CANDIDATE NUMBER

SYDNEY GRAMMAR SCHOOL

2017 Trial Examination

FORM VI

MATHEMATICS EXTENSION 2

Thursday 10th August 2017

General Instructions

- Reading time — 5 minutes
- Writing time — 3 hours
- Write using black pen.
- Board-approved calculators and templates may be used.

Total — 100 Marks

- All questions may be attempted.

Section I – 10 Marks

- Questions 1–10 are of equal value.
- Record your answers to the multiple choice on the sheet provided.

Section II – 90 Marks

- Questions 11–16 are of equal value.
- All necessary working should be shown.
- Start each question in a new booklet.

Collection

- Write your candidate number on each answer booklet and on your multiple choice answer sheet.
- Hand in the booklets in a single well-ordered pile.
- Hand in a booklet for each question in Section II, even if it has not been attempted.
- If you use a second booklet for a question, place it inside the first.
- Write your candidate number on this question paper and hand it in with your answers.
- Place everything inside the answer booklet for Question Eleven.

Checklist

- SGS booklets — 6 per boy
- Multiple choice answer sheet
- Reference sheet
- Candidature — 72 boys

Examiner

WJM

SECTION I - Multiple Choice

Answers for this section should be recorded on the separate answer sheet handed out with this examination paper.

QUESTION ONE

If $z = 4 + 2i$ and $w = 1 - i$, what is the value of $\dfrac{z}{w}$?

(A) $1 + 3i$

(B) $3 + 3i$

(C) $2 - 6i$

(D) $2 + 6i$

QUESTION TWO

What is the eccentricity of the hyperbola $3x^2 - y^2 = 24$?

(A) $\sqrt{2}$

(B) $\dfrac{2\sqrt{3}}{3}$

(C) 4

(D) 2

QUESTION THREE

What is the value of $\displaystyle\int_0^{\frac{\pi}{3}} \tan x \, dx$?

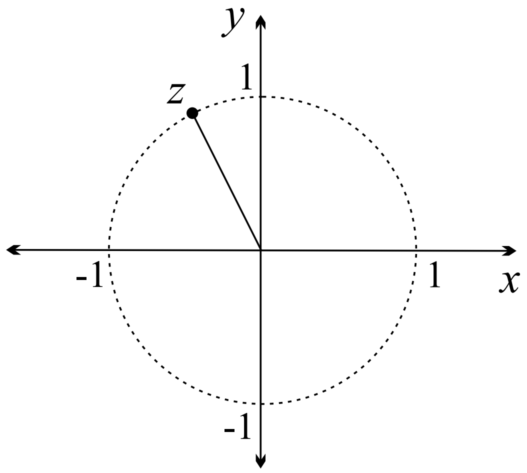
(A) 3

(B) $-\ln 2$

(C) $\ln 3$

(D) $\ln 2$

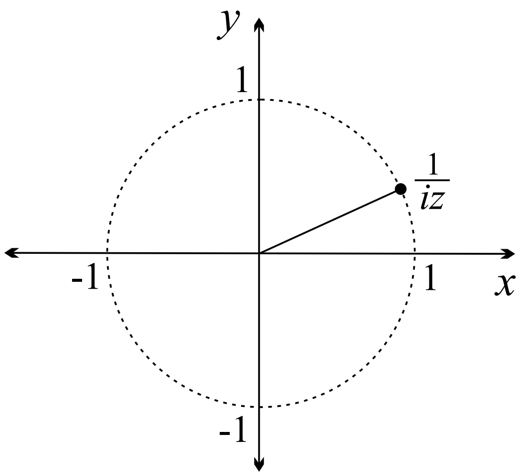
QUESTION FOUR



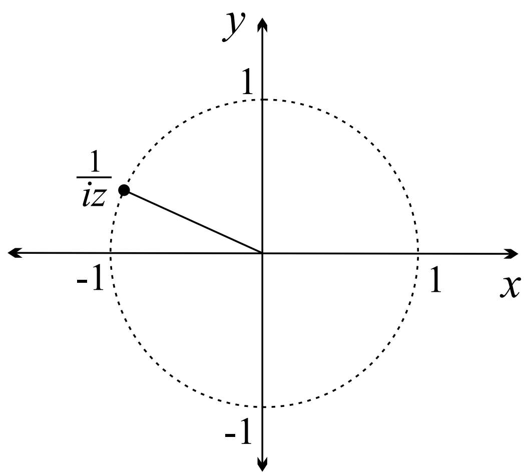
The complex number z is shown on the Argand diagram above.

Which of the following best represents the complex number $\dfrac{1}{iz}$?

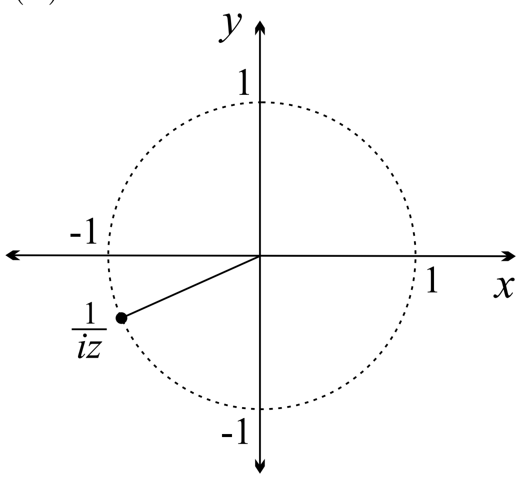
(A)



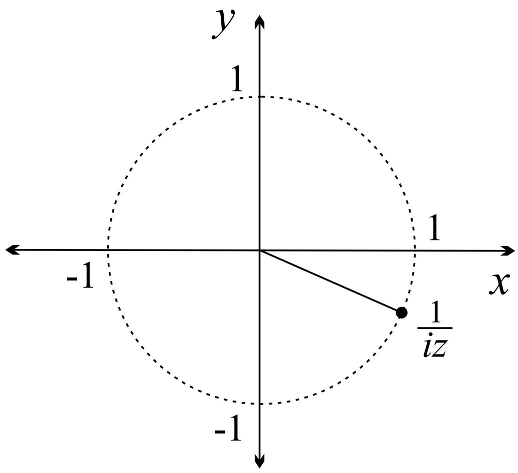
(B)



(C)



(D)



Examination continues overleaf ...

QUESTION FIVE

A skydiver jumps from a helicopter and accelerates toward the ground. It is known that when she opens her parachute, her equation of motion becomes

$$\ddot{x} = 10 - \frac{5v^2}{32}\,,$$

where v is the velocity of the skydiver and downwards is taken as positive.

The skydiver reaches $8\,\mathrm{ms}^{-1}$ when she opens her parachute.

Which of the following statements is TRUE after she opens her parachute?

(A) The skydiver's velocity will decrease.

(B) The skydiver's velocity will remain the same.

(C) The skydiver's velocity will increase.

(D) In order to analyse velocity, the mass of the skydiver must be known.

QUESTION SIX

When the polynomial $P(x)$ is divided by $x^2 + 9$, the remainder is $2x - 5$. What is the remainder when $P(x)$ is divided by $x - 3i$?

(A) $-18 + 15i$

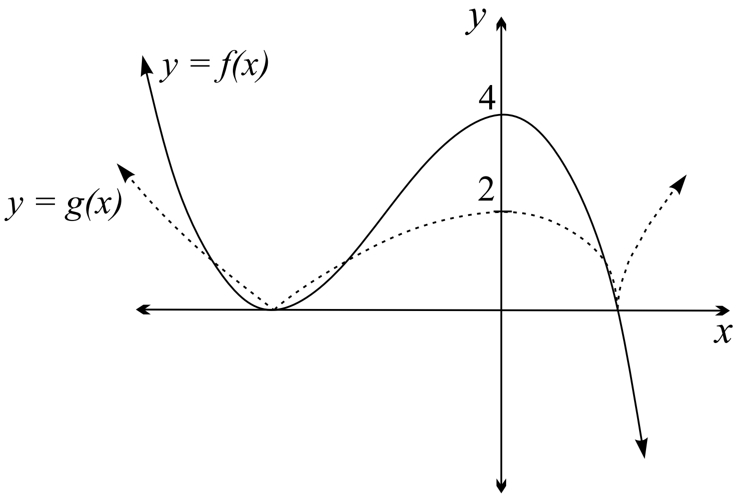
(B) $-18 - 15i$

(C) $-5 + 6i$

(D) $-5 - 6i$

Examination continues next page ...

QUESTION SEVEN



The diagram above shows the graphs of the functions $y = f(x)$ and $y = g(x)$. Which of the following could represent the relationship between $f(x)$ and $g(x)$?

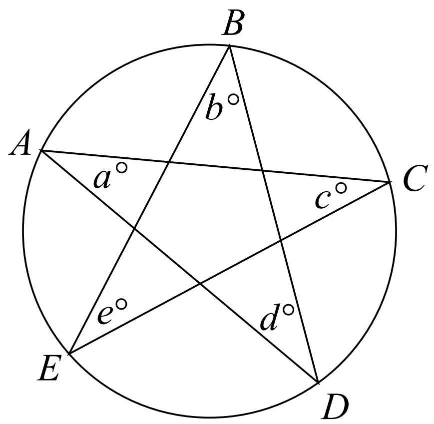
(A) $g(x) = \frac{1}{2}|f(x)|$

(B) $g(x) = \sqrt{f(x)}$

(C) $g(x) = \sqrt{|f(x)|}$

(D) $\big(g(x)\big)^2 = f(x)$

QUESTION EIGHT



The diagram above shows the five points A, B, C, D and E on the circumference of a circle. $\angle DAC = a^\circ$, $\angle EBD = b^\circ$, $\angle ACE = c^\circ$, $\angle BDA = d^\circ$, and $\angle CEB = e^\circ$.

Which of the following **must** be true?

(A) $a^\circ = b^\circ = c^\circ = d^\circ = e^\circ$

(B) $a^\circ + b^\circ + c^\circ + d^\circ + e^\circ = 180^\circ$

(C) $a^\circ + b^\circ + c^\circ + d^\circ + e^\circ = 270^\circ$

(D) $a^\circ + b^\circ + c^\circ - d^\circ - e^\circ = 90^\circ$

Examination continues overleaf ...

QUESTION NINE

A complex number z is defined such that $|z - 2ik| \leq k$, where k is real and positive.
If $-\pi < \arg(z) \leq \pi$, what is the maximum value of $\arg(z)$?

(A) $\dfrac{\pi}{6}$

(B) $\dfrac{\pi}{3}$

(C) $\dfrac{2\pi}{3}$

(D) $\dfrac{5\pi}{6}$

QUESTION TEN

The function $f(x)$ is odd and continuous. Given that $\displaystyle\int_0^a f(x)\,dx = b$, what is the value of $\displaystyle\int_0^a \Big(f(x-a) - f(a-x)\Big)\,dx$?

(A) 0

(B) b

(C) $2b$

(D) $-2b$

End of Section I

Examination continues next page ...

SECTION II - Written Response

Answers for this section should be recorded in the booklets provided.

Show all necessary working.

Start a new booklet for each question.

QUESTION ELEVEN (15 marks) Use a separate writing booklet. **Marks**

(a) Solve the quadratic equation $z^2 - 2iz + 3 = 0$. **2**

(b) Find:

(i) $\displaystyle\int xe^x\,dx$ **2**

(ii) $\displaystyle\int \frac{1}{x(\ln x)^2}\,dx$ **2**

(c) Given that $z = 1 - i\sqrt{3}$:

(i) Express z in modulus–argument form. **1**

(ii) Find z^6. **2**

(d) (i) Find the constants A, B and C such that **2**

$$\frac{3x^2 - 2x - 8}{(x^2+4)(x-3)} = \frac{Ax+B}{x^2+4} + \frac{C}{x-3}\,.$$

(ii) Hence find $\displaystyle\int \frac{3x^2 - 2x - 8}{(x^2+4)(x-3)}\,dx$. **2**

(e) A curve is implicitly defined by $x^3 + y^3 = x^2y^2$. **2**

Find an expression for $\dfrac{dy}{dx}$ in terms of x and y.

Examination continues overleaf ...

QUESTION TWELVE (15 marks) Use a separate writing booklet. **Marks**

(a) (i) Expand $(1+i)(1+2i)(1+3i)$. **1**

(ii) Hence show that $\tan^{-1}(2)+\tan^{-1}(3)=\dfrac{3\pi}{4}$. **2**

(b) (i) Sketch the ellipse $\dfrac{x^2}{16}+\dfrac{y^2}{12}=1$, clearly showing both foci, both directrices, and any intercepts with the axes. **3**

(ii) Find the equation of the tangent to the ellipse at $P(2,\,3)$. **2**

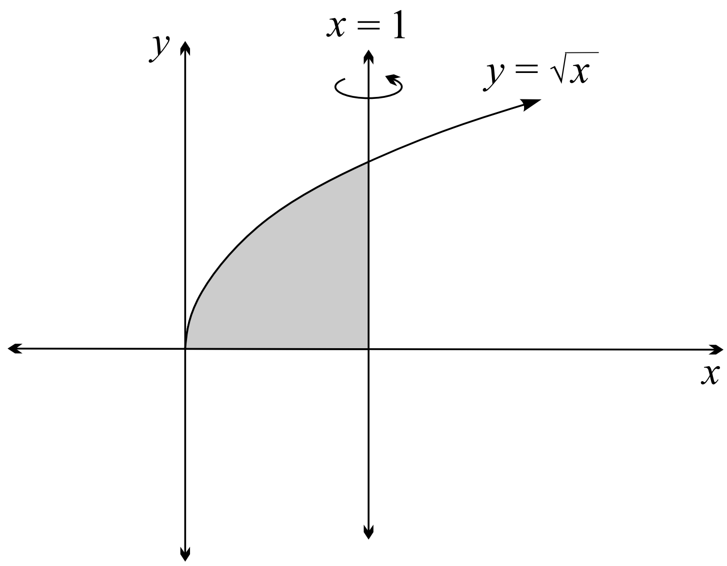
(iii) Show that the tangent at P and the x-axis intersect on one of the directrices of the ellipse. **1**

(c) Sketch the region in the complex plane which simultaneously satisfies **3**

$$\frac{\pi}{2}\le \arg(z)\le\frac{3\pi}{4}\quad\text{and}\quad |z|\le 2.$$

Clearly label the coordinates of any corners of the region, indicating if they are included in the region.

(d)



The diagram above shows the region bound by the curve $y=\sqrt{x}$, the x-axis, and the line $x=1$. This region is rotated about the line $x=1$ to form a solid. **3**

Use the method of cylindrical shells to find the volume of the solid.

Examination continues next page ...

QUESTION THIRTEEN (15 marks) Use a separate writing booklet. **Marks**

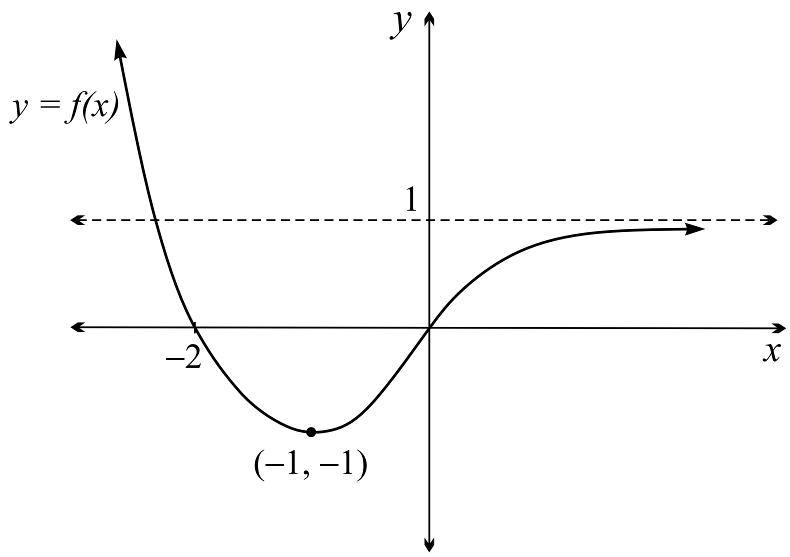
(a) The polynomial $P(x) = x^3 - 9x^2 + 11x + 21$ has zeroes α, β and γ.

(i) Find a simplified polynomial with zeroes $\alpha + 1$, $\beta + 1$ and $\gamma + 1$. **2**

(ii) Hence fully factorise $P(x)$. **2**

(b) Given that n is an integer, simplify $(1 + i)^{8n} + (1 - i)^{8n}$. **2**

(c)



The diagram above shows the graph of $y = f(x)$.

Copy or trace the graph onto three separate number planes, each one third of a page. Use your diagrams to sketch following graphs, clearly showing any intercepts with axes, turning points, and asymptotes.

(i) $y = f\left(|x|\right)$ **1**

(ii) $y = \left[f(x)\right]^2$ **2**

(iii) $y = e^{f(x)}$ **2**

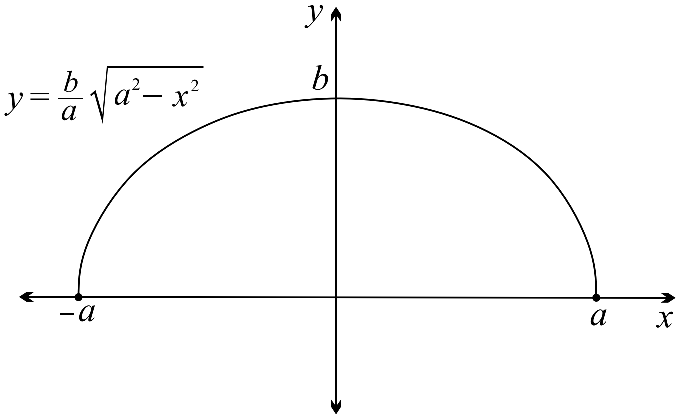
(d) Let $I_n = \displaystyle\int_1^e x^2 (\ln x)^n \, dx$, where n is an integer and $n \geq 0$.

(i) Show that $I_n = \frac{1}{3}\left(e^3 - nI_{n-1}\right)$ for $n \geq 1$. **2**

(ii) Hence find $\displaystyle\int_1^e x^2 \ln x \, dx$. **2**

QUESTION FOURTEEN (15 marks) Use a separate writing booklet. **Marks**

(a)



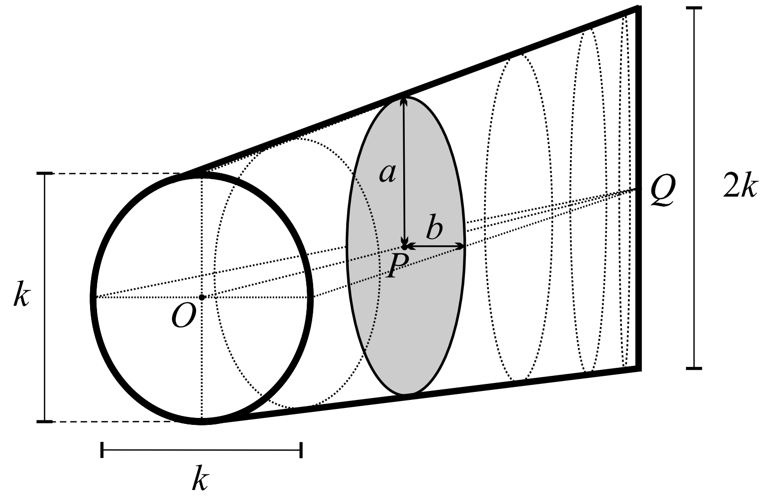
The diagram above shows the graph $y = \dfrac{b}{a}\sqrt{a^2 - x^2}$, that is, the section of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ where $y \geq 0$.

(i) Write down the value of $\displaystyle\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$. **1**

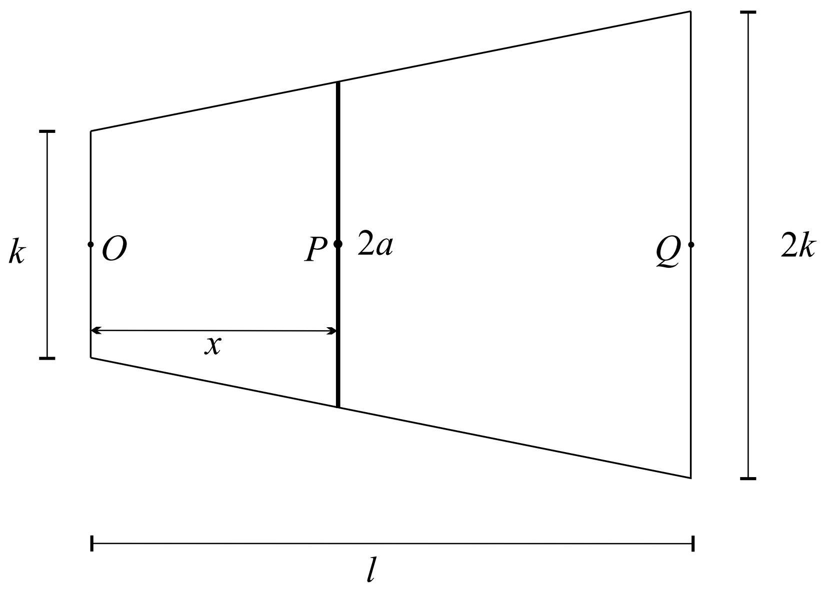
(ii) Deduce that the area of the ellipse $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ is πab units^2. **1**

QUESTION FOURTEEN (Continued)

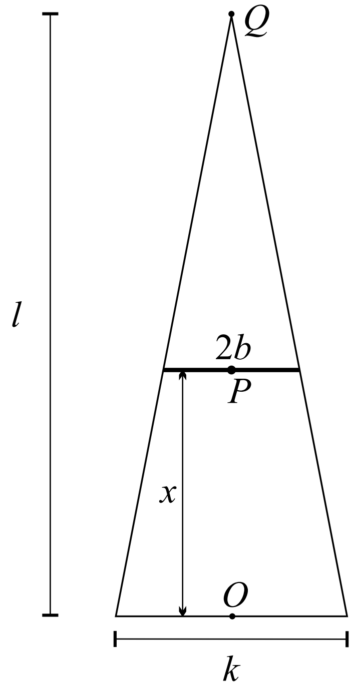
(b)



View from side:



View from above:



The diagram above represents a three-dimensional solid. The front-most face is a circle with centre O and diameter k, while the back of the solid is a straight edge of height $2k$. The point Q is the midpoint of the straight edge, and the solid has length l such that $OQ = l$. At a distance of x units from the circular face, a cross-section shaded grey is shown. The cross-section is an ellipse with centre P, such that $OP = x$. The semi-major axis length of the ellipse is a and the semi-minor axis length is b.

(i) Show that $a = \dfrac{k(l+x)}{2l}$. **2**

(ii) Find a similar expression for b. **1**

(iii) Use the result from (a) to find the volume of the solid. **3**

Examination continues overleaf ...

QUESTION FOURTEEN (Continued)

(c) Let α, β and γ be the distinct roots of the cubic equation $x^3 + bx^2 + cx - 216 = 0$, where b and c are real.

It is known that $\alpha^2 + \beta^2 = 0$ and $\alpha^2 + \gamma^2 = 0$.

(i) Show that $\beta + \gamma = 0$. **2**

(ii) Deduce that α is real. **1**

(iii) Explain why β and γ are both purely imaginary. **1**

(iv) Find b and c. **3**

Examination continues next page ...

QUESTION FIFTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) A stone with mass m kg is dropped from the top of a cliff. As the stone falls, it experiences a force due to gravity of $10m$ Newtons and air resistance of magnitude mkv Newtons, where v is the velocity of the stone in metres per second and k is a positive constant. Let the vertical displacement of the stone from the top of the cliff be y metres, such that

$$m\ddot{y} = 10m - mkv\,,$$

where the downwards direction is positive.

(i) Find v_T, the terminal velocity of the stone. **1**

(ii) Let t be the time after the stone is dropped in seconds. **2**

Show that $t = \dfrac{1}{k}\ln\left|\dfrac{10}{10-kv}\right|$.

(iii) Hence show that $v = \dfrac{10}{k}\left(1 - e^{-kt}\right)$. **1**

(iv) Use the result above to show that $y = \dfrac{10}{k}\left(t + \dfrac{1}{k}\left(e^{-kt} - 1\right)\right)$. **2**

(v) Five seconds after the first stone is dropped, an identical stone is thrown downward from the top of the cliff with a velocity of $\dfrac{15}{k}$ ms^{-1}. It can be shown that the displacement of the second stone is given by

$$y = \frac{5}{k}\left(2t - 10 - \frac{1}{k}\left(e^{-k(t-5)} - 1\right)\right).\text{(Do NOT prove this.)}$$

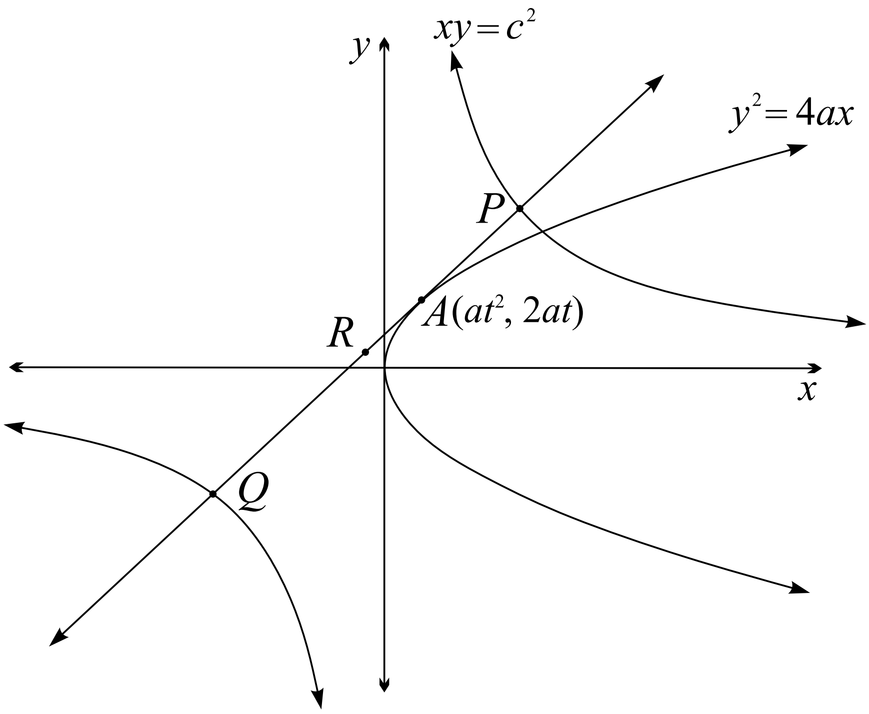
Note that the first stone is dropped when $t = 0$, and the second stone is thrown downward when $t = 5$.

(α) Describe the behaviour of the velocity of the second stone after it is thrown downwards. **1**

(β) Assuming that the cliff is sufficiently high, show that the second stone will only catch up to the first stone if $0 < k < \dfrac{3}{10}$. **3**

QUESTION FIFTEEN (Continued)

(b)



The diagram above shows the hyperbola $xy = c^2$ and the parabola $y^2 = 4ax$, where c and a are positive. The tangent to the parabola at the point $A(at^2, 2at)$ cuts the hyperbola at two distinct points P and Q. The diagram shows the situation when A is in the first quadrant. The midpoint of PQ is R. The tangent to the parabola at the point A is given by $x = yt - at^2$. (Do NOT prove this.)

(i) Find the coordinates of R. **2**

(ii) Show that R always lies on a fixed parabola, and find its equation. **1**

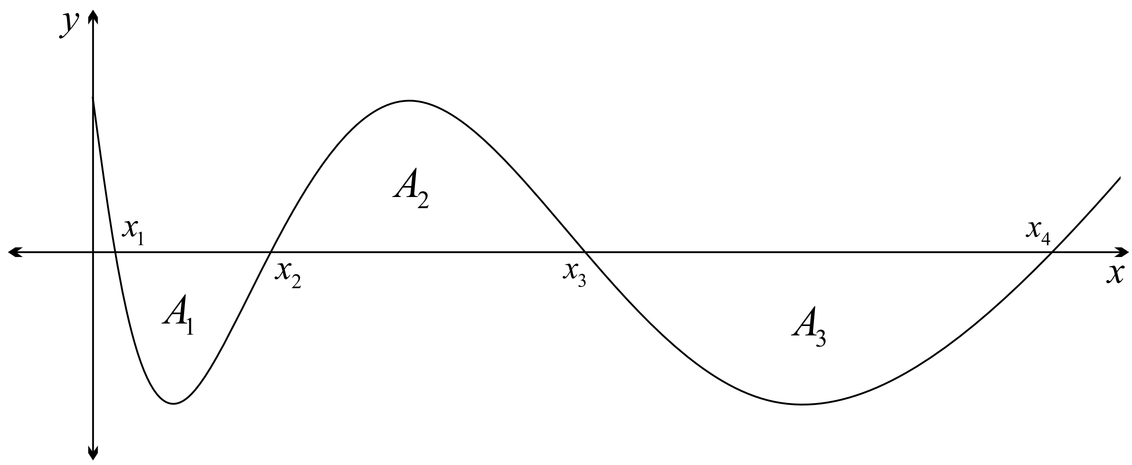
(iii) State any restrictions on the range of y-values that R can take. **2**

QUESTION SIXTEEN (15 marks) Use a separate writing booklet. **Marks**

(a) Use a suitable substitution to show that **2**

$$\int \cos\sqrt{x}\, dx = 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C.$$

(b)



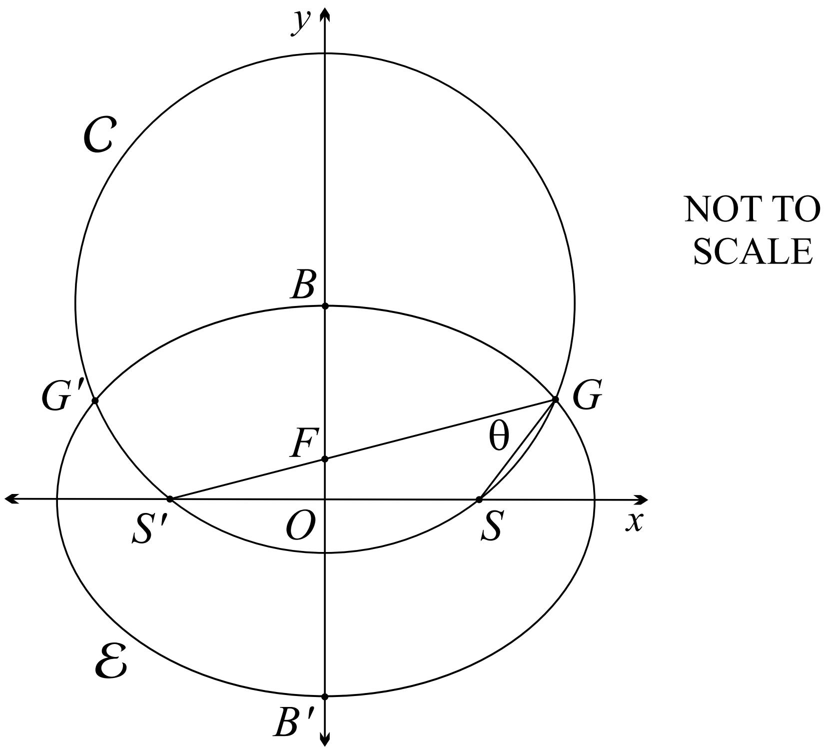
The diagram above shows the graph of $y = \cos\sqrt{x}$. The kth x-intercept of the graph is denoted by x_k, where k is a positive integer. The areas bounded by the curve and the x-axis are denoted by A_1, A_2, A_3, etc., as shown in the diagram above.

(i) Write down the value of x_k in terms of k. **1**

(ii) Use your answer to (a) to find the area of A_k, and hence show that the areas bounded by the curve and the x-axis form an arithmetic progression. **3**

QUESTION SIXTEEN (Continued)

(c)



The diagram above shows the ellipse $\mathcal{E}$ with equation $\dfrac{x^2}{a^2} + \dfrac{y^2}{b^2} = 1$ and foci S and S'. The point B has coordinates $(0, b)$, and a circle $\mathcal{C}$ with centre B is constructed that intersects the x-axis at S and S'. The circle and ellipse intersect at G and G'. The interval from S' to G intersects the y-axis at F, and $\angle SGS' = \theta$.

(i) Show that $BFSG$ is a cyclic quadrilateral. **3**

(ii) Show that $\cos\theta = \dfrac{b}{a}$. **1**

(iii) Suppose that for the ellipse $\mathcal{E}$, $S'B \,||\, SG$.

(α) Show that $S'G$ bisects $\angle BGS$. **2**

(β) Show that $S'G = 2b$. **1**

(γ) Use the geometric properties of an ellipse to find the exact value of the eccentricity of $\mathcal{E}$. **2**

End of Section II

END OF EXAMINATION

Extension 2 Maths Trial 2017

Multiple Choice

① $\dfrac{4+2i}{1-i} \times \dfrac{1+i}{1+i} = \dfrac{4+4i+2i-2}{1+1}$

$= 1+3i \Rightarrow$ (A)

② $3x^2 - y^2 = 24$

$\dfrac{x^2}{8} - \dfrac{y^2}{24} = 1$

$24 = 8(e^2-1)$

$3 = e^2 - 1$

$e^2 = 4$

$e = 2 \Rightarrow$ (D)

③ $\displaystyle\int_0^{\frac{\pi}{3}} \tan x \, dx = \int_0^{\frac{\pi}{3}} \frac{\sin x}{\cos x} \, dx$

$= \left[-\ln|\cos x|\right]_0^{\frac{\pi}{3}}$

$= -\ln(\cos\frac{\pi}{3}) + \ln(\cos 0)$

$= -\ln(\frac{1}{2}) + \ln 1$

$= \ln 2 \Rightarrow$ (D)

④ iz : Rotation of z anti-clockwise by $\frac{\pi}{2}$.

Since $|iz| = 1$, $\dfrac{1}{iz} = \overline{iz}$

$\Rightarrow$ (B)

(5) $\ddot{x} = 10 - \frac{5v^2}{32}$

When $v = 8$: $\ddot{x} = 10 - \frac{5 \times 8^2}{32}$

$= 0 \Rightarrow$ (B)

(6) $\frac{P(x)}{x^2+9} = Q(x) + \frac{2x-5}{x^2+9}$

$P(x) = Q(x)(x^2+9) + 2x - 5$

$P(3i) = 0 + 2 \times 3i - 5$

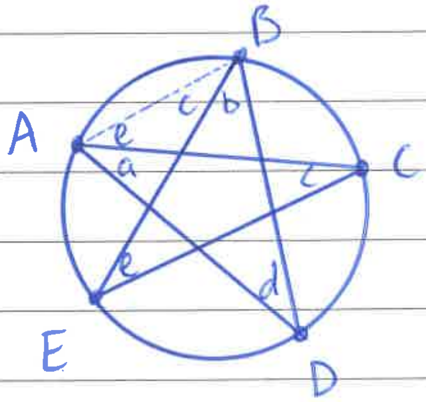
$= 6i - 5 \Rightarrow$ (C)

(7) * $g(x)$ defined when $f(x) < 0$

* $g(x) \geq 0$ for all x

$\Rightarrow$ (C)

(8)



Construct AB.

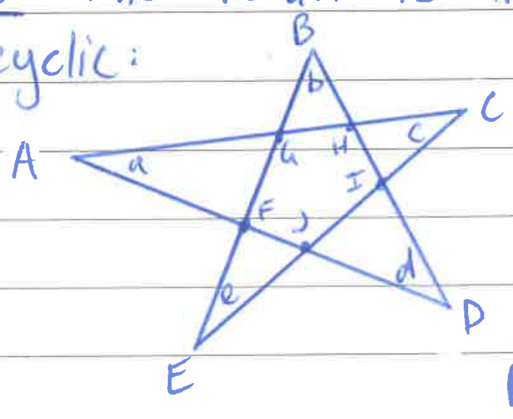
$\angle ABE = c^\circ$ ($\angle$s subtended by same arc on circumference)

Similarly, $\angle BAC = e^\circ$

$\angle$ sum of $\triangle ABD$:

$a^\circ + b^\circ + c^\circ + d^\circ + e^\circ = 180^\circ \Rightarrow$ (B)

<u>Note:</u> This result is independent of A, B, C, D, E being concyclic:



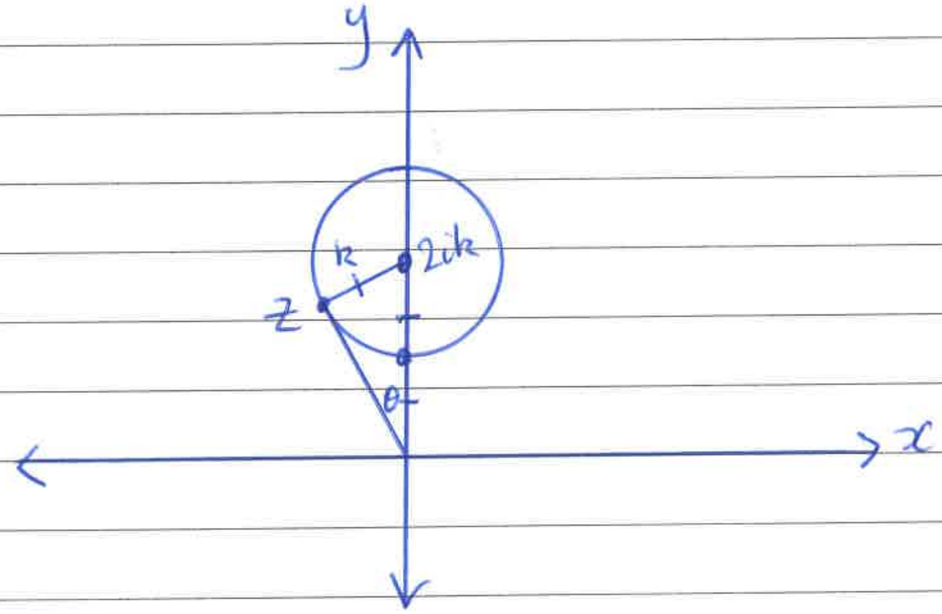
$\angle AGF = c + e$ (exterior $\angle$ = sum of opposite interior $\angle$s of $\triangle GEC$)

Similarly, $\angle AFG = b + d$

By $\angle$ sum of $\triangle AGF$: $a + (c+e) + (b+d) = 180^\circ$

(9) $|z - 2ik| \le k$

* Inside of a circle, radius k, centre $2ik$.



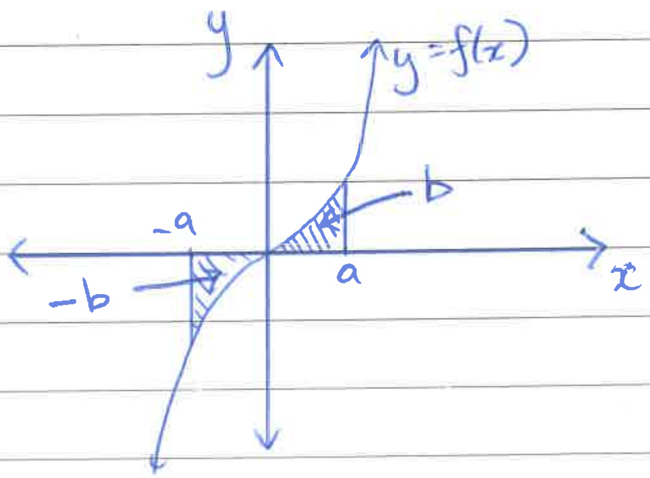
$$\theta = \sin^{-1}\left(\tfrac{1}{2}\right)$$
$$= \frac{\pi}{6}$$

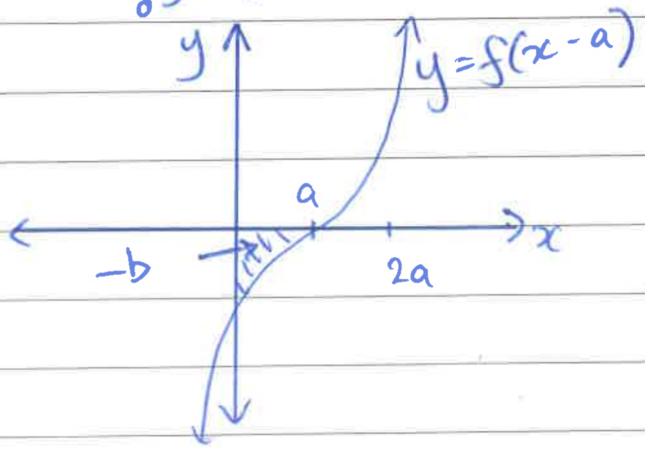
$\therefore$ max. $\arg(z) = \frac{\pi}{2} + \frac{\pi}{6}$

$= \frac{2\pi}{3} \Rightarrow$ (C)

(10) Since $f(x)$ is odd:

$$\int_0^a \big(f(x-a) - f(a-x)\big)\,dx = \int_0^a \big(f(x-a) + f(x-a)\big)\,dx$$
$$= 2\int_0^a f(x-a)\,dx$$





$$\therefore 2\int_0^a f(x-a)\,dx = 2 \times -b$$
$$= -2b \Rightarrow \text{(D)}$$

Question 11

(a) $z^2 - 2iz + 3 = 0$

$(z-i)^2 = -3 + i^2$ ✓ (or correct use of quadratic formula)

$= -4$

$z - i = \pm 2i$

$z = i \pm 2i$

$= -i$ or $3i$ ✓

(b) (i) $\int xe^x\,dx = \int x \cdot \frac{d}{dx}(e^x)\,dx$

$= xe^x - \int e^x \cdot 1\,dx$ ✓ (Applying IBP)

$= xe^x - e^x + c$ ✓

(ii) $\int \frac{1}{x(\ln x)^2}\,dx$ let $u = \ln x$

$du = \frac{dx}{x}$

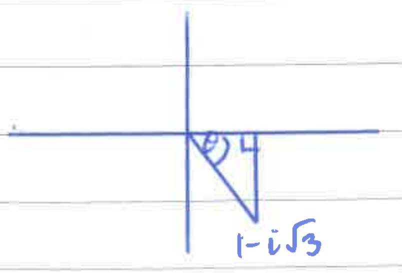
$= \int \frac{du}{u^2}$ ✓ (or correct use of reverse chain rule)

$= \frac{u^{-1}}{-1} + c$

$= -\frac{1}{\ln x} + c$ ✓

(c) (i) $|1 - i\sqrt{3}| = \sqrt{1^2 + (\sqrt{3})^2}$

$= 2$



$\theta = \tan^{-1}(\sqrt{3})$

$= \frac{\pi}{3}$ ✓

$\therefore \arg(z) = -\frac{\pi}{3}$ $\therefore 1 - i\sqrt{3} = 2\text{cis}\left(-\frac{\pi}{3}\right)$

(ii) $z^6 = \left(2\operatorname{cis}\left(-\frac{\pi}{3}\right)\right)^6$

$= 2^6 \operatorname{cis}\left(-\frac{6\pi}{3}\right)$ ✓ by De Moivre's theorem

$= 64 \operatorname{cis}(-2\pi)$

$= 64$ ✓

(d) (i) $\dfrac{3x^2 - 2x - 8}{(x^2+4)(x-3)} = \dfrac{Ax + B}{x^2+4} + \dfrac{C}{x-3}$

$3x^2 - 2x - 8 = (Ax+B)(x-3) + C(x^2+4)$

let $x = 3$:

$3\times 9 - 2\times 3 - 8 = C(9+4)$

$13 = 13C$

$C = 1$ ✓ (Any correct value)

Equate coef. of x^2: $3 = A + C$

$= A + 1$

$A = 2$

let $x = 0$: $-8 = -3B + 4C$

$-8 = -3B + 4$

$3B = 12$

$B = 4$ ✓

(ii) $\displaystyle\int \frac{3x^2 - 2x - 8}{(x^2+4)(x-3)}\,dx = \int\left(\frac{2x+4}{x^2+4} + \frac{1}{x-3}\right)dx$

$\displaystyle = \int\left(\frac{2x}{x^2+4} + \frac{4}{x^2+4} + \frac{1}{x-3}\right)dx$ ✓ (any correct)

$= \ln(x^2+4) + 4\times\frac{1}{2}\tan^{-1}\left(\frac{x}{2}\right) + \ln|x-3| + C$

$= \ln(x^2+4) + 2\tan^{-1}\left(\frac{x}{2}\right) + \ln|x-3| + C$ ✓

(e) $x^3 + y^3 = x^2y^2$

$$3x^2 + 3y^2\frac{dy}{dx} = 2xy^2 + 2yx^2 \cdot \frac{dy}{dx}$$

$$\frac{dy}{dx}(3y^2 - 2x^2y) = 2xy^2 - 3x^2$$

$$\frac{dy}{dx} = \frac{2xy^2 - 3x^2}{3y^2 - 2x^2y}$$

Question 12

(a) (i) $(1+i)(1+2i)(1+3i) = (1+3i-2)(1+3i)$

$= (-1+3i)(1+3i)$

$= -1+9i^2$

$= -10$ ✓

(ii) $\arg[(1+i)(1+2i)(1+3i)] = \arg(-10)$ ✓

$\therefore \arg(1+i) + \arg(1+2i) + \arg(1+3i) = \pi$

$\tan^{-1}(1) + \tan^{-1}(2) + \tan^{-1}(3) = \pi$

$\therefore \tan^{-1}(2) + \tan^{-1}(3) = \pi - \frac{\pi}{4}$

$= \frac{3\pi}{4}$ ✓

(b) (i) $\frac{x^2}{16} + \frac{y^2}{12} = 1$

$a^2 = 16 \qquad b^2 = 12$

$a = 4 \qquad b = 2\sqrt{3}$

$12 = 16(1-e^2)$

$1-e^2 = \frac{3}{4}$

$e^2 = \frac{1}{4}$

$e = \frac{1}{2}$ ✓

$\therefore$ Foci: $(\pm ae, 0) \rightarrow (\pm 4\times\frac{1}{2}, 0)$

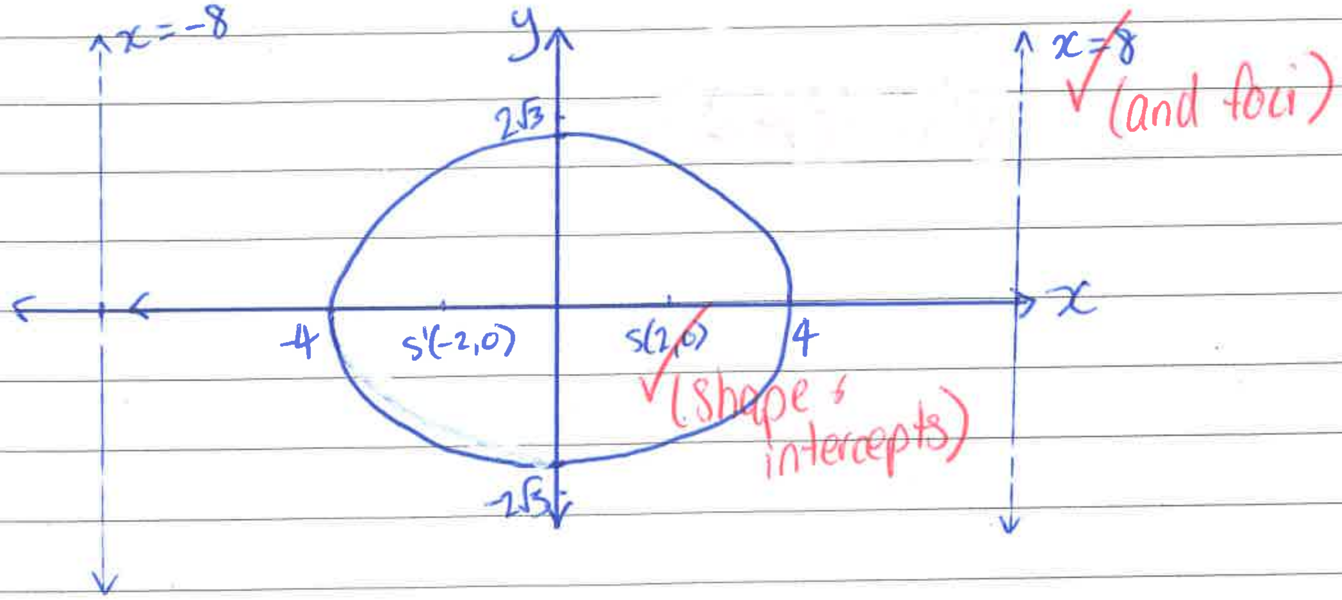
$(\pm 2, 0)$

Directrices: $x = \pm\frac{a}{e}$

$= \pm\frac{4}{\frac{1}{2}}$

$= \pm 8$

For this question, 3-Unit methods were awarded 1 mark. Full marks could not be achieved without incorporating the result from (a)(i).



(ii) $\dfrac{x^2}{16} + \dfrac{y^2}{12} = 1$

$\dfrac{2x}{16} + \dfrac{2y}{12} \cdot \dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx} = -\dfrac{x}{8} \times \dfrac{6}{y}$

$= -\dfrac{3x}{4y}$ ✓

at $P(2,3)$, $\dfrac{dy}{dx} = \dfrac{-3\times 2}{4\times 3}$

$= -\frac{1}{2}$

$\therefore$ tangent has eqn: $y - 3 = -\frac{1}{2}(x-2)$ ✓

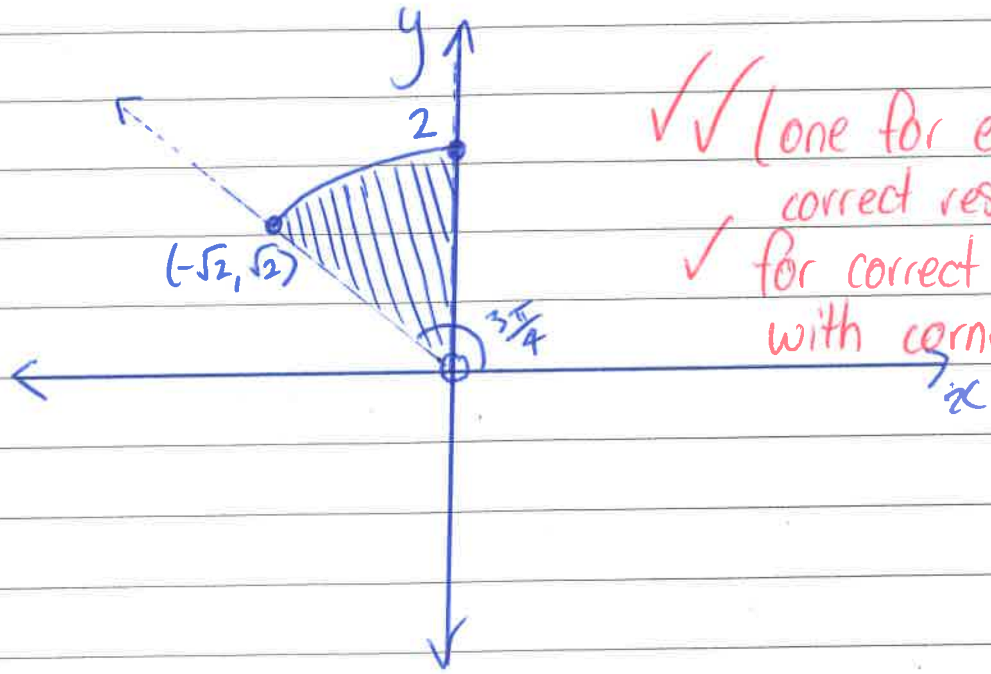
$2y - 6 = -x + 2$

$x + 2y = 8$

(iii) When $y = 0$, $x = 8$.

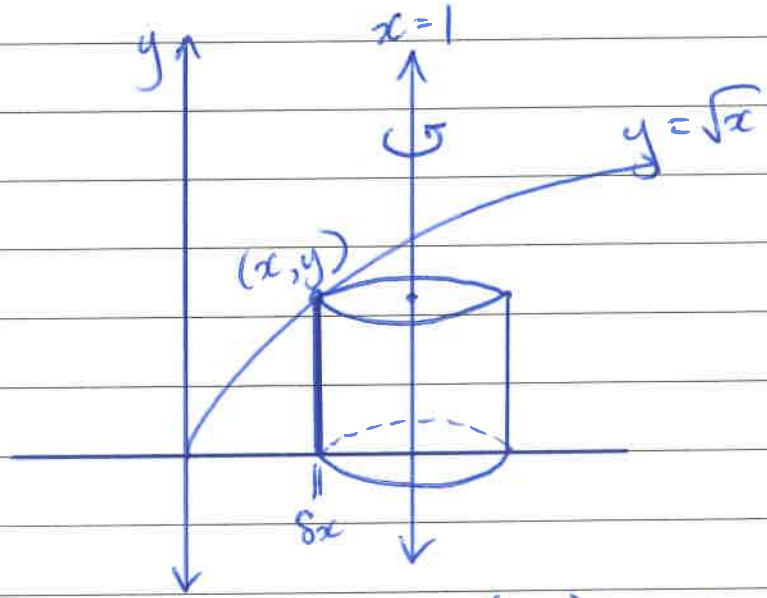
$\therefore$ tangent cuts x-axis at $(8, 0)$, which is on the directrix $x = 8$. ✓

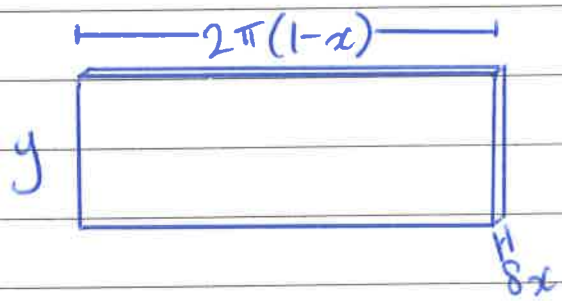
(c)



✓✓ (one for each correct restriction)
✓ for correct intersection with corners

(d)





$$\delta V = 2\pi(1-x).y.\delta x$$

$$= 2\pi(1-x).\sqrt{x}\,\delta x$$

$$V = 2\pi \int_0^1 \left(x^{\frac{1}{2}} - x^{3/2}\right) dx$$

$$= 2\pi\left[\frac{2x^{3/2}}{3} - \frac{2x^{5/2}}{5}\right]_0^1$$

$$= 2\pi\left[\left(\frac{2}{3} - \frac{2}{5}\right) - 0\right]$$

$$= 2\pi \times \frac{4}{15}$$

$$= \frac{8\pi}{15} \text{ units}^3$$

Question 13

(a) $P(x) = x^3 - 9x^2 + 11x + 21$

(i) Let $y = x + 1$

$\Rightarrow \quad x = y - 1$

$\therefore$ A polynomial with zeroes $\alpha+1, \beta+1, \gamma+1$ is:

$(y-1)^3 - 9(y-1)^2 + 11(y-1) + 21$

$= y^3 - 3y^2 + 3y - 1 - 9y^2 + 18y - 9 + 11y - 11 + 21$

$= y^3 - 12y^2 + 32y$

So a polynomial with zeroes $\alpha+1, \beta+1, \gamma+1$ can be written: $Q(x) = x^3 - 12x^2 + 32x$

(ii) $Q(x) = x(x^2 - 12x + 32)$

$= x(x-4)(x-8)$

$\therefore \quad \alpha + 1 = 0, \quad \beta + 1 = 4, \quad \gamma + 1 = 8$

$\alpha = -1, \quad \beta = 3, \quad \gamma = 7$

$\therefore \quad P(x) = (x+1)(x-3)(x-7)$

(b) $1 + i = \sqrt{2} \operatorname{cis} \frac{\pi}{4}$, $\quad 1 - i = \sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)$

$(1+i)^{8n} + (1-i)^{8n} = \left(\sqrt{2} \operatorname{cis} \frac{\pi}{4}\right)^{8n} + \left(\sqrt{2} \operatorname{cis}\left(-\frac{\pi}{4}\right)\right)^{8n}$

$= (\sqrt{2})^{8n}\left(\operatorname{cis} \frac{8n\pi}{4} + \operatorname{cis}\left(-\frac{8n\pi}{4}\right)\right)$ by DeMoivre's theorem

$= 2^{4n}\left(\operatorname{cis}(2n\pi) + \operatorname{cis}(-2n\pi)\right)$

$= 2^{4n} \times (1 + 1)$

$= 2^{4n+1}$

Alternatively:

$$(1+i)^{8n} + (1-i)^{8n} = \left[(1+i)^2\right]^{4n} + \left[(1-i)^2\right]^{4n}$$

$$= [2i]^{4n} + [-2i]^{4n}$$

$$= \left[(2i)^4\right]^n + \left[(-2i)^4\right]^n$$

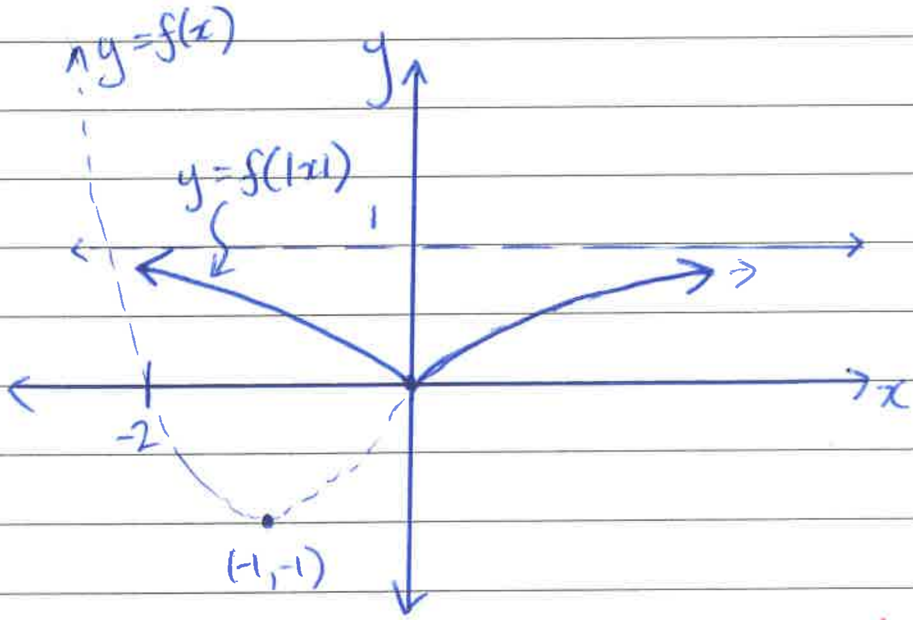
$$= 16^n + 16^n$$

$$= 2 \times 16^n$$

$$= 2 \times (2^4)^n$$

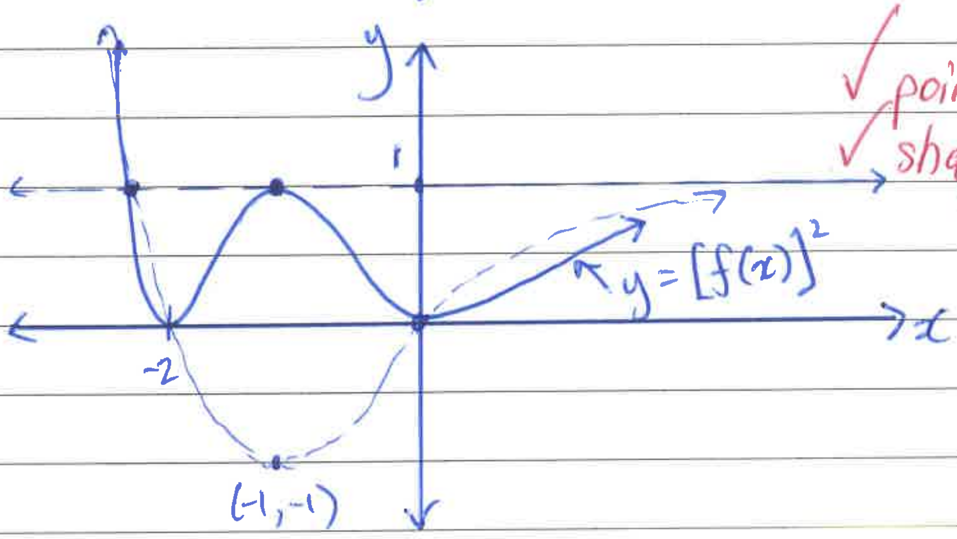
$$= 2^{4n+1}$$

(c)(i)



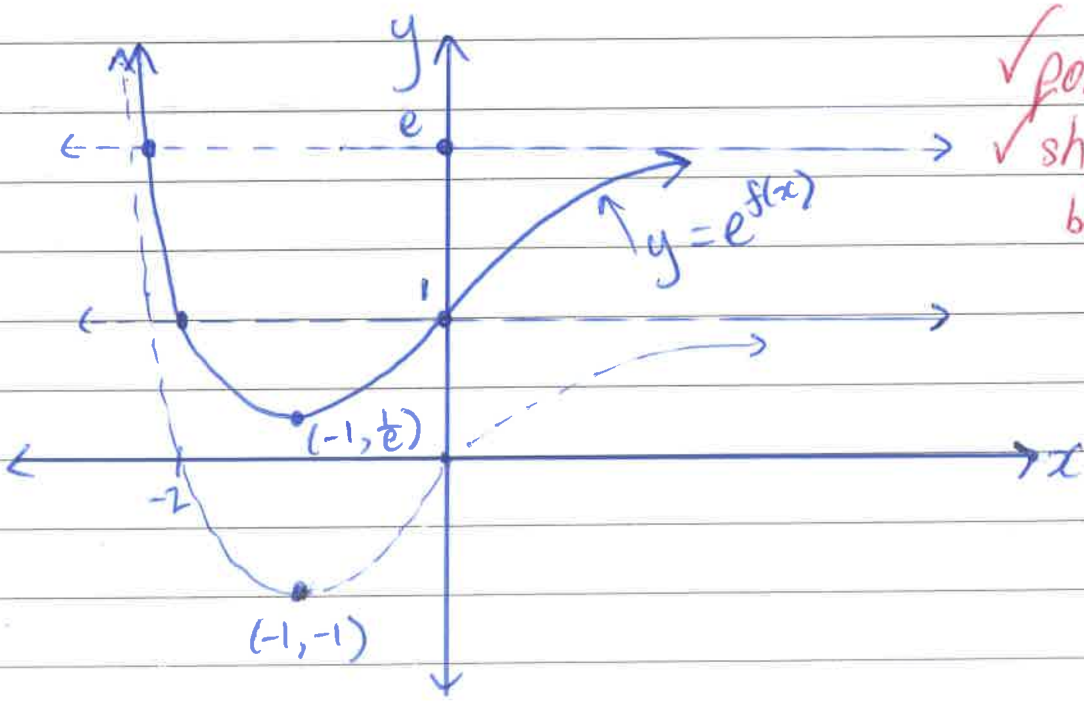
✓

(ii)



✓ points
✓ shape/behaviour

(iii)



✓ points
✓ shape/behaviour

(d) $I_n = \int_1^e x^2 (\ln x)^n \, dx$

(i) $I_n = \int_1^e (\ln x)^n \times \frac{d}{dx}\left(\frac{x^3}{3}\right) dx$

$= \left[\frac{x^3}{3}(\ln x)^n\right]_1^e - \frac{n}{3}\int_1^e (\ln x)^{n-1} \times \frac{1}{x} \times x^3 \, dx$

$= \left(\frac{e^3}{3}(\ln e)^n - 0\right) - \frac{n}{3}\int_1^e x^2 (\ln x)^{n-1} \, dx$

$= \frac{e^3}{3} - \frac{n}{3} I_{n-1}$

$\therefore I_n = \frac{1}{3}(e^3 - n I_{n-1})$

(ii) $\int_1^e x^2 (\ln x) \, dx = I_1$

$I_0 = \int_1^e x^2 \, dx$

$= \left[\frac{x^3}{3}\right]_1^e$

$= \frac{e^3 - 1}{3}$

$I_1 = \frac{1}{3}(e^3 - 1 \times I_0)$

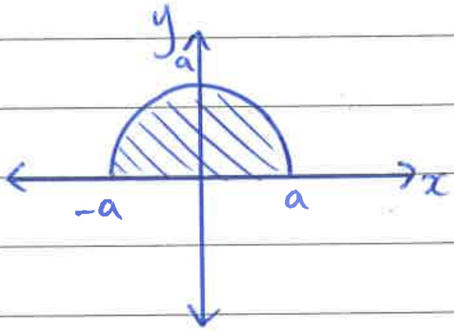
$= \frac{1}{3}\left(e^3 - \left(\frac{e^3 - 1}{3}\right)\right)$

$= \frac{1}{3} \times \frac{2e^3 + 1}{3}$

$= \frac{2e^3 + 1}{9}$

Question 14

(a) (i) $\int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$



$\dfrac{\pi a^2}{2}$

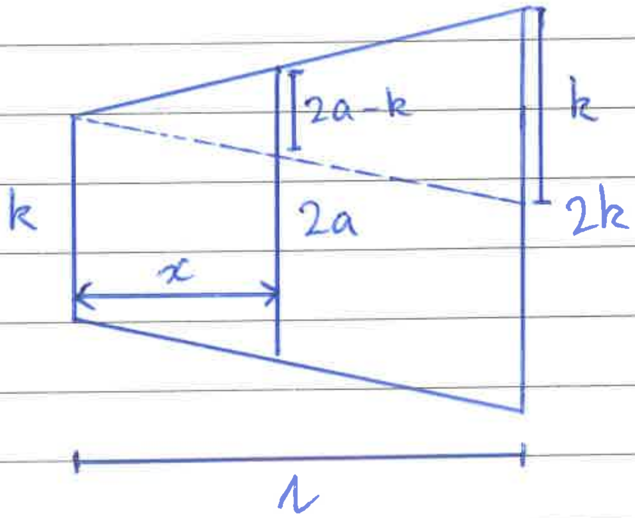
(ii) Area of ellipse $= 2 \times \int_{-a}^{a} \dfrac{b}{a}\sqrt{a^2 - x^2}\, dx$

$= \dfrac{2b}{a} \int_{-a}^{a} \sqrt{a^2 - x^2}\, dx$

$= \dfrac{2b}{a} \times \dfrac{\pi a^2}{2}$

$= \pi ab$ units^2

(b) (i)

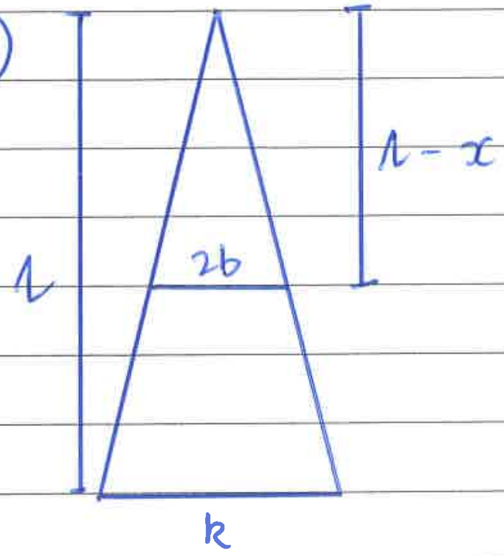


By similarity: $\dfrac{2a-k}{k} = \dfrac{x}{h}$

$2a = k + \dfrac{kx}{h}$

$a = \dfrac{kh + kx}{2h} = \dfrac{k(h+x)}{2h}$

(ii)



By similarity: $\dfrac{2b}{k} = \dfrac{h-x}{h}$

$\therefore \; b = \dfrac{k(h-x)}{2h}$

(iii) From (a)(ii). Area of ellipse $= \pi ab$

$$\therefore \; \delta V = \pi \times \frac{k(h+x)}{2h} \times \frac{k(h-x)}{2h} \times \delta x$$

$$= \frac{\pi k^2}{4h^2}\left(h^2 - x^2\right)\delta x$$

$$V = \frac{\pi k^2}{4h^2}\int_0^h \left(h^2 - x^2\right)dx$$

$$= \frac{\pi k^2}{4h^2}\left[h^2 x - \frac{x^3}{3}\right]_0^h$$

$$= \frac{\pi k^2}{4h^2}\left(h^3 - \frac{h^3}{3}\right)$$

$$= \frac{\pi k^2}{4h^2} \times \frac{2h^3}{3}$$

$$= \frac{\pi k^2 h}{6} \text{ units}^3$$

(c) (i) $\alpha^2 + \beta^2 = 0$ ①

$\alpha^2 + \gamma^2 = 0$ ②

① − ②:

$$\beta^2 - \gamma^2 = 0$$

$$(\beta + \gamma)(\beta - \gamma) = 0$$ ✓

Now β and γ are distinct, $\therefore \beta - \gamma \neq 0$

$\therefore \beta + \gamma = 0$ ✓

(Explanation as to why $\beta - \gamma \neq 0$ required for 2nd mark)

(ii) Sum of roots: $\alpha + (\beta + \gamma) = -b$

$\therefore \quad \alpha = -b$

Since b is real, α is real. ✓

Note: Alternative explanations were accepted here, however responses that made assumptions without justifications were not awarded the mark.

Care should be taken when referring to "real", "imaginary", and "complex" numbers, noting that real numbers and imaginary numbers are both subsets of $\mathbb{C}$.

Example of alternate explanation:

Since $\beta = -\gamma$, β and γ are either both purely real, or both complex. Since the coefficients in the polynomial are real, any complex roots occur in conjugate pairs. Thus the polynomial has either 1 or 3 real roots. In either case, α must be real.

(iii) $\alpha^2 + \beta^2 = 0$

$\beta^2 = -\alpha^2$

Since α is real, $-\alpha^2 < 0$ (0 not a root, so $\alpha \neq 0$)

$\therefore \beta^2 < 0$

$\therefore \beta$ is purely imaginary.

Similarly, $\gamma^2 = -\alpha^2$ and γ is purely imaginary.

Note: Alternative solutions were also accepted here if sufficient justification was provided, however assumptions made without justification were, again, not awarded the mark.

Assertions that $\beta = -\gamma$, $\therefore \beta$ and γ must be imaginary were not sufficient, as this relies on an underlying assumption that β and γ are complex conjugates, which needed its own justification.

(iv) Since $\beta^2 = -\alpha^2$ and $\gamma^2 = -\alpha^2$

$\beta = i\alpha$, $\gamma = -i\alpha$ without loss of generality.

Product of roots: $\alpha \times i\alpha \times -i\alpha = -(-216)$

$\alpha^3 = 216$

$\underline{\alpha = 6}$

From (ii), $\alpha = -b$

$\therefore \underline{b = -6}$

Sum of roots 2 at a time:

$\alpha \cdot i\alpha + \alpha(-i\alpha) + i\alpha \cdot (-i\alpha) = c$

$\alpha^2 = c$

$\therefore \underline{c = 36}$

Question 15

(a) (i) At $v = v_T$, $\ddot{y} = 0$: $10m - mkv_T = 0 \Rightarrow v_T = \frac{10}{k}$

(ii)
$$m\ddot{y} = 10m - mkv$$
$$\ddot{y} = 10 - kv$$
$$\frac{dv}{dt} = 10 - kv$$
$$\frac{dt}{dv} = \frac{1}{10-kv}$$
$$t = -\frac{1}{k}\int \frac{-k\,dv}{10-kv}$$
$$= -\frac{1}{k}\ln|10-kv| + c_1$$

When $t = 0$, $v = 0$ (stone dropped)

$$\therefore c_1 = \frac{1}{k}\ln(10)$$
$$\therefore t = \frac{1}{k}\ln\left|\frac{10}{10-kv}\right|$$

(iii)
$$kt = \ln\left|\frac{10}{10-kv}\right|$$
$$e^{kt} = \frac{10}{10-kv}$$ since initial velocity $= 0$, $10 - kv > 0$
$$10 - kv = 10e^{-kt}$$
$$kv = 10(1 - e^{-kt})$$
$$v = \frac{10}{k}(1 - e^{-kt})$$

(iv)
$$\frac{dy}{dt} = \frac{10}{k}(1 - e^{-kt})$$
$$y = \frac{10}{k}\int (1 - e^{-kt})\,dt$$

$$y = \frac{10}{k}\left(t + \frac{1}{k}e^{-kt}\right) + c_2$$ ✓

When $t=0$, $y=0$

$$\therefore \quad c_2 = -\frac{10}{k}\left(0 + \frac{1}{k}\right)$$
$$= -\frac{10}{k^2}$$

$$\therefore \quad y = \frac{10}{k}\left(t + \frac{1}{k}e^{-kt}\right) - \frac{10}{k^2}$$
$$= \frac{10}{k}\left(t + \frac{1}{k}e^{-kt} - \frac{1}{k}\right)$$
$$= \frac{10}{k}\left(t + \frac{1}{k}\left(e^{-kt} - 1\right)\right)$$ ✓

(V) (α) Since stone thrown downwards with velocity $\frac{15}{k} > V_T$, the velocity of the stone will decrease, ✓ approaching $\frac{10}{k}$ from above.

(β) Stone "catches up" when displacements are equal for a given t.
Let the stone catch up at $t = T$.

$$\underbrace{\frac{10}{k}\left(T + \frac{1}{k}\left(e^{-kT} - 1\right)\right)}_{\text{Displacement of 1st stone after } T \text{ seconds}} = \underbrace{\frac{5}{k}\left(2T - 10 - \frac{1}{k}\left(e^{-k(T-5)} - 1\right)\right)}_{\text{Displacement of 2nd stone after } T \text{ seconds.}}$$ ✓

$$\cancel{2T} + \frac{2}{k}e^{-kT} - \frac{2}{k} = \cancel{2T} - 10 - \frac{1}{k}e^{-kT+5k} + \frac{1}{k}$$

$$\frac{2}{k}e^{-kT} + \frac{1}{k}e^{-kT} \times e^{5k} = \frac{3}{k} - 10$$

$$e^{-kT}\left(2 + e^{5k}\right) = 3 - 10k$$ ✓

$$e^{kT} = \frac{2 + e^{5k}}{3 - 10k}$$

$$T = \frac{1}{k}\ln\left(\frac{2 + e^{5k}}{3 - 10k}\right)$$

Since $2 + e^{5k} > 0$, a solution for T only exists when $3 - 10k > 0$.

$$\Rightarrow k < \frac{3}{10}.$$ ✓ (with brief explanation)

Since we know k is positive:

$$0 < k < \frac{3}{10}$$

(b) (i) $x = ty - at^2$ intersects $xy = c^2$ at P & Q.

Solving simultaneously:

$$y(ty - at^2) = c^2$$

$$ty^2 - at^2y - c^2 = 0$$

As a quadratic in y, let the roots be y_1 and y_2 : The y-values of P and Q.

$$y_1 + y_2 = -\frac{-at^2}{t}$$

$$= at$$

$$\therefore \frac{y_1 + y_2}{2} = \frac{at}{2} \checkmark$$

$$x = t \times \frac{at}{2} - at^2$$

$$= -\frac{at^2}{2} \checkmark$$

$\therefore$ R has coordinates $\left(-\frac{at^2}{2}, \frac{at}{2}\right)$

(ii) $x = -\frac{at^2}{2}$, $y = \frac{at}{2} \longrightarrow t = \frac{2y}{a}$

$$\therefore x = -\frac{a}{2}\left(\frac{2y}{a}\right)^2$$

$$= -\frac{2y^2}{a}$$

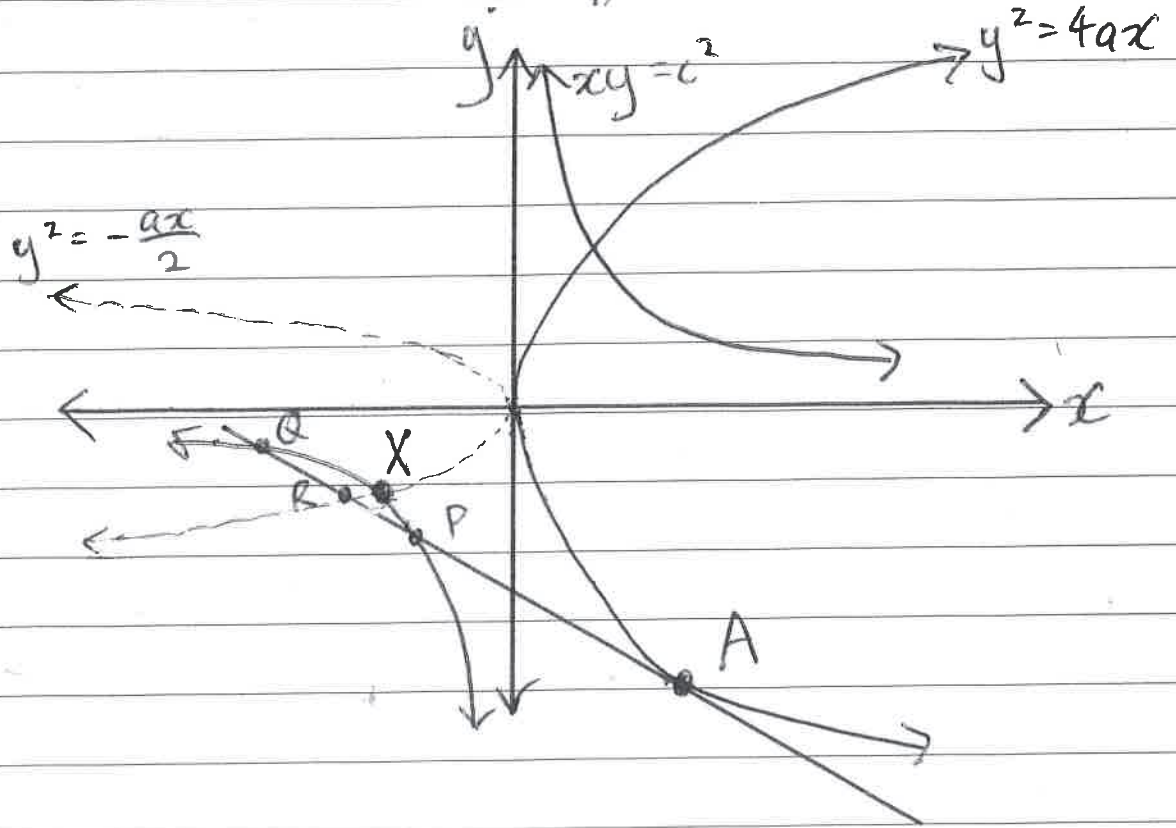
$\therefore$ Without restrictions, the locus of R is

$$y^2 = -\frac{ax}{2} \checkmark$$

Which is a parabola with vertex at the origin, in the 2nd & 3rd quadrants.

(iii) If A is in the first quadrant, $t > 0$.
$R\left(-\frac{at^2}{2}, \frac{at}{2}\right) \Rightarrow y > 0$ ✓ (Justification required)

If A is in the 4th quadrant:



Clearly the y-value of R < y-value of X, where X is the intersection of $y^2 = -\frac{ax}{2}$ and $xy = c^2$

Solving simultaneously: $x = \frac{c^2}{y}$

$$y^2 = -\frac{a}{2} \times \frac{c^2}{y}$$
$$y^3 = -\frac{ac^2}{2}$$
$$y = -\sqrt[3]{\frac{ac^2}{2}}$$

✓ (Range restrictions)

$\therefore$ The locus of R is the parabola $y^2 = -\frac{ax}{2}$, $y < -\sqrt[3]{\frac{ac^2}{2}}$, or $y > 0$

(iii) Alternative: for 2nd case
When A is in the 4th quadrant, consider the case where the tangent at A is a tangent to $xy = c^2$:

$x = ty - at^2$ is tangent to $xy = c^2$
Solving simultaneously:

$$ty^2 - at^2 y - c^2 = 0$$

$$\Delta = (-at^2)^2 - 4 \times t \times (-c^2)$$
$$= a^2t^4 + 4tc^2$$

When $\Delta = 0$, $t(a^2t^3 + 4c^2) = 0$

$\ast\ t \neq 0$, so $a^2t^3 + 4c^2 = 0$

$$t = \sqrt[3]{\frac{-4c^2}{a^2}}$$

y-value of R is $\frac{at}{2} = \frac{a}{2}\sqrt[3]{-\frac{4c^2}{a^2}}$

$$\therefore \quad y < \frac{a}{2}\sqrt[3]{\frac{-4c^2}{a^2}}$$

$$< \sqrt[3]{\frac{a^3}{8} \times \frac{-4c^2}{a^2}}$$

$$< \sqrt[3]{-\frac{ac^2}{2}}$$

Question 16

(a) $\int \cos\sqrt{x}\,dx$ let $u = \sqrt{x}$

$u^2 = x$

$2u\,du = dx$

$= \int \cos u \cdot 2u\,du$

$= 2\int u \cdot \frac{d}{du}(\sin u)\,du$

$= 2u\sin u - 2\int \sin u\,du$

$= 2u\sin u + 2\cos u + C$

$= 2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} + C$

(b)(i) x_k : kth x-intercept of $y = \cos\sqrt{x}$

$\cos\sqrt{x} = 0$

$\sqrt{x} = \frac{\pi}{2}, \frac{3\pi}{2}, \frac{5\pi}{2}, \ldots, \frac{(2k-1)\pi}{2}$ where k is an integer.

$\therefore x_k = \frac{(2k-1)^2\pi^2}{4}$

(ii) $A_k = \left| \int_{x_k}^{x_{k+1}} \cos\sqrt{x}\,dx \right|$

$* \; x_k = \frac{(2k-1)^2\pi^2}{4}$

$x_{k+1} = \frac{(2(k+1)-1)^2\pi^2}{4}$

$= \frac{(2k+1)^2\pi^2}{4}$

$$\therefore A_k = \left| \left[2\sqrt{x}\sin\sqrt{x} + 2\cos\sqrt{x} \right]_{\frac{(2k-1)^2\pi^2}{4}}^{\frac{(2k+1)^2\pi^2}{4}} \right|$$ ✓

$$= 2\left| \left(\frac{(2k+1)\pi}{2} \times \sin\left(\frac{(2k+1)\pi}{2}\right) + \cos\left(\frac{(2k+1)\pi}{2}\right)\right) - \left(\frac{(2k-1)\pi}{2} \times \sin\left(\frac{(2k-1)\pi}{2}\right) + \cos\left(\frac{(2k-1)\pi}{2}\right)\right)\right|$$

$$= 2\left| \frac{(2k+1)\pi}{2} \times (-1)^k + 0 - \left(\frac{(2k-1)\pi}{2} \times (-1)^{k+1} + 0\right)\right|$$ ✓

$$= 2\left| \frac{(2k+1)\pi}{2}(-1)^k + \frac{(2k-1)\pi}{2}(-1)^k \right|$$

(Evaluating trig terms)

$$= 2 \times \left| (-1)^k \times \frac{4k\pi}{2} \right|$$

$$= 4k\pi$$

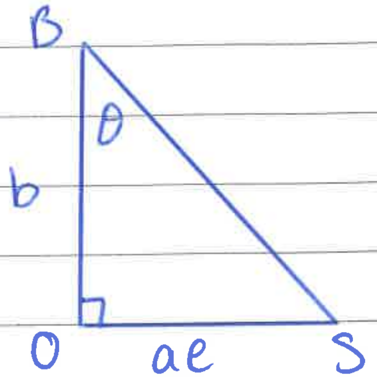
$$* \; A_{k+1} - A_k = 4(k+1)\pi - 4k\pi$$
$$= 4\pi \quad (\text{constant})$$

$\therefore$ the areas form an arithmetic progression, with common difference 4π. ✓

(Justification of arithmetic progression required)

(c) (i) $\angle S'BS = 2\theta$ ($\angle$ at centre $= 2 \times \angle$ at circumference)
Since ellipse is symmetrical about the y-axis,
$\angle S'BO = \angle SBO = \theta$
$\therefore \angle FBS = \angle FGS = \theta$
$\therefore$ BFSG is a cyclic quadrilateral
(FS subtends equal $\angle$s at B and G)

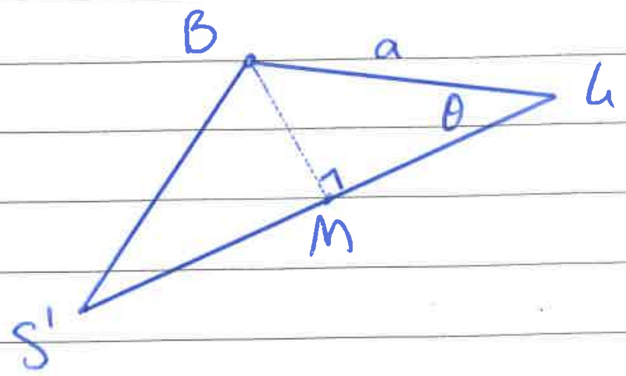
(ii) Consider $\triangle OBS$:



$BS^2 = b^2 + a^2e^2$
$= a^2(1-e^2) + a^2e^2$
$= a^2$
$\therefore BS = a$
$\therefore \cos\theta = \dfrac{b}{a}$

(iii) (α) $\angle BS'G = \angle S'GS = \theta$ (alternate $\angle$s, $SG \parallel S'B$)
Now $BS' = BG = BS = a$ (equal radii)
$\therefore \triangle BS'G$ is isosceles
$\therefore \angle BGS = \angle BS'G = \theta$ (base $\angle$s of isosceles triangle)
$\therefore \angle BGS' = \angle S'GS = \theta$
$\therefore S'G$ bisects $\angle BGS$.

(β) Consider $\Delta BGS'$:



Construct M, the midpoint of S'G.

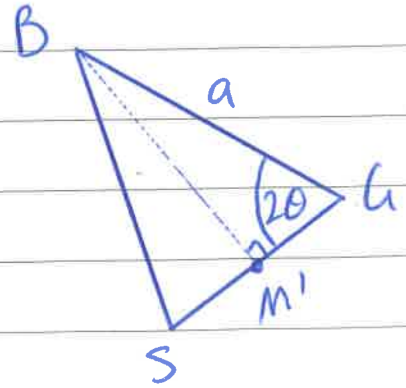
$$\cos\theta = \frac{MG}{a}$$

$$\therefore \frac{MG}{a} = \frac{b}{a}$$

$$MG = b$$

$$\therefore S'G = 2b$$

(γ) Consider ΔBSG:



Construct M', the midpoint of SG.

$$\cos 2\theta = \frac{M'G}{a}$$

$$\therefore \frac{M'G}{a} = 2\cos^2\theta - 1$$

$$\frac{M'G}{a} = 2 \times \frac{b^2}{a^2} - 1$$

$$M'G = \frac{2b^2}{a} - a$$

$$\therefore SG = \frac{4b^2}{a} - 2a$$

$S'G + SG = 2a$ (sum of the focal lengths of an ellipse)

$$\therefore 2b + \frac{4b^2}{a} - 2a = 2a$$

$$2ab + 4b^2 = 4a^2$$

$* \; b^2 = a^2(1-e^2)$

$b = a\sqrt{1-e^2}$:

$$a \times a\sqrt{1-e^2} + \cancel{4}2a^2(1-e^2) = 2a^2$$

$$a^2\sqrt{1-e^2} + 2a^2 - 2a^2e^2 = 2a^2$$

$$\sqrt{1-e^2} = 2e^2$$

$$1-e^2 = 4e^4$$

$$4e^4 + e^2 - 1 = 0$$

$$e^2 = \frac{-1 \pm \sqrt{1^2 - 4 \times 4 \times -1}}{2 \times 4}$$

Since $e^2 > 0$:

$$e^2 = \frac{-1 + \sqrt{17}}{8}$$

and since $e > 0$:

$$e = \sqrt{\frac{\sqrt{17} - 1}{8}}$$

(γ) Alternatively,

$$SG = 2a - S'G \quad \text{(sum of focal lengths of an ellipse)}$$
$$= 2(a-b)$$

Applying the cosine rule to $\Delta S'GS$:

$$(S'S)^2 = (S'G)^2 + (SG)^2 - 2 \times S'G \times SG \times \cos\theta \checkmark \quad \text{(Use result for } SG\text{)}$$
$$(2ae)^2 = (2b)^2 + (2(a-b))^2 - 2 \times 2b \times 2(a-b) \times \frac{b}{a}$$
$$4a^2e^2 = 4b^2 + 4(a^2 - 2ab + b^2) - \cancel{8}(a-b) \cdot \frac{b^2}{a}$$
$$a^2e^2 = \cancel{b^2} + a^2 - 2ab + \cancel{b^2} - \cancel{2b^2} + \frac{2b^3}{a}$$
$$0 = a^2(1-e^2) - 2ab + \frac{2b^3}{a}$$
$$0 = b^2 - 2ab + \frac{2b^3}{a}$$

$$0 = 1 - 2\frac{a}{b} + 2\frac{b}{a}$$

$$* \quad \frac{b}{a} = \sqrt{1-e^2}$$

$$0 = 1 - \frac{2}{\sqrt{1-e^2}} + 2\sqrt{1-e^2}$$

$$0 = \sqrt{1-e^2} - 2 + 2(1-e^2)$$
$$2e^2 = \sqrt{1-e^2}$$
$$4e^4 = 1 - e^2$$
$$4e^4 + e^2 - 1 = 0$$

$$e^2 = \frac{-1 \pm \sqrt{1^2 - 4 \times 4 \times -1}}{2 \times 4}$$

$e^2 > 0 \quad \therefore \quad e^2 = \dfrac{\sqrt{17} - 1}{8}$

$e > 0 \quad \therefore \quad e = \sqrt{\dfrac{\sqrt{17} - 1}{8}}$ ✓