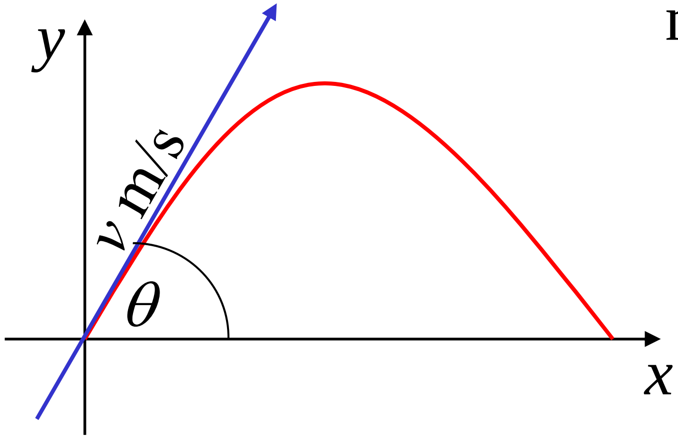
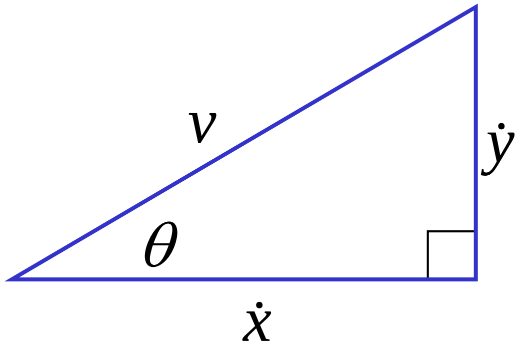
Projectile Motion



maximum range

$\theta = 45^\circ$

Initial conditions when $t = 0$



$$\frac{\dot{x}}{v} = \cos\theta \qquad \frac{\dot{y}}{v} = \sin\theta$$

$$\dot{x} = v\cos\theta \qquad \dot{y} = v\sin\theta$$

$$x = 0 \qquad y = 0$$

$$\ddot{x} = 0$$

$$\dot{x} = c_1$$

$$\text{when } t = 0, \dot{x} = v\cos\theta$$

$$c_1 = v\cos\theta$$

$$\dot{x} = v\cos\theta$$

$$x = vt\cos\theta + c_3$$

$$\text{when } t = 0, x = 0$$

$$c_3 = 0$$

$$x = vt\cos\theta$$

$$\ddot{y} = -g$$

$$\dot{y} = -gt + c_2$$

$$\dot{y} = v\sin\theta$$

$$c_2 = v\sin\theta$$

$$\dot{y} = -gt + v\sin\theta$$

$$y = -\frac{1}{2}gt^2 + vt\sin\theta + c_4$$

$$y = 0$$

$$c_4 = 0$$

$$y = -\frac{1}{2}gt^2 + vt\sin\theta$$

Note: parametric coordinates of a parabola

$$t = \frac{x}{v\cos\theta}$$

$$y = -\frac{gx^2}{2v^2\cos^2\theta} + \frac{x\sin\theta}{\cos\theta}$$

$$y = -\frac{gx^2}{2v^2}\sec^2\theta + x\tan\theta$$

$$y = -\frac{gx^2}{2v^2}\left(\tan^2\theta + 1\right) + x\tan\theta$$

Common Questions

(1) When does the particle hit the ground?

Particle hits the ground when $y = 0$

(2) What is the range of the particle?

(i) find when $y = 0$

(ii) substitute into x

roots of the quadratic

(3) What is the greatest height of the particle?

(i) find when $\dot{y} = 0$

(ii) substitute into y

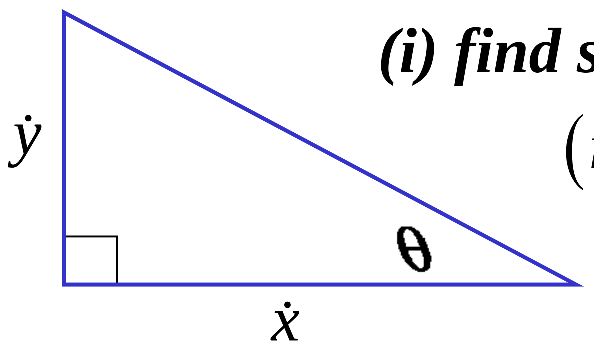
vertex of the parabola

(4) What angle does the particle make with the ground?

(i) find when $y = 0$

(ii) substitute into $\dot{y}$

(iii) $\tan\theta = \dfrac{\dot{y}}{\dot{x}}$



(i) find slope of the tangent

(ii) $m = \tan\theta$

Summary

A particle undergoing projectile motion obeys

$$\ddot{x} = 0 \qquad \text{and} \qquad \ddot{y} = -g$$

with initial conditions

$$\dot{x} = v\cos\theta \qquad \text{and} \qquad \dot{y} = v\sin\theta$$

e.g. A ball is thrown with an initial velocity of 25 m/s at an angle of $\theta = \tan^{-1}\frac{3}{4}$ to the ground. Determine;.

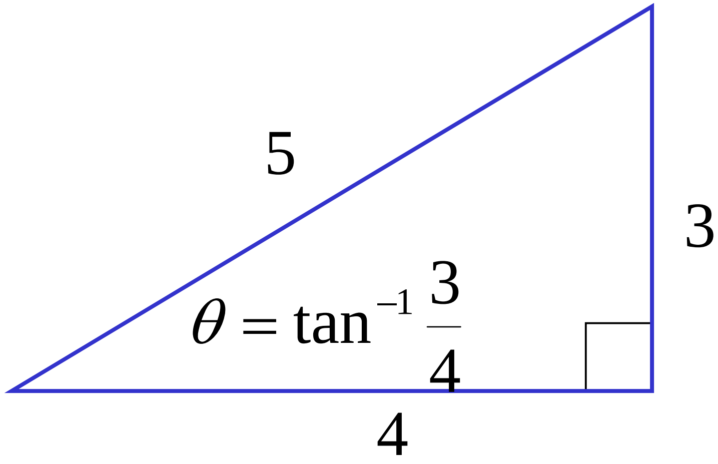
a) greatest height obtained

Initial conditions

$$\dot{x} = v\cos\theta \qquad \dot{y} = v\sin\theta$$

$$\dot{x} = 25\left(\frac{4}{5}\right) \qquad \dot{y} = 25\left(\frac{3}{5}\right)$$

$$= 20\text{m/s} \qquad = 15\text{m/s}$$



$$\ddot{x} = 0$$

$$\dot{x} = c_1$$

when $t = 0, \dot{x} = 20$

$$c_1 = 20$$

$$\dot{x} = 20$$

$$x = 20t + c_3$$

when $t = 0, x = 0$

$$c_3 = 0$$

$$x = 20t$$

$$\ddot{y} = -10$$

$$\dot{y} = -10t + c_2$$

$$\dot{y} = 15$$

$$c_2 = 15$$

$$\dot{y} = -10t + 15$$

$$y = -5t^2 + 15t + c_4$$

$$y = 0$$

$$c_4 = 0$$

$$y = -5t^2 + 15t$$

Using definite integrals

$$\ddot{x} = 0$$

$$\frac{d\dot{x}}{dt} = 0$$

$$\int_{20}^{\dot{x}} d\dot{x} = 0$$

$$\left[\dot{x}\right]_{20}^{\dot{x}} = 0$$

$$\dot{x} - 20 = 0$$

$$\dot{x} = 20$$

$$\frac{dx}{dt} = 20$$

$$\int_{0}^{x} dx = \int_{0}^{t} 20dt$$

$$\left[x\right]_{0}^{x} = \left[20t\right]_{0}^{t}$$

$$x = 20t$$

$$\ddot{y} = -10$$

$$\frac{d\dot{y}}{dt} = -10$$

$$\int_{15}^{\dot{y}} d\dot{y} = -\int_{0}^{t} 10dt$$

$$\left[\dot{y}\right]_{15}^{\dot{y}} = \left[10t\right]_{t}^{0}$$

$$\dot{y} - 15 = 0 - 10t$$

$$\dot{y} = 15 - 10t$$

$$\frac{dy}{dt} = 15 - 10t$$

$$\int_{0}^{y} dy = \int_{0}^{t} 15 - 10tdt$$

$$\left[y\right]_{0}^{y} = \left[15t - 5t^2\right]_{0}^{t}$$

$$y = 15t - 5t^2$$

greatest height occurs when $\dot{y} = 0$

$$-10t + 15 = 0$$

$$t = \frac{3}{2}$$

when $t = \frac{3}{2}, y = -5\left(\frac{3}{2}\right)^2 + 15\left(\frac{3}{2}\right)$

$$= \frac{45}{4}$$

$\therefore$ greatest height is $11\frac{1}{4}$ m above the ground

b) range

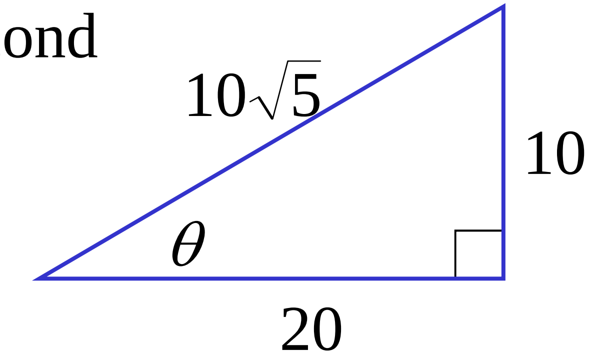
time of flight is 3 seconds

when $t = 3, x = 20(3)$

$$= 60$$

$\therefore$ range is 60m

c) velocity and direction of the ball after $\frac{1}{2}$ second



when $t=\frac{1}{2}, \dot{x}=20$ $\quad\quad$ $\dot{y}=-10\left(\frac{1}{2}\right)+15$

$$=10$$

$$\tan\theta=\frac{1}{2}$$

$$\theta=26^\circ 34'$$

$\therefore$ after $\frac{1}{2}$ second, velocity $=10\sqrt{5}$ m/s and it is traveling at an angle of $26^\circ 34'$ to the horizontal

d) cartesian equation of the path

$$x=20t \qquad\qquad y=-5t^2+15t$$

$$t=\frac{x}{20} \qquad\qquad y=-5\left(\frac{x}{20}\right)^2+15\left(\frac{x}{20}\right)$$

$$y=\frac{-x^2}{80}+\frac{3x}{4}$$

Using the cartesian equation to solve the problem

a) greatest height is y value of the vertex

$$\triangle = \left(\frac{3}{4}\right)^2 - 4\left(-\frac{1}{80}\right)(0)$$

$$= \frac{9}{16}$$

$$y = -\frac{\triangle}{4a}$$

$$= -\frac{9}{16} \times -\frac{20}{1}$$

$$= \frac{45}{4}$$

$\therefore$ greatest height is $11\frac{1}{4}$ m above the ground

b) range

$$y = \frac{-x^2}{80} + \frac{3x}{4}$$

$$= \frac{x}{4}\left(3 - \frac{x}{20}\right)$$

roots are 0 and 60

$\therefore$ range is 60m

c) velocity and direction of the ball after $\frac{1}{2}$ second

$$y = \frac{-x^2}{80} + \frac{3x}{4}$$

$$\frac{dy}{dx} = \frac{-x}{40} + \frac{3}{4}$$

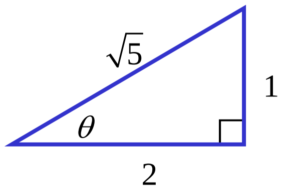
when $t = \frac{1}{2}, x = 10$

$$\frac{dy}{dx} = \frac{-10}{40} + \frac{3}{4}$$

$$= \frac{1}{2}$$

$$\tan\theta = \frac{1}{2}$$

$$\theta = 26^\circ 34'$$



$$v = \frac{x}{t\cos\theta}$$

$$= \frac{10}{\frac{1}{2} \times \frac{2}{\sqrt{5}}}$$

$$= 10\sqrt{5}$$

Exercise 3G; 1ac, 2ac, 4, 6, 8, 9, 11, 13, 16, 18

Exercise 3H; 2, 4, 6, 7, 10, 11

$\therefore$ after $\frac{1}{2}$ second, velocity $= 10\sqrt{5}$ m/s and it is traveling

at an angle of $26^\circ 34'$ to the horizontal