1)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$.

(i) Show that the equation of the tangent to the ellipse at P is given by **2**

$$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1.$$

(ii) The tangent intersects the lines $x = a$ and $x = -a$ at Q and R respectively. **3**

Show that QR subtends a right angle at the focus S.

(iii) Hence, explain why the points Q, R, S and S' are concyclic. **1**

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) A number sequence is defined by $u_1 = 0$, $u_2 = 32$, and $u_{n+2} = 6u_{n+1} - 8u_n$ for $n \geq 1$. **3**

Use mathematical induction to prove that $u_n = 4^{n+1} - 2^{n+3}$ for all positive integers n.

(b) You are given that $I_n = \displaystyle\int_1^5 \frac{x^n}{\sqrt{2x-1}}\,dx, \quad n \geq 0$.

(i) Prove that $(2n+1)I_n = n\,I_{n-1} + 3 \times 5^n - 1$, for $n \geq 1$. **3**

(ii) Using the result in part (i), find the exact value of I_1. **1**

Question 15 continues on Page 14

Question 15 (continued)

(c) (i) Let $f(x)$ be a continuous function such that $f(k-x)=f(x)$, **2**

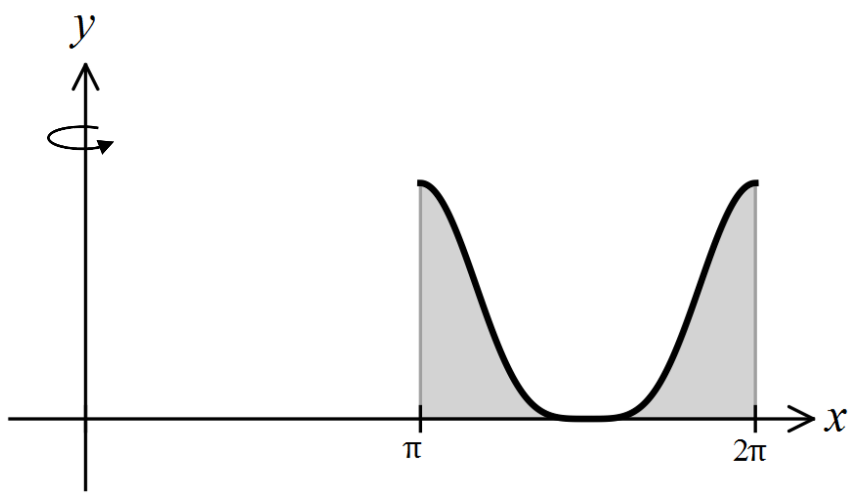
where k is a constant.

Prove that $\int_0^k xf(x)\,dx = \frac{k}{2}\int_0^k f(x)\,dx$.

(ii) If $z=\cos\theta + i\sin\theta$, then $z^n + \frac{1}{z^n} = 2\cos n\theta$. (DO NOT SHOW THIS). **2**

Use this result to show that $\cos^4\theta = \frac{1}{8}(\cos 4\theta + 4\cos 2\theta + 3)$.

(iii) The diagram shows the shaded region bounded by the curve $y=\cos^4 x$ and the x-axis for $\pi \le x \le 2\pi$. **4**



The shaded region is rotated about the y-axis to form a solid of revolution.

Use the method of cylindrical shells and the results in parts (i) and (ii) to find the volume of this solid.

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $x^3 + y^3 + z^3 - 3xyz = (x+y+z)(x^2 + y^2 + z^2 - xy - yz - xz)$ for all real x, y, z. **1**

(ii) Hence or otherwise, if $x + y + z = 1$ prove that $xy + yz + xz - 3xyz \le \frac{1}{4}$ where x, y, z are positive real numbers. **3**

(b) Let $f(x)$ be an odd function for $-a \le x \le a$, where a is a positive constant.

(i) By using a suitable substitution, prove that $\int_0^{2a} f(x-a)\,dx = 0$. **1**

(ii) Show that $\dfrac{\sqrt{3} + \tan\left(x - \frac{\pi}{6}\right)}{\sqrt{3} - \tan\left(x - \frac{\pi}{6}\right)} = \dfrac{1 + \sqrt{3}\tan x}{2}$. **2**

(iii) Hence, by using the results in (i) and (ii), show that **3**

$$\int_0^{\frac{\pi}{3}} \ln\left(1 + \sqrt{3}\tan x\right) dx = \frac{\pi \ln 2}{3}.$$

(c) Consider the function $f(x) = 2\log_e x - \dfrac{x^2 - 1}{x}$, $x > 0$.

(i) Show that the only zero of $f(x)$ is at $x = 1$. **2**

(ii) Let $g(x) = \dfrac{x \log_e x}{x^2 - 1}$, $x > 0$ and $x \ne 1$. **3**

Explain why $0 < g(x) < \frac{1}{2}$ for all $x > 0, x \ne 1$.

End of paper

NSGHS 2016 Extension 2 Trial – Suggested Solutions

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1–10.

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1 Which of the following is a focus of $\frac{x^2}{11} - \frac{y^2}{25} = 1$?

(A) $(0,5)$

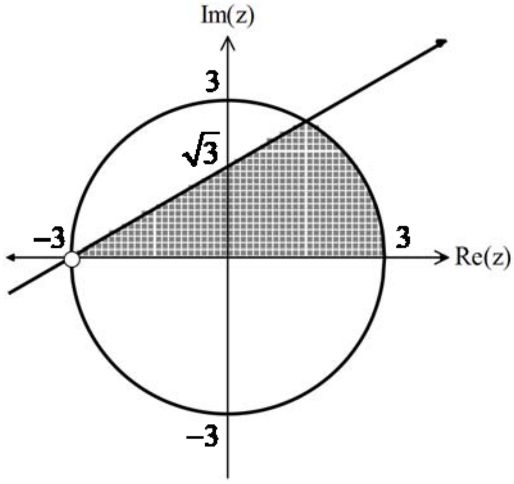
(B) $(5,0)$

(C) $(0,6)$

(D) $(6,0)$

$$b^2 = a^2\left(e^2 - 1\right)$$
$$a^2e^2 = a^2 + b^2$$
$$= 11 + 25 = 36$$
$$\therefore ae = 6$$

2 Which of the following describes the locus shown below?



(A) $|z| \le 3$ and $0 \le \arg(z) \le \frac{\pi}{6}$

(B) $|z| \le 3$ and $0 \le \arg(z) \le \frac{\pi}{3}$

(C) $|z| \le 3$ and $0 \le \arg(z+3) \le \frac{\pi}{6}$

(D) $|z| \le 3$ and $0 \le \arg(z+3) \le \frac{\pi}{3}$

3 In which quadrant is the complex number $(-3+3i)^3$ located on the Argand plane?

(A) the first quadrant

(B) the second quadrant

(C) the third quadrant

(D) the fourth quadrant

$$\arg(-3+3i)=\frac{3\pi}{4}$$
$$\arg(-3+3i)^3=3\times\frac{3\pi}{4}=\frac{9\pi}{4}=\frac{\pi}{4}$$

4 z_1 and z_2 are complex numbers such that z_1+z_2 lies on the real axis. Which of the following statements can be inferred?

(A) $z_1=\overline{z_2}$

(B) $\arg(z_1)=-\arg(z_2)$

(C) $|\mathrm{Im}\, z_1|=|\mathrm{Im}\, z_2|$

(D) $|z_1|=|z_2|$

5 Consider the equation $z^3-2z^2+bz+c=0$, where b and c are real numbers. If one of the roots of the equation is $2-i$, what is the value of b?

(A) -3

(B) -19

(C) 3

(D) 19

$2+i+2-i+\alpha=2$ (sum of roots)

$\alpha=-2$

$b=-2(2+i)-2(2-i)+(2+i)(2-i)$ (sum of pairs of roots)

$b=-8+5=-3$

6 Which of the following correctly describes some of the features of the graph of $y = \frac{x^2 - 4x + 3}{x^2 - x - 6}$?

(A) x-intercepts at 3 and 1 and horizontal asymptote at $y = 1$

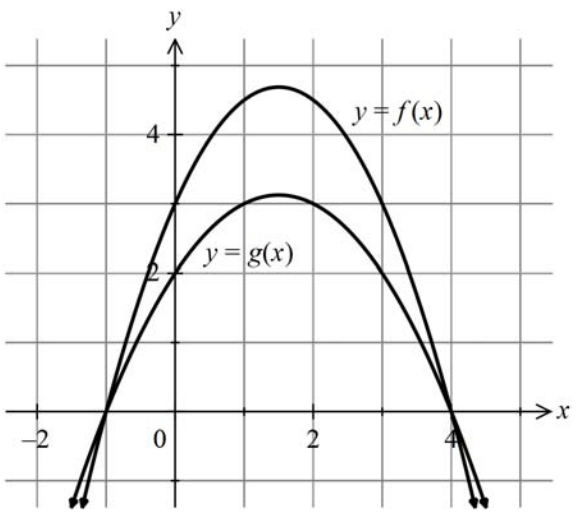
(B) Vertical asymptotes at $x = 3$ and $x = -2$ and horizontal asymptote at $y = 0$

(C) Vertical asymptotes at $x = 3$ and $x = -2$ and horizontal asymptote at $y = 1$

(D) Vertical asymptote at $x = -2$ and horizontal asymptote at $y = 1$

$y = \frac{\cancel{(x-3)}(x-1)}{\cancel{(x-3)}(x+2)}$ Therefore, the graph has an x-intercept at $x = 1$, vertical asymptote at $x = -2$, horizontal asymptote at $y = 1$ and discontinuity at $x = 3$

7 Shown below are the graphs of $y = f(x)$ and $y = g(x)$.



Given that $f(g(x)) = 3$, what are all the possible values of x?

(A) $x = 0,\ 3$

(B) $x = 1,\ 2$

(C) $x = -1,\ 4$

(D) $x = -1,\ 1,\ 2,\ 4$

$f(\alpha) = 3 \Rightarrow \alpha = 0, 3$
$g(x) = 0 \Rightarrow x = -1, 4$
$g(x) = 3 \Rightarrow x = 1, 2$

8 Which of the following statements is NOT true?

(A) $\displaystyle\int_{-1}^{1}\left(e^{x}-e^{-x}\right)dx=0$

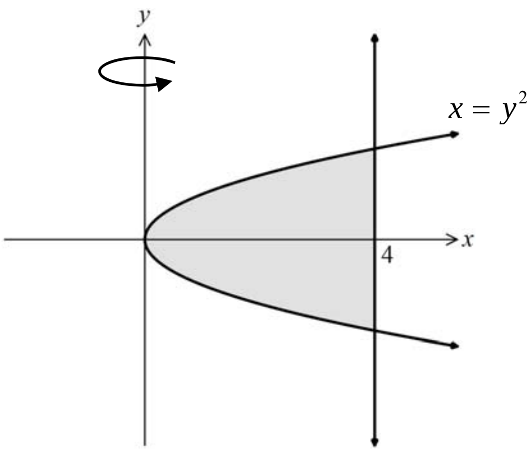
(B) $\displaystyle\int_{-\frac{\pi}{2}}^{\frac{\pi}{2}} x\sin x\,dx=0$

(C) $\displaystyle\int_{0}^{\pi}\cos 2x\,dx=0$

(D) $\displaystyle\int_{-1}^{1}\tan^{-1}x\,dx=0$

(A) and (D) are true because the functions are odd. (C) is true as the integral includes equal areas above and below the x-axis. The function in (B) is even, so this statement is not true.

9 The region enclosed by the curve $x=y^2$ and the line $x=4$ is rotated around the y-axis. Which of the following gives an expression for the volume of the resulting solid of revolution?



(A) $V=\pi\displaystyle\int_{-2}^{2}\left(4-y^2\right)^2 dy$

(B) $V=2\pi\displaystyle\int_{0}^{2}\left(16-y^4\right)dy$

(C) $V=\pi\displaystyle\int_{-2}^{2}\left(4-y^2\right)dy$

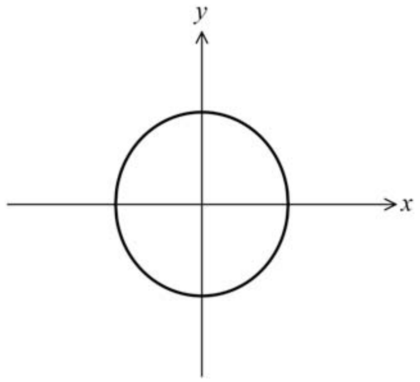
(D) $V=2\pi\displaystyle\int_{-2}^{2}\left(16-y^4\right)dy$

The cross-section perpendicular to the axis of rotation is an annular disk with outer radius 4 and inner radius x. Cross-section of the annular disk is $\pi\left(4^2-x^2\right)=\pi\left(16-y^4\right)$. y values range from -2 to 2. As the integrand is even, the volume is

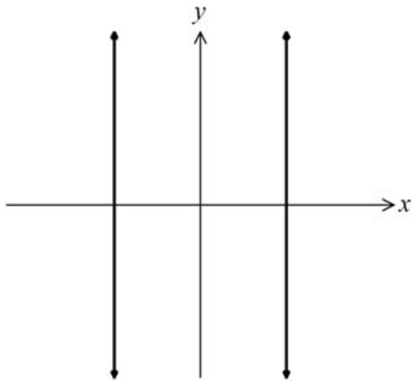
$$2\pi\int_{0}^{2}\left(16-y^4\right)dy.$$

10 Consider the conic $\frac{x^2}{p}+\frac{y^2}{q}=1$ with an eccentricity e, where $p>0$. Which of the following describes the limiting shape of the conic for a given value of p, as $e \to 1^+$?

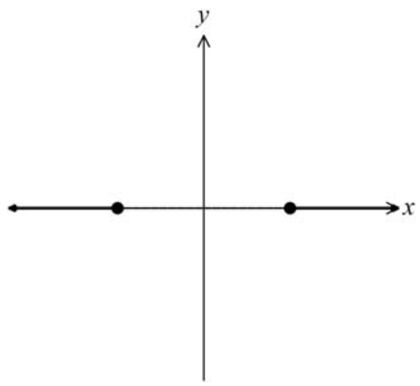
(A)



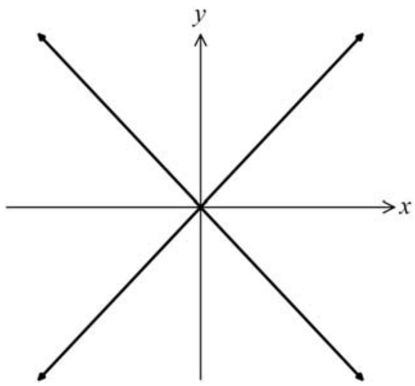
(B)



(C)



(D)



As e approaches 1 from above, $e>1$ so the conic is initially a hyperbola. $q=p(e-1)$. As $e \to 1^+$, $q \to 0$. As the asymptotes of the hyperbola are $y=\pm\frac{b}{a}x$, the branches fold and collapse onto the x-axis in the limiting position.

Section II

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) If $u = -\sqrt{3} + i$ and $v = 2 + 2i$:

(i) Find $\dfrac{u}{v}$ in the form $x + iy$. **2**

$$\begin{aligned}\frac{u}{v} &= \frac{-\sqrt{3}+i}{2+2i} = \frac{1}{2} \times \frac{-\sqrt{3}+i}{1+i} \times \frac{1-i}{1-i} \\ &= \frac{-\sqrt{3}+i\sqrt{3}+i+1}{4} \\ &= \frac{1-\sqrt{3}}{4} + i\frac{1+1\sqrt{3}}{4}\end{aligned}$$

(ii) Write u and v in modulus-argument form. **2**

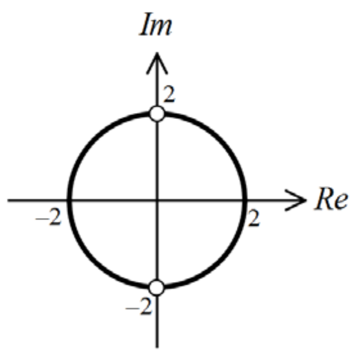
$$u = 2\text{cis}\frac{5\pi}{6} \text{ and } v = 2\sqrt{2}\text{cis}\frac{\pi}{4}$$

(iii) Write down the argument of uv . **1**

$$\arg(uv) = \frac{5\pi}{6} + \frac{\pi}{4} = \frac{13\pi}{12} = -\frac{11\pi}{12}$$

(b) Sketch the locus of z if $\dfrac{z+2i}{z-2i}$ is purely imaginary. **2**

$\arg\left(\dfrac{z+2i}{z-2i}\right) = \pm\dfrac{\pi}{2} \Rightarrow \arg(z+2i) - \arg(z-2i) = \pm\dfrac{\pi}{2}$. Therefore the locus is two semicircular arcs between $z = -2i$ and $z = 2i$ excluding the points $(0, 2i)$ and $(0, -2i)$.



(c) Consider the curve with the equation $x^2 - xy + y^2 - 21 = 0$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{y-2x}{2y-x}$. **1**

$$2x - y - x\frac{dy}{dx} + 2y\frac{dy}{dx} = 0$$

$$(2y - x)\frac{dy}{dx} = y - 2x$$

$$\frac{dy}{dx} = \frac{y-2x}{2y-x}$$

(ii) Find the points on the curve where the tangents are parallel to the line $y = 3x + 1$. **2**

Differentiating implicitly

$$\frac{dy}{dx} = 3 \Rightarrow \frac{y-2x}{2y-x} = 3$$

$$y - 2x = 6y - 3x$$

$$5y = x$$

Sub into equation of curve

$$(5y)^2 - (5y)y + y^2 - 21 = 0 \Rightarrow 21y^2 = 21$$

$y = \pm 1$ so $x = \pm 5$

Points are $(5,1)$ and $(-5,-1)$

(d) (i) Using partial fraction decomposition, show that $\displaystyle\int_0^1 \frac{10dt}{(1+3t)(3-t)} = \log_e 6$. **3**

$$\int_0^1 \frac{10dt}{(1+3t)(3-t)} = \int_0^1 \left(\frac{3}{(1+3t)} + \frac{1}{3-t}\right) dt$$

$$= \left[\ln(1+3t) - \ln(3-t)\right]_0^1$$

$$= \left[\ln\left(\frac{1+3t}{3-t}\right)\right]_0^1$$

$$= \ln 2 - \ln\frac{1}{3} = \ln 6$$

$$\frac{10}{(1+3t)(3-t)} = \frac{A}{1+3t} + \frac{B}{3-t}$$

$$10 = A(3-t) + B(1+3t)$$

$$t = 3 \Rightarrow B = 1$$

$$t = -\frac{1}{3} \Rightarrow A = 3$$

(ii) Hence, by using the substitution $t = \tan\frac{\theta}{2}$, evaluate $\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta + 3\cos\theta}$. **2**

$t = \tan\frac{\theta}{2} \Rightarrow \theta = 2\tan^{-1} t$

$d\theta = \frac{2dt}{1+t^2} \quad \theta = 0, t = 0 \quad \theta = \frac{\pi}{2}, t = 1$

$\sin\theta = \frac{2t}{1+t^2}$ and $\cos\theta = \frac{1-t^2}{1+t^2}$

$$\int_0^{\frac{\pi}{2}} \frac{d\theta}{4\sin\theta + 3\cos\theta} = \int_0^1 \frac{1}{4\left(\frac{2t}{1+t^2}\right) + 3\left(\frac{1-t^2}{1+t^2}\right)} . \frac{2dt}{1+t^2}$$

$$= \int_0^1 \frac{2dt}{8t + 3 - 3t^2}$$

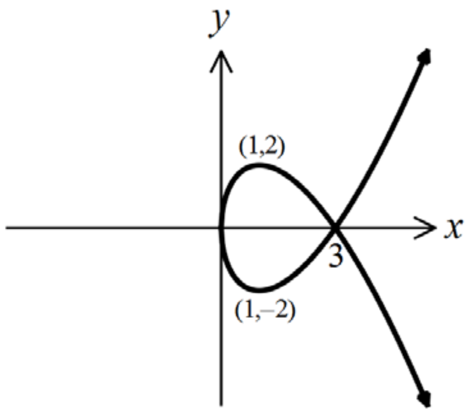
$$= \frac{1}{5}\int_0^1 \frac{10dt}{(1+3t)(3-t)}$$

$$= \frac{1}{5}\ln 6 \qquad \text{(using part (i))}$$

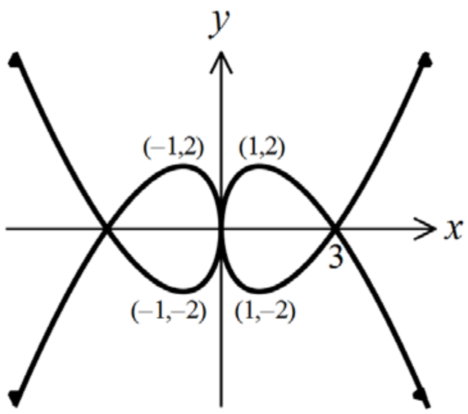
Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The graph of the curve $y = x(x-3)^2$ is shown below.

(i) Neatly sketch the graph of the curve $y^2 = x(x-3)^2$ on one-third of a page, clearly showing the coordinates of any turning points. **2**



(ii) Hence, on a new number plane, sketch the graph of $y^2 = |x|\left(|x|-3\right)^2$. **1**



(b) The equation $x^3 - 2x^2 + 3x - 1 = 0$ has roots α, β and γ.

(i) Find a polynomial equation whose roots are α^2, β^2 and γ^2. **3**

An equation whose roots are α^2, β^2 and γ^2 is given by:

$$\left(\sqrt{x}\right)^3 - 2\left(\sqrt{x}\right)^2 + 3\sqrt{x} - 1 = 0$$

$$x\sqrt{x} - 2x + 3\sqrt{x} - 1 = 0$$

$$\sqrt{x}\left(x+3\right) = 2x + 1$$

$$x\left(x^2 + 6x + 9\right) = 4x^2 + 4x + 1$$

$$x^3 + 2x^2 + 5x - 1 = 0$$

(ii) Hence or otherwise, find a polynomial equation whose roots are $\alpha^2\beta^2$, $\beta^2\gamma^2$ and $\gamma^2\alpha^2$.	**2**

$\alpha^2\beta^2\gamma^2 = 1$ (product of roots)

$\alpha^2\beta^2 = \dfrac{1}{\gamma^2}, \alpha^2\gamma^2 = \dfrac{1}{\beta^2}$ and $\beta^2\gamma^2 = \dfrac{1}{\alpha^2}$

Thus, these roots are reciprocals of the roots of the equation obtained in (i).

Therefore, an equation whose roots are $\alpha^2\beta^2$, $\beta^2\gamma^2$ and $\gamma^2\alpha^2$ is given by:

$$\left(\frac{1}{x}\right)^3 + 2\left(\frac{1}{x}\right)^2 + 5\left(\frac{1}{x}\right) - 1 = 0 \quad \times x^3$$

$1 + 2x + 5x^2 - x^3 = 0$ or $x^3 - 5x^2 - 2x - 1 = 0$

(c) Given that the polynomial equation $x^3 - ax^2 + b = 0$ $(a, b \neq 0)$ has a non-zero multiple root, show that $4a^3 - 27b = 0$.	**3**

Let $P(x) = x^3 - ax^2 + b$ and let ω be the multiple root of $P(x) = 0$.

Then, $P(\omega) = P'(\omega) = 0$ (Multiple Root theorem)

$\alpha^3 - a\omega^2 + b = 0$ (1) and $3\omega^2 - 2a\omega = 0$ (2)

From (2) $\omega(3\omega - 2a) = 0 \Rightarrow \omega = \dfrac{2a}{3}$ as $\omega \neq 0$

$$\left(\frac{2a}{3}\right)^3 - a\left(\frac{2a}{3}\right)^2 + b = 0$$

Sub into (1) $\dfrac{8a^3}{27} - \dfrac{4a^3}{9} + b = 0 \Rightarrow b = \dfrac{4a^3}{27}$

$27b = 4a^3$ or $4a^3 - 27b = 0$ as required

(d) *OABC* is a parallelogram and *D* is the midpoint of *BC*. *OB* and *AD* intersect at *P*. Let $\overrightarrow{OA}$ represent the complex number z_1 and $\overrightarrow{OC}$ represent the complex number z_2. (i) Given that $\overrightarrow{AP} = m \times \overrightarrow{AD}$, for some real number *m*, explain why $\overrightarrow{AP}$ is equal to $m\left(z_2 - \dfrac{z_1}{2}\right)$.	**1**

$\overrightarrow{AD} = \overrightarrow{AB} + \overrightarrow{BD} = z_2 - \dfrac{1}{2}z_1$

$\overrightarrow{AP} = m\left(z_2 - \dfrac{1}{2}z_1\right)$

(ii) Given that $\overrightarrow{OP} = n \times \overrightarrow{OB}$, for some real number n, find another expression for $\overrightarrow{AP}$ in terms of n, z_1 and z_2. **1**

$\overrightarrow{OB} = \overrightarrow{OA} + \overrightarrow{AB} = z_1 + z_2$
$\overrightarrow{OP} = n(z_1 + z_2)$
$\overrightarrow{AP} = \overrightarrow{OP} - \overrightarrow{OA} = n(z_1 + z_2) - z_1 = (n-1)z_1 + nz_2$

(iii) Hence, deduce the values of m and n. **2**

Equating expressions for $\overrightarrow{AP}$ from (i) and (ii)

$$m\left(z_2 - \frac{1}{2}z_1\right) = (n-1)z_1 + nz_2$$

$$(m-n)z_2 = \left(n + \frac{m}{2} - 1\right)z_1$$

As z_1 and z_2 and vectors in different directions, this is only possible if the real multipliers are zero.

$$m - n = 0 \text{ and } n + \frac{m}{2} - 1 = 0$$

$$\therefore m = n \text{ and so } m + \frac{m}{2} - 1 = 0$$

$$m = n = \frac{2}{3}$$

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Let $P\left(cp, \frac{c}{p}\right)$ be a variable point on the hyperbola $xy = c^2$.

(i) Show that the equation of the tangent to the hyperbola at P is $x + p^2 y = 2cp$. **2**

$$\frac{dy}{dx} = \frac{dy}{dp} \times \frac{dp}{dx} = -\frac{c}{p^2} \times \frac{1}{c} = -\frac{1}{p^2}$$

Equation of tangent is:

$$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$$

$$p^2 y - cp = -x + cp$$

$$x + p^2 y = 2cp$$

(ii) The tangent meets the x-axis at Q. Show that the equation of the line through Q perpendicular to the tangent at P is $p^2 x - y = 2cp^3$. **2**

For Q, sub $y = 0$ in the equation of tangent. $Q = (2cp, 0)$.

Line through Q perpendicular to tangent is given by:

$$y - 0 = p^2(x - 2cp)$$

$$p^2 x - y = 2cp^3$$

(iii) The line through Q in part (ii) cuts the hyperbola at two points R and S. Find the Cartesian equation of the locus of M, the midpoint of RS, noting any restrictions that may apply. **3**

From (ii) equation of RS is $y = p^2 x - 2cp^3$.

For intersections of this line with the hyperbola, solve simultaneously with $xy = c^2$.

$$x\left(p^2 x - 2cp^3\right) = c^2$$

$$p^2 x^2 - 2cp^3 x - c^2 = 0$$

Sum of roots: $\alpha + \beta = \frac{2cp^3}{p^2} = 2cp$.

$\therefore x_M = \left(\frac{\alpha + \beta}{2}\right) = \frac{2cp}{2} = cp$. Sub into equation of line for y_M.

$$y_M = p^2(cp) - 2cp^3 = -cp^3$$

Now, $p = \frac{x_M}{c}$. So, $y_M = -c\left(\frac{x_M}{c}\right)^3 = -\frac{{x_M}^3}{c^2}$

Therefore, the equation of the locus of M is $y = -\frac{x^3}{c}$. As $p \neq 0$, then restriction is $M \neq (0,0)$.

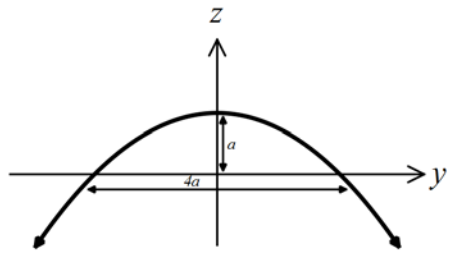
(b) A solid has as its base in the xy plane, the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{4}=1$.

Cross-sections taken perpendicular to the x-axis and the base are inverted parabolas with latus rectums in the base.

The *latus rectum* is the focal chord of the parabola which is perpendicular to the axis of the parabola.

(i) Explain why the height of any cross-sectional slice is equal to $\dfrac{\sqrt{16-x^2}}{4}$. **1**

The length of the latus rectum $=4a$ where a is the focal length.
The distance from the vertex of the parabola to this line is a units.



From the ellipse, the length of the latus rectum is $2y$

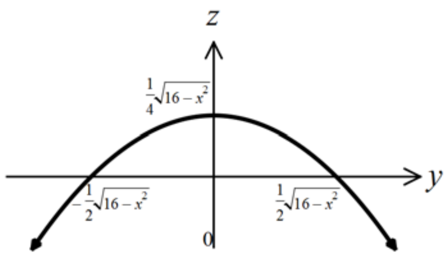
so the height of the parabola is $\dfrac{y}{2}$.

$$\frac{y^2}{4}=1-\frac{x^2}{16}=\frac{16-x^2}{16}$$

$$\frac{y}{2}=\pm\frac{\sqrt{16-x^2}}{4}$$

(ii) By first using Simpson's rule to find an expression for the area of the cross-section, find the volume of the solid. **3**

Using Simpson's rule with three function values, the area under the curve is given by:



$$A=\frac{\sqrt{16-x^2}}{6}\left(0+4.\frac{\sqrt{16-x^2}}{4}+0\right)=\frac{16-x^2}{6}$$

Note that this is an exact value for the area under the curve.

Volume of typical cross-sectional slice $=\delta V=\dfrac{16-x^2}{6}\delta x$

$$V=\lim_{\delta x\to 0}\sum_{x=-4}^{4}\delta V=\int_{-4}^{4}\frac{16-x^2}{6}\,dx$$

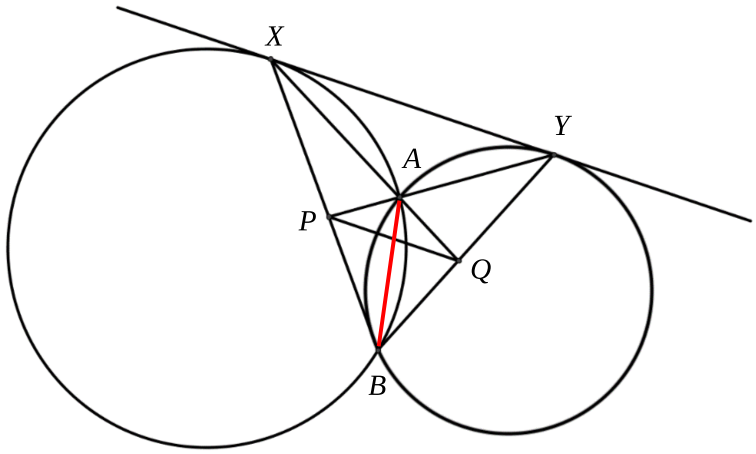
$$=2\int_0^4\frac{16-x^2}{6}\,dx \quad \text{(even function)}$$

$$=\frac{2}{3}\left[16x-\frac{x^3}{3}\right]_0^4$$

$$=\frac{2}{3}\left[\left(64-\frac{64}{3}\right)-0\right]=\frac{128}{9}\ \text{units}^3$$

(c) Two circles meet at the points A and B. The line XY is a common tangent to the two circles as shown. XA produced meets BY at Q and YA produced meets BX at P.

(i) Prove that $PAQB$ is a cyclic quadrilateral. **3**



Join AB. Let $\angle XYA = \alpha$ and $\angle YXA = \beta$

$\angle XYA = \angle YBA = \alpha$ (angle in alternate segment)

$\angle YXA = \angle XBA = \beta$ (angle in alternate segment)

So, $\angle XBY = \angle YBA + \angle XBA = \alpha + \beta$ (adjacent angles)

Now $\angle XAY = 180 - (\alpha + \beta)$ (angle sum ΔXYA)

$\angle XAY = \angle PAQ = 180 - (\alpha + \beta)$ (vertically opposite angles)

$\therefore \angle PAQ + \angle PBQ = (\alpha + \beta) + 180 - (\alpha + \beta) = 180$

$\therefore PAQB$ is a cyclic quadrilateral (opposite angles are supplementary)

(ii) Hence, or otherwise, prove that $PQ \parallel XY$. **1**

$\angle APQ = \angle ABQ = \alpha$ (angles on same arc in circle $PAQB$)

Then, $\angle APQ = \angle XYA$. But these are alternate angles.

Therefore, $PQ \parallel XY$ as alternate angles are equal.

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Show that $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right)+C$, where C is a constant and $x > a > 0$. **2**

$$\frac{d}{dx}\left(\ln\left(x+\sqrt{x^2-a^2}\right)\right) = \frac{1}{x+\sqrt{x^2-a^2}}\cdot\left(1+\frac{1}{\cancel{2}\sqrt{x^2-a^2}}\cdot\cancel{2}x\right)$$

$$= \frac{1}{\cancel{x+\sqrt{x^2-a^2}}}\cdot\left(\frac{\cancel{\sqrt{x^2-a^2}+x}}{\sqrt{x^2-a^2}}\right)$$

$$= \frac{1}{\sqrt{x^2-a^2}}$$

Therefore, $\displaystyle\int \frac{dx}{\sqrt{x^2-a^2}} = \ln\left(x+\sqrt{x^2-a^2}\right)+C$

Alternately,

$$\int \frac{dx}{\sqrt{x^2-a^2}} = \int \frac{dx}{\sqrt{x^2-a^2}} \times \frac{x+\sqrt{x^2-a^2}}{x+\sqrt{x^2-a^2}} = \int \frac{x+\sqrt{x^2-a^2}}{\sqrt{x^2-a^2}} \times \frac{1}{x+\sqrt{x^2-a^2}}dx$$

$$= \int\left(\frac{x}{\sqrt{x^2-a^2}}+1\right)\times\frac{1}{x+\sqrt{x^2-a^2}}dx = \int \frac{d\left(x+\sqrt{x^2-a^2}\right)}{x+\sqrt{x^2-a^2}}$$

$$=\ln\left(x+\sqrt{x^2-a^2}\right)+C$$

Can also be done using substitution $x = a\sec\theta$

(b) Using the substitution $u = \tan x$, or otherwise, evaluate $\displaystyle\int_0^{\frac{\pi}{3}} \frac{\tan x\, dx}{7\sin^2 x + 5\cos^2 x}$. **3**

$$\int_0^{\frac{\pi}{3}} \frac{\tan x\, dx}{7\sin^2 x + 5\cos^2 x} == \int_0^{\sqrt{3}} \frac{udu}{\left(7u^2+5\right)}$$

$$= \frac{1}{14}\int_0^{\sqrt{3}} \frac{14udu}{\left(7u^2+5\right)}$$

$$= \frac{1}{14}\Big[\ \ln\left(7u^2+5\right)\Big]_0^{\sqrt{3}}$$

$$= \frac{1}{14}\left(\ln 26 - \ln 5\right) = \frac{1}{14}\ln\left(\frac{26}{5}\right)$$

$u = \tan x$

$du = \sec^2 x\, dx$

$u = 0 \Rightarrow x = 0$

$u = \dfrac{\pi}{3} \Rightarrow x = \sqrt{3}$

$$\frac{\tan xdx}{7\sin^2 x + 5\cos^2 x} = \frac{\tan xdx}{\cos^2 x\left(7\tan^2 x + 5\right)}$$

$$= \frac{udu}{7u^2+5}$$

(c)	(i)	If a and b are real numbers, prove that $a^2 + b^2 \geq 2ab$.	**1**

Consider $(a-b)^2 \geq 0$

$a^2 - 2ab + b^2 \geq 0$

$a^2 + b^2 \geq 2ab$

	(ii)	Show that $a^2 + b^2 + c^2 \geq ab + ac + bc$, where a, b and c are real.	**1**

From (i) $a^2 + b^2 \geq 2ab$

Similarly, $b^2 + c^2 \geq 2bc$

and $a^2 + c^2 \geq 2ac$

Adding we get $2a^2 + 2b^2 + 2c^2 \geq 2ab + 2ac + 2bc$

$\therefore a^2 + b^2 + c^2 \geq ab + ac + bc$

	(iii)	Hence or otherwise, show that the equation $x^3 - 3x^2 + 4x + k = 0$ cannot have three real roots for any real constant k.	**2**

$\alpha + \beta + \gamma = 3$ and $\alpha\beta + \beta\gamma + \alpha\gamma = 4$

$$\begin{aligned}\alpha^2 + \beta^2 + \gamma^2 &= \left(\sum \alpha\right)^2 - 2\sum \alpha\beta \\ &= 3^2 - 2(4) \\ &= 1 \\ &< \alpha\beta + \beta\gamma + \alpha\gamma\end{aligned}$$

Then from part (i) α, β, γ cannot all be real.

(d)		The point $P(x_1, y_1)$ lies on the ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$, where $a > b$.	
	(i)	Show that the equation of the tangent to the ellipse at P is given by $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$.	**2**

$\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$

$\frac{dy}{dx} = -\frac{b^2 x}{a^2 y}$

At $P(x_1, y_1)$ $\frac{dy}{dx} = -\frac{b^2 x_1}{a^2 y_1}$.

Equation of tangent is:

$$y - y_1 = -\frac{b^2 x_1}{a^2 y_1}(x - x_1)$$

$$a^2 y_1 y - a^2 y_1{}^2 = -b^2 x_1 x + b^2 x_1{}^2$$

$$b^2 x_1 x + a^2 y_1 y = b^2 x_1{}^2 + a^2 y_1{}^2 \quad \text{(divide by } a^2 b^2\text{)}$$

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = \frac{x_1{}^2}{a^2} + \frac{y_1{}^2}{b^2}$$

$$\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1 \qquad \text{as } (x_1, y_1) \text{ lies on the ellipse}$$

(ii)	The tangent intersects the lines $x = a$ and $x = -a$ at Q and R respectively. Show that QR subtends a right angle at the focus S.	**3**

For Q: sub $x = a$ into equation of tangent

$$\frac{\cancel{a} x_1}{a \cancel{^2}} + \frac{y y_1}{b^2} = 1$$

$$y = \frac{b^2}{y_1}\left(1 - \frac{x_1}{a}\right)$$

$Q = \left(a, \frac{b^2}{y_1}\left(1 - \frac{x_1}{a}\right)\right)$ Replacing a with $-a$ throughout, $R = \left(-a, \frac{b^2}{y_1}\left(1 + \frac{x_1}{a}\right)\right)$

$$m_{QS}.m_{RS} = \frac{\frac{b^2}{y_1}\left(1 - \frac{x_1}{a}\right)}{a - ae} \times \frac{\frac{b^2}{y_1}\left(1 + \frac{x_1}{a}\right)}{-a - ae} = -\frac{b^4}{y_1{}^2} \cdot \frac{\left(1 - \frac{x_1{}^2}{a^2}\right)}{a^2 - a^2 e^2}$$

$$= -\frac{b^4}{y_1{}^2} \cdot \frac{\frac{y_1{}^2}{b^2}}{b^2} = -\frac{b^4}{y_1{}^2} \frac{y_1{}^2}{b^4}$$

$$= -1$$

(iii)	Hence, explain why the points Q, R, S and S' are concyclic.	**1**

By symmetry, QR also subtends a right angle at S'. $\angle QSR = \angle QS'R = 90$. Hence, Q, R, S and S' are concyclic (converse of angles in the same segment).

Alternately,

QR is the diameter of the circle through Q, R and S (converse of angle in a semicircle) and the centre of this circle is the mid-point of QR which lies on the y-axis. As the y-axis is the perpendicular bisector of SS', the centre of the circle is equidistant from S and S'. Hence, the circle passes through S' and the points Q, R, S and S' are concyclic.

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a)	A number sequence is defined by $u_1 = 0$, $u_2 = 32$, and $u_{n+2} = 6u_{n+1} - 8u_n$ for $n \geq 1$. Use mathematical induction to prove that $u_n = 4^{n+1} - 2^{n+3}$ for all positive integers n.	**3**

<u>Test for $n = 1, 2$</u>

$u_1 = 4^2 - 2^4 = 16 - 16 = 0$

$u_2 = 4^3 - 2^5 = 64 - 32 = 32$

$\therefore$ true for $n = 1, 2$

<u>Assume true for $n = k, k+1$</u>

i.e assume that $u_k = 4^{k+1} - 2^{k+3}$ and $u_{k+1} = 4^{k+2} - 2^{k+4}$

<u>Assume true for $n = k+2$</u>

ie. to prove that $u_{k+2} = 4^{k+3} - 2^{k+5}$

$$\begin{aligned} u_{k+2} &= 6u_{k+1} - 8u_k \\ &= 6\left(4^{k+2} - 2^{k+4}\right) - 8\left(4^{k+1} - 2^{k+3}\right) \\ &= 24.4^{k+1} - 12.2^{k+3} - 8.4^{k+1} + 8.2^{k+3} \\ &= 16.4^{k+1} - 4.2^{k+3} \\ &= 4^{k+3} - 2^{k+5} \end{aligned}$$

Therefore the statement is true by mathematical induction.

(b)	You are given that $I_n = \displaystyle\int_1^5 \frac{x^n}{\sqrt{2x-1}}\, dx, \quad n \geq 0$. (i) Prove that $(2n+1) I_n = n\, I_{n-1} + 3 \times 5^n - 1$, for $n \geq 1$.	**3**

$$\begin{aligned} I_n &= \int_1^5 \frac{x^n}{\sqrt{2x-1}}\, dx = \int_1^5 x^n d\left(\sqrt{2x-1}\right) \\ &= \left[x^n \sqrt{2x-1}\right]_1^5 - \int_1^5 nx^{n-1}\sqrt{2x-1}\, dx \\ &= 3 \times 5^n - 1 - n\int_1^5 x^{n-1} \frac{2x-1}{\sqrt{2x-1}}\, dx \\ &= 3 \times 5^n - 1 - 2n\int_1^5 \frac{x^n}{\sqrt{2x-1}}\, dx + n\int_1^5 \frac{x^{n-1}}{\sqrt{2x-1}} dx \\ &= 3 \times 5^n - 1 - 2nI_n + nI_{n-1} \end{aligned}$$

$\therefore (2n+1) I_n = nI_{n-1} + 3 \times 5^n - 1$

(ii) Using the result in part (i), find the exact value of I_1.		**1**

$$I_0=\int_1^5 \frac{1}{\sqrt{2x-1}}\,dx=\int_1^5 (2x-1)^{-\frac{1}{2}}\,dx=\left[(2x-1)^{\frac{1}{2}}\right]_1^5=3-1=2$$

Using part (i) $3I_1=I_0+5\times 3-1=2+15-1=16$ So, $I_1=\dfrac{16}{3}$

(c) (i) Let $f(x)$ be a continuous function such that $f(k-x)=f(x)$, where k is a constant. Prove that $\displaystyle\int_0^k xf(x)\,dx=\frac{k}{2}\int_0^k f(x)\,dx$.		**2**

$$\int_0^k xf(x)\,dx=\int_k^0 (k-u)\,f(k-u)(-du)$$

$$=\int_0^k (k-u)\,f(u)\,du \qquad \text{as } f(k-u)=f(u)$$

$$=\int_0^k (k-x)\,f(x)\,dx \qquad \text{as } u \text{ is a dummy variable}$$

$$=\int_0^k kf(x)\,dx-\int_0^k xf(x)\,dx$$

$$2\int_0^k xf(x)\,dx=k\int_0^k f(x)\,dx$$

$$\int_0^k xf(x)\,dx=\frac{k}{2}\int_0^k f(x)\,dx$$

Let $u=k-x$

$du=-dx$

$x=k-u$

$x=0, u=k$ and $x=k, u=0$

(ii) If $z=\cos\theta+i\sin\theta$, then $z^n+\dfrac{1}{z^n}=2\cos n\theta$. (DO NOT SHOW THIS). Use this result to show that $\cos^4\theta=\dfrac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$.		**2**

$$\cos^4\theta=\left(\frac{1}{2}\left(z+\frac{1}{z}\right)\right)^4=\frac{1}{16}\left(z^4+4z^2+6+\frac{4}{z^2}+\frac{1}{z^4}\right)$$

$$=\frac{1}{16}\left(\left(z^4+\frac{1}{z^4}\right)+4\left(z^2+\frac{1}{z^2}\right)+6\right)$$

$$=\frac{1}{16}(2\cos 4\theta+8\cos 2\theta+6)$$

$$=\frac{1}{8}(\cos 4\theta+4\cos 2\theta+3)$$

(iii)	The diagram shows the shaded region bounded by the curve $y = \cos^4 x$ and the x-axis for $\pi \le x \le 2\pi$. The shaded region is rotated about the y-axis to form a solid of revolution. Use the method of cylindrical shells and the results in parts (i) and (ii) to find the volume of this solid.	**4**

Typical element is a cylindrical shell of radius x and height $y = \cos^4 x$ and thickness δx.

$$\delta V = 2\pi x \cos^4 x \delta x$$

$$V = 2\pi \int_{\pi}^{2\pi} x \cos^4 x dx$$

$$= 2\pi \int_{0}^{2\pi} x \cos^4 x dx - 2\pi \int_{0}^{\pi} x \cos^4 x dx$$

Now, $\cos^4 (2\pi - x) = (\cos x)^4 = \cos^4 x$

And $\cos^4 (\pi - x) = (-\cos x)^4 = \cos^4 x$

Hence, using part (i)

$$V = 2\pi \left[\pi \int_{0}^{2\pi} \cos^4 x dx - \frac{\pi}{2} \int_{0}^{\pi} \cos^4 x dx \right]$$

$$= \frac{\pi^2}{4} \int_{0}^{2\pi} (\cos 4x + 4\cos 2x + 3) dx - \frac{\pi^2}{8} \int_{0}^{\pi} (\cos 4x + 4\cos 2x + 3) dx \quad \text{using part (ii)}$$

$$= \frac{\pi^2}{4} \left[\frac{\sin 4x}{4} + 4\frac{\sin 2x}{2} + 3x \right]_0^{2\pi} - \frac{\pi^2}{8} \left[\frac{\sin 4x}{4} + 4\frac{\sin 2x}{2} + 3x \right]_0^{\pi}$$

$$= \frac{\pi^2}{4}(6\pi) - \frac{\pi^2}{8}(3\pi)$$

$$= \frac{3\pi^3}{2} - \frac{3\pi^3}{8}$$

$$= \frac{9\pi^3}{8} \text{ units}^3$$

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that $x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-xz)$ for all real x, y, z. **1**

$$\text{RHS}=(x+y+z)(x^2+y^2+z^2-xy-yz-xz)=x^3+xy^2+xz^2-x^2y-xyz-x^2z$$
$$+x^2y+y^3+yz^2-xy^2-y^2z-xyz$$
$$+x^2z+y^2z+z^3-xyz-yz^2-xz^2$$
$$=x^3+y^3+z^3-3xyz=\text{LHS}$$

(ii) Hence or otherwise, if $x+y+z=1$ prove that $xy+yz+xz-3xyz\le\frac{1}{4}$ where x, y, z are positive real numbers. **3**

$$x^3+y^3+z^3-3xyz=(x+y+z)(x^2+y^2+z^2-xy-yz-xz) \quad \text{from (i)}$$
$$=x^2+y^2+z^2-xy-yz-xz \quad \text{as } x+y+z=1$$
$$xy+yz+xz-3xyz=x^2+y^2+z^2-x^3-y^3-z^3$$
$$=x^2-x^3+y^2-y^3+z^2-z^3$$
$$=x\left[x-x^2\right]+y\left[y-y^2\right]+z\left[z-z^2\right]$$

$$x-x^2=\frac{1}{4}-\left(x-\frac{1}{2}\right)^2$$
$$\le\frac{1}{4} \quad \text{as } \left(x-\frac{1}{2}\right)^2\ge 0$$

$$xy+yz+xz-3xyz=x\left[x-x^2\right]+y\left[y-y^2\right]+z\left[z-z^2\right]$$
$$\le x\left(\frac{1}{4}\right)+y\left(\frac{1}{4}\right)+z\left(\frac{1}{4}\right)$$
$$=\frac{1}{4}(x+y+z)=\frac{1}{4} \quad \text{as } x+y+z=1$$

Therefore, $xy+yz+xz-3xyz\le\frac{1}{4}$

(b) Let $f(x)$ be an odd function for $-a\le x\le a$, where a is a positive constant.

(i) By using a suitable substitution, prove that $\int_0^{2a} f(x-a)\,dx=0$. **1**

$$\int_0^{2a} f(x-a)\,dx=\int_{-a}^{a} f(u)\,du$$
$$=0 \quad \text{as } f(x) \text{ is odd}$$

$u=x-a$
$du=dx$
$x=0, u=-a$
$x=2a, u=a$

(ii) Show that $\dfrac{\sqrt{3}+\tan\left(x-\dfrac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\dfrac{\pi}{6}\right)}=\dfrac{1+\sqrt{3}\tan x}{2}$. **2**

$$\tan\left(x-\frac{\pi}{6}\right)=\frac{\tan x-\dfrac{1}{\sqrt{3}}}{1+\tan x\dfrac{1}{\sqrt{3}}}=\frac{\sqrt{3}\tan x-1}{\sqrt{3}+\tan x}$$

$$\begin{aligned}\frac{\sqrt{3}+\tan\left(x-\dfrac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\dfrac{\pi}{6}\right)}&=\frac{\sqrt{3}+\dfrac{\sqrt{3}\tan x-1}{\sqrt{3}+\tan x}}{\sqrt{3}-\dfrac{\sqrt{3}\tan x-1}{\sqrt{3}+\tan x}}\\&=\frac{\sqrt{3}\left(\sqrt{3}+\tan x\right)+\sqrt{3}\tan x-1}{\sqrt{3}\left(\sqrt{3}+\tan x\right)-\sqrt{3}\tan x+1}\\&=\frac{3+\sqrt{3}\tan x+\sqrt{3}\tan x-1}{3+\sqrt{3}\tan x-\sqrt{3}\tan x+1}\\&=\frac{2+2\sqrt{3}\tan x}{4}\\&=\frac{1+\sqrt{3}\tan x}{2}\end{aligned}$$

(iii) Hence, by using the results in (i) and (ii), show that **3**

$$\int_0^{\frac{\pi}{3}}\ln\left(1+\sqrt{3}\tan x\right)dx=\frac{\pi\ln 2}{3}.$$

$$\int_0^{\frac{\pi}{3}}\ln\left(1+\sqrt{3}\tan x\right)dx=\int_0^{\frac{\pi}{3}}\ln\left(2\times\frac{\sqrt{3}+\tan\left(x-\dfrac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\dfrac{\pi}{6}\right)}\right)dx \quad \text{using part (ii)}$$

$$=\int_0^{\frac{\pi}{3}}\ln 2+\int_0^{\frac{\pi}{3}}\ln\left(\frac{\sqrt{3}+\tan\left(x-\dfrac{\pi}{6}\right)}{\sqrt{3}-\tan\left(x-\dfrac{\pi}{6}\right)}\right)$$

Consider $f(x) = \ln\left(\dfrac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right)$.

$$f(-x) = \ln\left(\frac{\sqrt{3} + \tan(-x)}{\sqrt{3} - \tan(-x)}\right)$$

$$= \ln\left(\frac{\sqrt{3} - \tan x}{\sqrt{3} + \tan x}\right) \quad \text{as } \tan x \text{ is odd}$$

$$= -\ln\left(\frac{\sqrt{3} + \tan(x)}{\sqrt{3} - \tan(x)}\right) = -f(x)$$

Therefore, $f(x) = \ln\left(\dfrac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right)$ is odd

Using part (i) with $f(x) = \ln\left(\dfrac{\sqrt{3} + \tan x}{\sqrt{3} - \tan x}\right)$ and $a = \dfrac{\pi}{6}$, $\displaystyle\int_0^{\frac{\pi}{3}} \ln\left(\frac{\sqrt{3} + \tan\left(x - \frac{\pi}{6}\right)}{\sqrt{3} - \tan\left(x - \frac{\pi}{6}\right)}\right) dx = 0$

$$\int_0^{\frac{\pi}{3}} \ln\left(1 + \sqrt{3}\tan x\right) dx = \int_0^{\frac{\pi}{3}} \ln 2 \, dx + 0$$

$$= \ln 2\left(\frac{\pi}{3} - 0\right)$$

$$= \frac{\pi \ln 2}{3}$$

(c) Consider the function $f(x) = 2\log_e x - \dfrac{x^2 - 1}{x}$, $x > 0$.

(i) Show that the only zero of $f(x)$ is at $x = 1$. **2**

$f(1) = 2\ln 1 - \dfrac{1^2 - 1}{1} = 0$. Therefore, $x = 1$ is a zero.

$$f(x) = 2\ln x - x + \frac{1}{x}$$

$$f'(x) = \frac{2}{x} - 1 - \frac{1}{x^2}$$

$$= \frac{2x - 1 - x^2}{x^2}$$

$$= -\frac{(x-1)^2}{x^2}$$

$$\leq 0 \qquad \text{for all } x, x \neq 0, 1$$

i.e. $f(x)$ is monotonic decreasing.

$\therefore x = 1$ is the only zero

(ii) Let $g(x) = \dfrac{x\log_e x}{x^2 - 1}$, $x > 0$ and $x \neq 1$. **3**

Explain why $0 < g(x) < \dfrac{1}{2}$ for all $x > 0, x \neq 1$.

From (i) $2\ln x - \dfrac{x^2-1}{x}$ > 0 for $0 < x < 1$

< 0 for $x > 1$

<u>Case 1</u>: $0 < x < 1$

$$2\ln x - \frac{x^2-1}{x} > 0$$

$$2\ln x > \frac{x^2-1}{x}$$

$$x\ln x > \frac{x^2-1}{2} \quad \text{as } x > 0$$

$$\frac{x\ln x}{x^2-1} < \frac{1}{2} \quad \text{as } x^2 - 1 < 0$$

Also, $\dfrac{x\ln x}{x^2-1} > 0$ as $x > 0, \ln x < 0, x^2 - 1 < 0$

So, $0 < g(x) < \dfrac{1}{2}$

<u>Case 2</u>: $x > 1$

$$2\ln x - \frac{x^2-1}{x} < 0$$

$$2\ln x < \frac{x^2-1}{x}$$

$$x\ln x < \frac{x^2-1}{2} \quad \text{as } x > 0$$

$$\frac{x\ln x}{x^2-1} < \frac{1}{2} \quad \text{as } x^2 - 1 > 0$$

Also, $\dfrac{x\ln x}{x^2-1} > 0$ as $x > 0, \ln x > 0, x^2 - 1 > 0$

So, $0 < g(x) < \dfrac{1}{2}$

Therefore, $0 < g(x) < \dfrac{1}{2}$ for all $x > 0, x \neq 1$

End of solutions