Caringbah High School

Year 12 2017
Mathematics Extension 2
HSC Course
Assessment Task 4

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black or blue pen
- Board-approved calculators may be used
- The reference sheet from the Board of Studies is provided.
- In Questions 11-16, show relevant mathematical reasoning and/or calculations
- Marks may not be awarded for partial or incomplete answers

Total marks – 100

Section I **10 marks**
Attempt Questions 1-10
Mark your answers on the answer sheet provided. You may detach the sheet and write your name on it.

Section II **90 marks**
Attempt Questions 11-16
Write your answers in the answer booklets provided. Ensure your name or student number is clearly visible.

Name: ______________________________ **Class:** __________

Marker's Use Only

Section I	Section II						Total
Q 1-10	**Q11**	**Q12**	**Q13**	**Q14**	**Q15**	**Q16**	
/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10

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1. For the hyperbola $\frac{x^2}{9}-\frac{y^2}{72}=1$ the eccentricity is

(A) $\frac{3}{2\sqrt{2}}$

(B) 3

(C) $2\sqrt{2}$

(D) 8

2. If ω is one of the complex roots of $z^3-1=0$, then the value of $\frac{1}{1+\omega}+\frac{1}{1+\omega^2}$ is

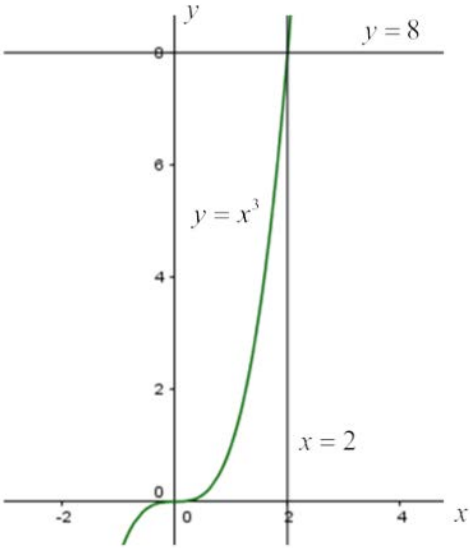
(A) -1

(B) 2

(C) 0

(D) 1

3. The volume of the solid generated when the region bounded by $y = x^3, x = 0, y = 8$ is rotated about the line $x = 2$ is given by



(A) $\pi\int_0^8 4x - x^2\,dx$

(B) $\pi\int_0^8 4y^{\frac{1}{3}} - y^{\frac{2}{3}}\,dy$

(C) $\pi\int_0^2 4x - x^2\,dx$

(D) $\pi\int_0^2 4y^{\frac{1}{3}} - y^{\frac{2}{3}}\,dy$

4. The polynomial equation $x^3 - 2x^2 + 3 = 0$ has roots α, β and γ.
What is the value of $\alpha^3 + \beta^3 + \gamma^3$?

(A) -2

(B) -1

(C) -8

(D) 8

5. If $x = \theta - \sin\theta$ and $y = 1 - \cos\theta$ which of the following is an expression for $\frac{dy}{dx}$?

(A) $cot^2 \frac{\theta}{2}$

(B) $cot \frac{\theta}{2}$

(C) $\tan \frac{\theta}{2}$

(D) $\tan^2 \frac{\theta}{2}$

6. Let α, β and γ be roots of the equation $x^3 + 3x^2 + 4 = 0$. Which of the following polynomial equations have roots α^2, β^2 and γ^2 ?

(A) $x^3 - 9x^2 - 24x - 4 = 0$

(B) $x^3 - 9x^2 - 12x - 4 = 0$

(C) $x^3 - 9x^2 - 24x - 16 = 0$

(D) $x^3 - 9x^2 - 12x - 16 = 0$

7. Which of the following is an expression for $\int xe^{\frac{x}{2}} dx$?

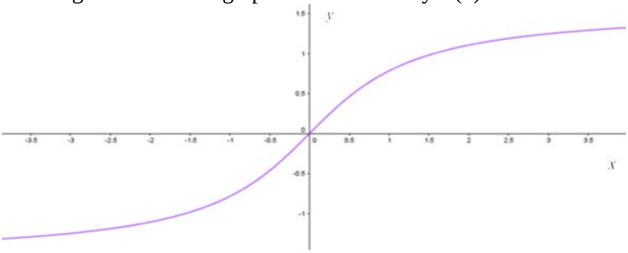
(A) $\frac{1}{2}xe^{\frac{x}{2}} - \frac{1}{4}e^{\frac{x}{2}} + c$

(B) $\frac{1}{2}xe^{\frac{x}{2}} - \frac{1}{2}e^{\frac{x}{2}} + c$

(C) $2xe^{\frac{x}{2}} - 2e^{\frac{x}{2}} + c$

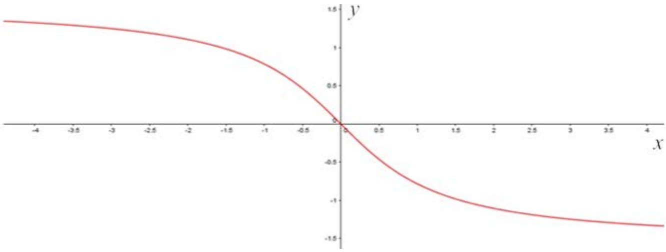
(D) $2xe^{\frac{x}{2}} - 4e^{\frac{x}{2}} + c$

8. The diagram shows the graph of the function y=f(x)

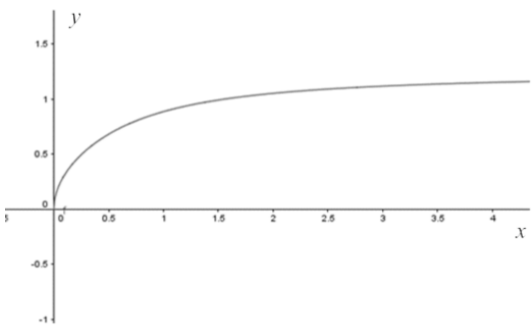


Which of the following is the graph of $y^2 = f(x)$

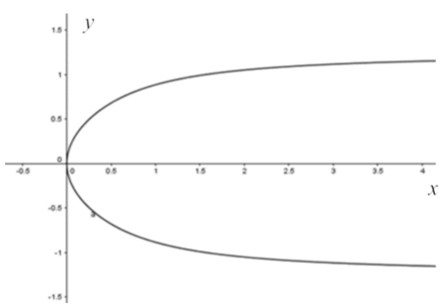
(A)



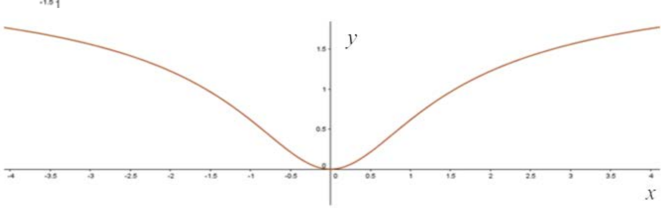
(B)



(C)



(D)



9. It is given that $3+i$ is a root of $P(z)=z^3+az^2+bz+10$, where a and b are real numbers. Which expression factorises $P(z)$ over the set of real numbers?

(A) $(z+1)(z^2-6z+10)$

(B) $(z-1)(z^2-6z-10)$

(C) $(z+1)(z^2+6z+10)$

(D) $(z+1)(z^2+6z-10)$

10. z and w are two complex numbers. Which of the following statements is always true?

(A) $|z|-|w|\geq|z+w|$

(B) $|z|+|w|\leq|z+w|$

(C) $|z|+|w|\leq|z+w|$

(D) $|z+w|+|z|\geq|w|$

End of Section I

Section II

60 marks
Attempt Questions 11–16
Allow about 2 hours 45 minutes for this section

Answer each question in a SEPARATE writing booklet.

Your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Use the substitution $u = 1 + \sin^2 x$ to find $\displaystyle\int \frac{\sin 2x}{\sqrt{1+\sin^2 x}}\,dx$ **2**

(b) Evaluate $\displaystyle\int_{\sqrt{2}}^{\sqrt{3}} \frac{2e^{\frac{2}{x^2}}}{x^3}\,dx$ **3**

(c) **(i)** Find real numbers a, b and c such that **2**

$$\frac{9-x}{(1+x^2)(1+x)} = \frac{ax+b}{1+x^2} + \frac{c}{1+x}$$

(ii) Hence, or otherwise, find **3**

$$\int \frac{9-x}{(1+x^2)(1+x)}\,dx$$

(d) Evaluate $\displaystyle\int_0^{\frac{\pi}{2}} e^x \sin 2x\,dx$ **5**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

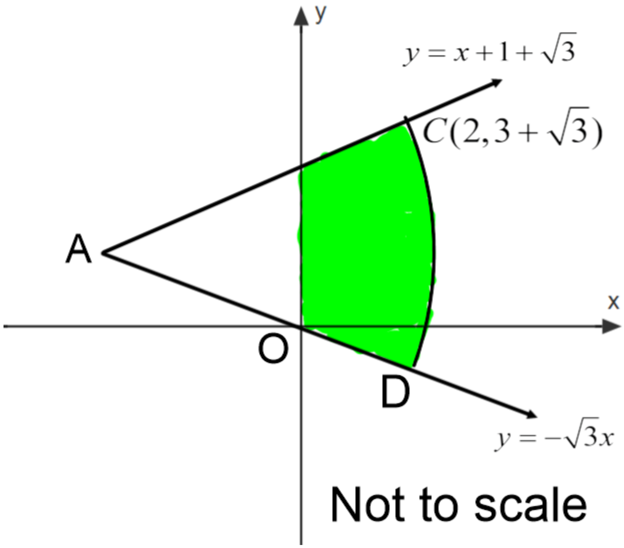
(a) **(i)** Express $z = -1 + \sqrt{3}i$ in mod-arg form. **1**

(ii) Hence find $\left(-1+\sqrt{3}i\right)^5$, giving your answer in the form $a + ib$ **2**

(b) **(i)** Find the square roots of the complex number i.
Express your answers in the form $z = a + ib$,
where a and b are real numbers. **2**

(ii) Given that $z = \dfrac{2\left(\sqrt{2} - 2\sqrt{i}\right)}{1-i}$ and $\operatorname{Im}(z) < 0$, using the solution to (i), express z in the form $a + ib$, where a and b are real numbers. **3**

(c) **4**



In the Argand diagram above, the shaded region is part of a circle centred at A, with radii AD and AC. Find the conditions that should be satisfied by the complex numbers z which are in this region.

(d) If x, y and z are positive and unequal real numbers, prove that

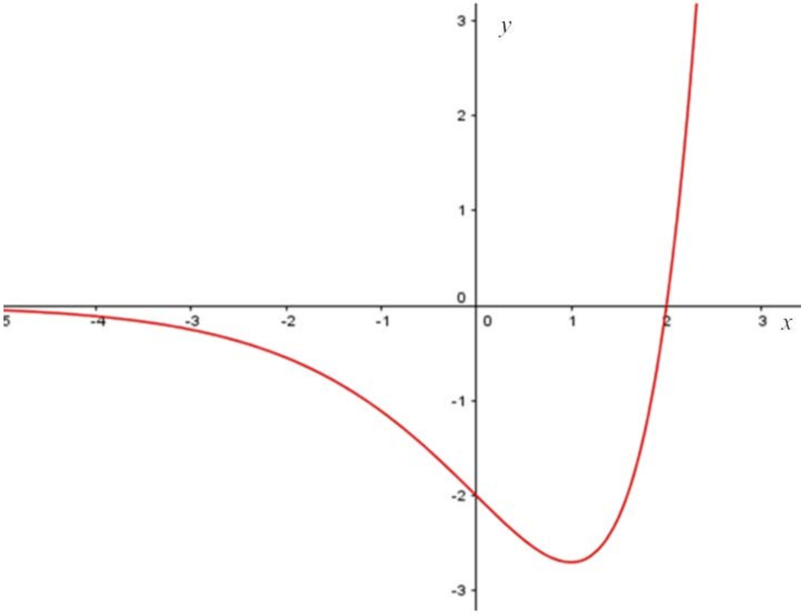
(i) $x + y - 2\sqrt{xy} > 0$ **1**

(ii) $(x+y)(y+z)(z+x) > 8xyz$ **2**

End of Question 12

Question 13 (5 marks) Use a SEPARATE writing booklet.

(a) The diagram show the graph of the function $y = f(x)$ where $f(x) = e^x(x-2)$



On separate $\frac{1}{2}$ page diagrams sketch the following graphs showing the coordinates of any turning points and the equations of any asymptotes and the x and y intercepts where possible.

(i) $y = f(|x|)$ **1**

(ii) $y = \ln[f(x)]$ **2**

(iii) $y = e^{f(x)}$ **2**

(iv) $y = f\left(\frac{1}{x}\right)$ **2**

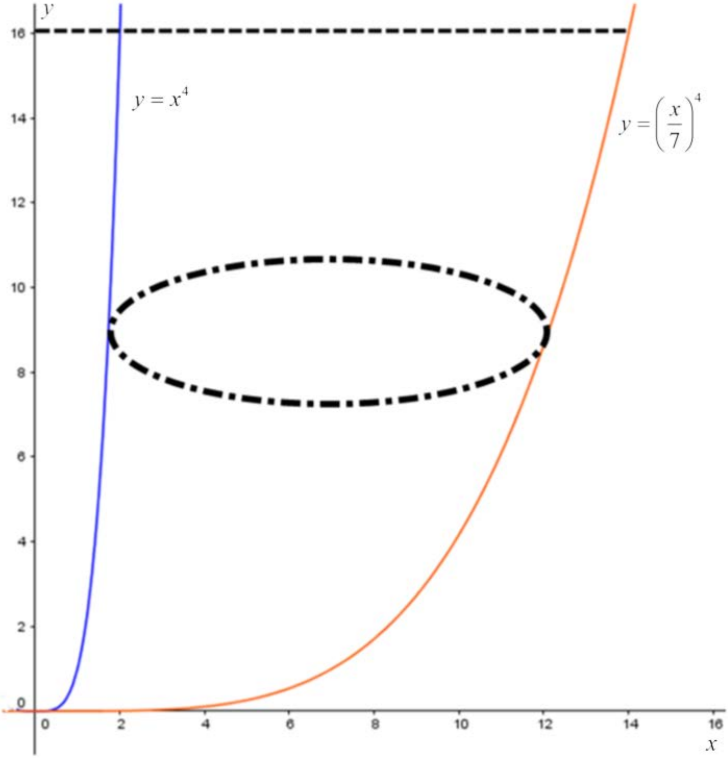
(b) Consider the curve defined by the equation $3x^2 + y^2 - 2xy - 8x + 2 = 0$

(i) Show that $\frac{dy}{dx} = \frac{3x - y - 4}{x - y}$ **2**

(ii) Find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y = 2x$ **2**

Question 13 continues on page 9

(c) A mould for a drinking horn is bounded by the curves $y = x^4$ and $y = \left(\frac{x}{7}\right)^4$ between $y = 0$ and $y = 16$. Each cross section perpendicular to the y axis is a circle. All measurements are in cm. Find the exact volume of the drinking horn in terms of π cm^3 **4**



End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) **(i)** Using the result of de Moivre's theorem that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$, solve the equation $8x^3 - 6x + 1 = 0$ **2**

(ii) Hence deduce that α) $\cos\dfrac{2\pi}{9} + \cos\dfrac{4\pi}{9} = \cos\dfrac{\pi}{9}$ **1**

and β) $\sec\dfrac{\pi}{9}\sec\dfrac{2\pi}{9}\sec\dfrac{4\pi}{9} = 8$ **1**

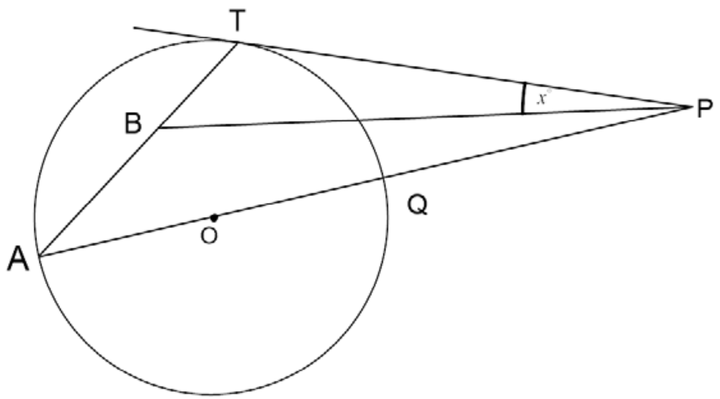
(b) Consider the equation $P(z) = z^4 + pz^3 + qz + r$ when $P(z) = 0$, p, q and r are real numbers. The sum of the roots of this equation is 6 more than the product of the roots. If $1+i$ is a root of the equation,

(i) form 2 equations in terms of p, q that satisfy $P(z) = 0$ **3**

(ii) Hence find the values of p, q and r. **2**

(iii) Find all the roots of the equation **2**

(c) PT is a tangent to a circle centre O. PO produced meets the circle at A and Q. The bisector of $\angle TPA$ meets the chord AT at B. Let $\angle TPB = x^\circ$ **4**

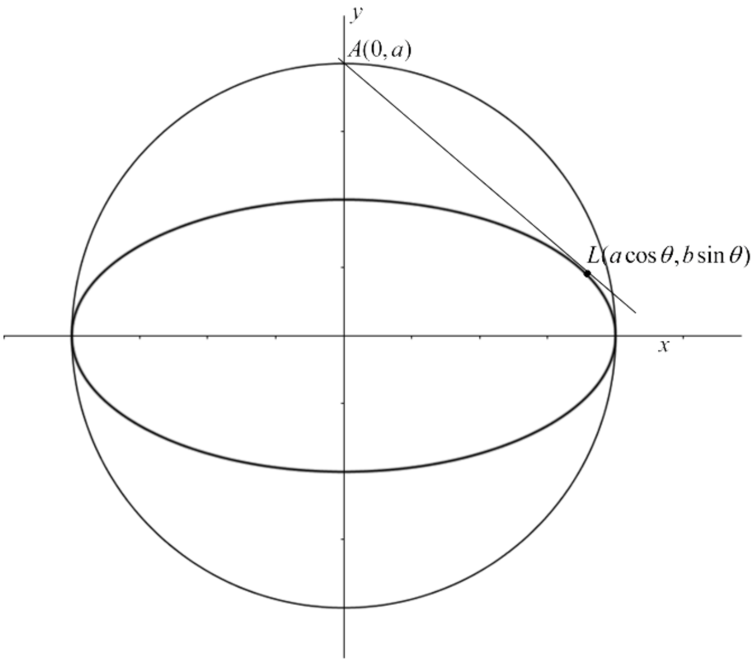


Use the template provided to prove that $\angle PBT = 45^\circ$.

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) $L(a\cos\theta, b\sin\theta)$ is a variable point on the ellipse with equation $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$.



(i) Show that the equation of the tangent at L is $\dfrac{x\cos\theta}{a}+\dfrac{y\sin\theta}{b}=1$ **2**

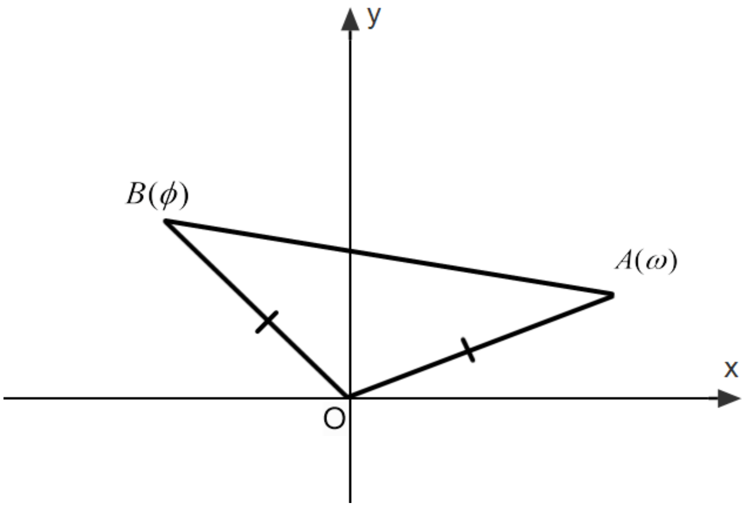
(ii) This tangent meets the auxiliary circle of the ellipse as shown on the minor axis at $A(0,a)$. Show that the line from L that is perpendicular to the x axis passes through a focus of the ellipse. **4**

(b) **(i)** Show that the equation of the normal to the rectangular hyperbola $xy=4$ at the point $P(2p,\dfrac{2}{p})$ is $py-p^3x=2(1-p^4)$. **2**

(ii) If this normal meets the hyperbola again at the point $Q(2q,\dfrac{2}{q})$ show that $p^3q=-1$ **2**

Question 15 continues on page 12

(b) The Argand diagram shows the points A and B which represent the complex numbers ω and ϕ respectively. Given that ΔBOA is isosceles and $\angle BOA = \frac{2\pi}{3}$, **5**



Show that
$(\omega + \phi)^2 = \omega\phi$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) **(i)** On the same set of axes sketch the graphs of **2**

$$y = \frac{1}{x+1} \text{ and } y = \frac{1}{x+2}$$

(ii) The region bounded by the curves $y = \frac{1}{x+1}$, $y = \frac{1}{x+2}$, the ordinates $x = 0$ and $x = 2$ is rotated about the y axis. Using the method of cylindrical shells, calculate the volume of the solid so generated. **5**

(b)

Let $I_n = \int_0^{\frac{1}{2}} \frac{1}{\left(1+4x^2\right)^n} dx$, where n is a positive integer.

(i) Find the value of I_1 **2**

(ii) Using integration by parts, show that **4**

$$I_n = \frac{2nI_{n+1}}{2n-1} + \frac{1}{2^{n+1}(1-2n)}$$

$$\left(\text{Hint :} \frac{m}{(1+m)^{n+1}} = \frac{1+m-1}{(1+m)^{n+1}} \right)$$

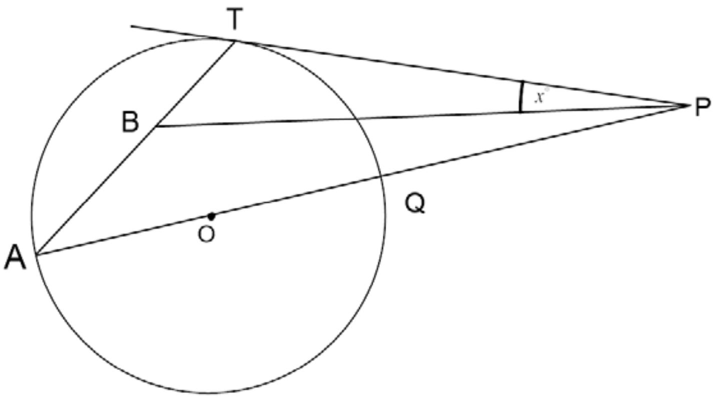
(iii) Hence evaluate $I_3 = \int_0^{\frac{1}{2}} \frac{1}{\left(1+4x^2\right)^3} dx$ **2**

End of Paper

Q14 (c) Answer Template

Name: ________________________

PT is a tangent to a circle centre O. PO produced meets the circle at A and Q. The bisector of $\angle TPA$ meets the chord AT at B. Let $\angle TPB = x^\circ$



Prove that $\angle PBT = 45^\circ$.

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Examination: Ext 2 2017 TRIAL.

Mult. Choice: 1 B 2 D 3 B 4 B 5 B 6 C 7 D 8 C 9 A 10 D

1. $e^2 = \frac{b^2}{a^2}+1 = \frac{72}{9}+1 = 9 \quad e = 3 \quad \boxed{B}$

2. $\frac{1}{\omega+1} + \frac{1}{1+\omega^2} = \frac{1+\omega^2+1+\omega}{(1+\omega)(1+\omega^2)} \qquad 1+\omega+\omega^2 = 0$

$= \frac{1}{1+\omega+\omega^2+\omega^3} = \frac{1}{1} = D$

3. $\pi \int_0^8 2-(2-x)^2\,dy = \pi\int_0^8 4-4+4x-x^2\,dx = \pi \int 4x-x^2\,dy$

$= \pi \int_0^8 4y^{\frac{1}{3}} - y^{\frac{2}{3}}\,dy \quad \boxed{B}$

4. $x^3-2x^2+3=0$

$\alpha+\beta+\gamma = 2 \quad \alpha\beta+\alpha\gamma+\beta\gamma = 0 \quad \alpha\beta\gamma = -3$

$\alpha^3 = 2\alpha^2 - 3$ so $\alpha^3+\beta^3+\gamma^3 = 2(\alpha^2+\beta^2+\gamma^2) - 9$

$\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2 - 2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 4$

$\therefore \alpha^3+\beta^3+\gamma^3 = 2\times4 - 9 = -1. \quad \boxed{B}$

5. $x = \theta - \sin\theta \quad y = 1-\cos\theta$

$\frac{dx}{d\theta} = 1-\cos\theta \quad \frac{dy}{d\theta} = \sin\theta$

$\frac{dy}{dx} = \frac{dy}{d\theta}\times\frac{d\theta}{dx} = \frac{\sin\theta}{1-\cos\theta} = \frac{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}{1-(1-2\sin^2\frac{\theta}{2})} = \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}} = \cot\frac{\theta}{2} \quad \boxed{B}$

6. $x = \alpha^2$ so $\alpha = \sqrt{x}$

$x^{\frac{3}{2}} + 3x + 4 = 0 \quad \boxed{C}$

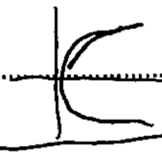
$3x+4 = -x^{\frac{3}{2}}$

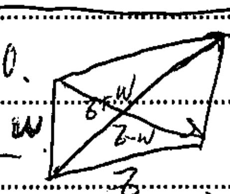
$9x^2+24x+16 = x^3$

$\therefore x^3 - 9x^2 - 24x - 16 = 0$

7. $\int xe^{\frac{x}{2}}dx = 2xe^{\frac{x}{2}} - \int 2e^{\frac{x}{2}}dx \quad \boxed{D}$

$= 2xe^{\frac{x}{2}} - 4e^{\frac{x}{2}} + C$

8.  $\boxed{C}$

10. 

$|z+w|+|z| \geq |w|$

$\boxed{D}$

9. $(3+i)(3-i)\times\gamma = -10$

$10\gamma = -10$

$\gamma = -1 \quad \boxed{A}$

$\therefore (z+1)(z^2-6z+10)$

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Examination:.................................

11 a) $u = 1+\sin^2 x$ $\quad du = 2\sin x\cos x$
$= \sin 2x$

$$\int \frac{\sin 2x}{\sqrt{1+\sin^2 x}}\,dx$$

$$\int \frac{du}{\sqrt{u}} = 2u^{\frac{1}{2}}+c = 2\sqrt{1+\sin^2 x} + c$$

b) $$\int_{\sqrt{2}}^{\sqrt{3}} \frac{2e^{\frac{2}{x^2}}}{x^3}\,dx = \int_{\sqrt{2}}^{\sqrt{3}} 2\left(e^{2x^{-2}}\right)x^{-3}\,dx$$

$$= -\frac{e^{2x^{-2}}}{2}\Bigg]_{\sqrt{2}}^{\sqrt{3}} = \frac{-e^{\frac{2}{3}}+e^{1}}{2} = \frac{e-e^{\frac{2}{3}}}{2}$$

c) i) $$\frac{9-x}{(1+x^2)(1+x)} = \frac{ax+b}{1+x^2} + \frac{c}{1+x}$$

$(ax+b)(1+x) + c(1+x^2) = 9-x$

let $x=-1$ $\quad c=5$ $\quad$ let $x=0$ $\quad b=4$ $\quad$ let $x=1$ $\quad a=-5$

$$\frac{9-x}{(1+x^2)(1+x)} = \frac{-5x+4}{x^2+1} + \frac{5}{x+1}$$

ii) $$\int \frac{9-x}{(1+x^2)(1+x)}\,dx = \int \frac{4}{x^2+1}\,dx - \int \frac{5x}{x^2+1}\,dx + \int \frac{5}{x+1}$$

$$= 4\tan^{-1}x - \frac{5}{2}\ln(x^2+1) + 5\ln(x+1) + C$$

$$= 4\tan^{-1}x - \frac{5}{2}\left(\ln\frac{x^2+1}{(x+1)^2}\right) + C$$

or $$= 4\tan^{-1}x + \frac{5}{2}\ln\left(\frac{(x+1)^2}{x^2+1}\right) + C$$

d) $$\int_0^{\frac{\pi}{2}} e^x\sin 2x\,dx = -\frac{1}{2}e^x\cos 2x\Big]_0^{\frac{\pi}{2}} + \frac{1}{2}\int e^x\cos 2x\,dx$$

$$= \frac{e^{\frac{\pi}{2}}}{2} + \frac{1}{2} + \frac{1}{4}\underbrace{e^x\sin 2x\Big]_0^{\frac{\pi}{2}}}_{=0} - \frac{1}{4}\int e^x\sin 2x\,dx$$

$$\therefore \frac{5}{4}\int_0^{\frac{\pi}{2}} e^x\sin 2x\,dx = \frac{e^{\frac{\pi}{2}}+1}{2}$$

$$\therefore \int_0^{\frac{\pi}{2}} e^x\sin 2x\,dx = \frac{2}{5}\left(e^{\frac{\pi}{2}}+1\right)$$

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Examination:..............................

Q12 a)(i) $z = -1+\sqrt{3}i$ $\quad \frac{2\pi}{3}$ $\quad 2\operatorname{cis}\frac{2\pi}{3}$

$r = \sqrt{1^2+3} = \sqrt{4} = 2.$

ii) $(-1+\sqrt{3})^5 = 32\operatorname{cis}\frac{10\pi}{3} = 32\operatorname{cis}\frac{4\pi}{3} = 32\times\left(\frac{-1}{2} - \frac{\sqrt{3}i}{2}\right) = -16-16\sqrt{3}i$

b) $\sqrt{i} = \operatorname{cis}\left(\frac{\frac{\pi}{2}+2k\pi}{2}\right) \quad k=0,1$ $\qquad$ or $(x+iy)^2 = i$

$= \operatorname{cis}\frac{\pi}{4}, \operatorname{cis}\frac{5\pi}{4}$ $\qquad x^2-y^2=0 \quad 2xy=1$

$= \frac{1+i}{\sqrt{2}}, \frac{-1-i}{\sqrt{2}}$ $\qquad x^2-\frac{1}{4x^2}=0 \quad y=\frac{1}{2x}$

$4x^4-1=0 \quad x = \pm\frac{1}{\sqrt{2}}, \; y = \pm\frac{1}{\sqrt{2}}$

$(2x^2-1)(2x^2+1)=0$

$z = \pm\left(\frac{1+i}{\sqrt{2}}\right)$

ii) $z = \frac{2(\sqrt{2}-2(\sqrt{i})}{1-i}$

$= 2\left(\sqrt{2} - 2\left(\pm\frac{1+i}{\sqrt{2}}\right)\right)/(1-i)$

$= 2(\sqrt{2} \pm (\sqrt{2}+\sqrt{2}i))/(1-i)$

$= \frac{2(-\sqrt{2}i)(1+i)}{(1-i)(1+i)}$ or $\frac{2(2\sqrt{2}+\sqrt{2}i)(1+i)}{(1-i)(1+i)}$

$= \frac{2(\sqrt{2}-\sqrt{2}i)}{2}, \frac{2(\sqrt{2}+3\sqrt{2}i)}{2}$

$= \sqrt{2}-\sqrt{2}i \qquad , \sqrt{2}+3\sqrt{2}i$

~~but~~ $Im(z) < 0$ so ~~the~~ only solution

is $\sqrt{2}-\sqrt{2}i$

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Examination:..................................

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c) $x+1+\sqrt{3} = -\sqrt{3}x$

$x + \sqrt{3}x = -(1+\sqrt{3})$

$x(1+\sqrt{3}) = -(1+\sqrt{3})$

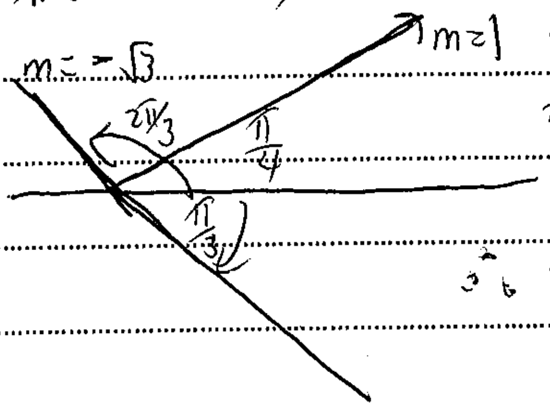
$\therefore x = -1$, $y = -\sqrt{3}x - 1 = \sqrt{3}$

$A(-1, \sqrt{3})$ $C(2, 3+\sqrt{3})$

circle centre $(-1, \sqrt{3})$ radius AC: $AC = \sqrt{3^2+3^2} = 3\sqrt{2}$.

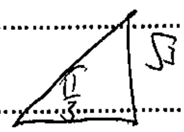
$\therefore \left|z - (-1+\sqrt{3}i)\right| \le 3\sqrt{2}$.

And $Re(z) \ge 0$ (or $x \ge 0$)



$m=1$ $\tan\theta = 1$ $\theta = \frac{\pi}{4}$

$\tan\theta = -\sqrt{3}$ $\theta = \frac{2\pi}{3}$



$\therefore -\frac{\pi}{3} \le Arg\left(z - (-1+\sqrt{3}i)\right) \le \frac{\pi}{4}$.

i) $x + y - 2\sqrt{xy} > 0$

$x, y > 0$ and $x \neq y$

so $(\sqrt{x} - \sqrt{y})^2 > 0$

$\therefore x - 2\sqrt{xy} + y > 0$

$\therefore x + y - 2\sqrt{xy} > 0$

ii) similarly

$y + z \geq 2\sqrt{yz}$

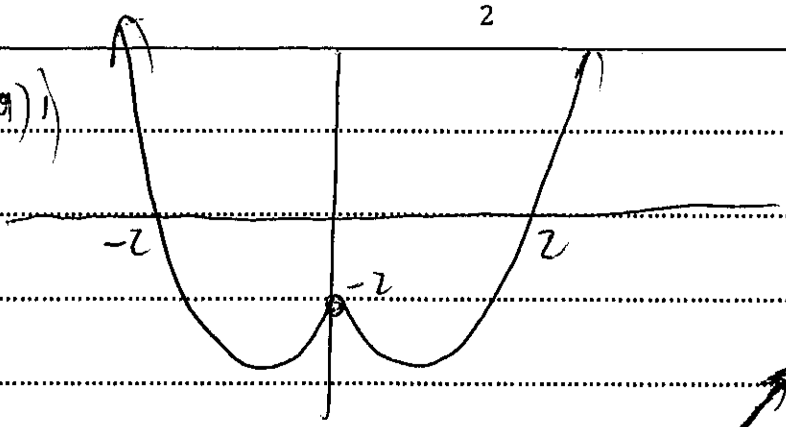
$z + x \geq 2\sqrt{zx}$

$\therefore (x+y)(y+z)(z+x) \geq 2\sqrt{xy} \times 2\sqrt{xz} \times 2\sqrt{yz}$

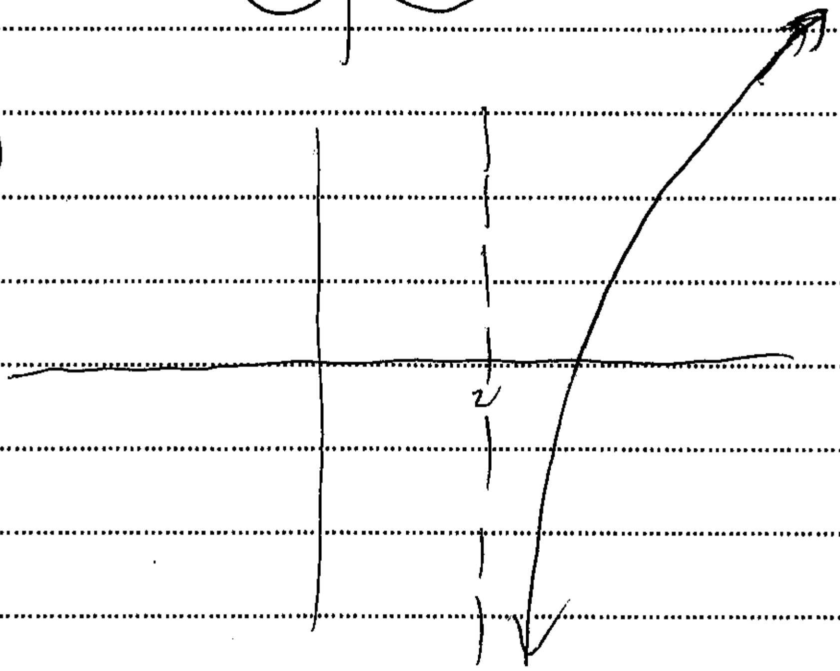
$\geq 8\sqrt{x^2y^2z^2}$

$\geq 8xyz$

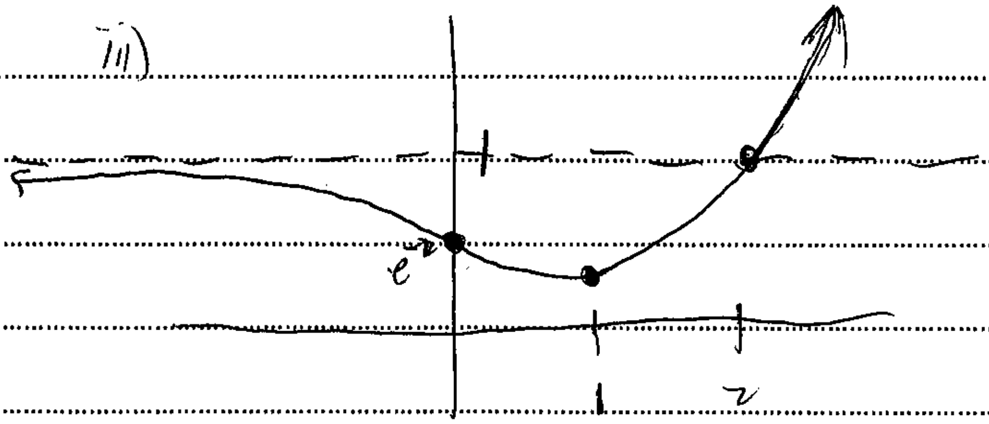
13) a) i)



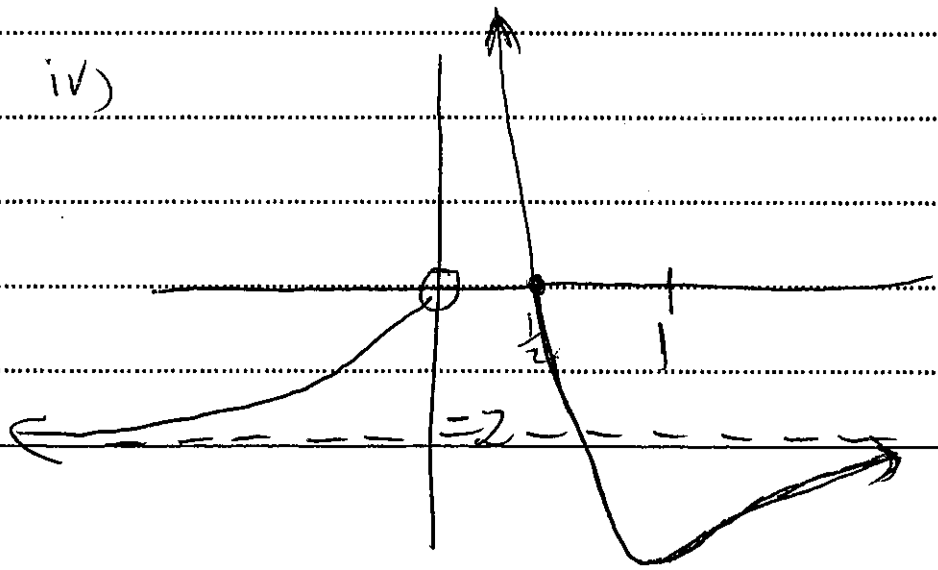
ii)



iii)



iv)



b) $3x^2 + y^2 - 2xy - 8x + 2 = 0$

$$6x + 2y\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} - 8 = 0$$

$$6x - 2y - 8 + (2y - 2x)\frac{dy}{dx} = 0$$

$$(x - y)\frac{dy}{dx} = 3x - y - 4$$

$$\therefore \frac{dy}{dx} = \frac{3x - y - 4}{x - y}$$

$$\cancel{y = 2x} \quad \cancel{\frac{dy}{dx} = \frac{3x - 2x - 4}{x - 2x} = 2}$$

$$\cancel{x - 4 = -2x}$$

$$\cancel{3x = 4}$$

$$\cancel{x = \frac{4}{3}}$$

$$\cancel{y = \frac{8}{3}}$$

$$\frac{3x - y - 4}{x - y} = 2$$

$$3x - y - 4 = 2x - 2y$$

$$y = 4 - x$$

$$\therefore 3x^2 + (4 - x)^2 + 2x(4 - x) - 8x + 2 = 0$$

$$6x^2 - 24x + 18 = 0$$

$$6(x - 3)(x - 1) = 0$$

$\therefore$ At P: $y = 4 - x$ $\quad x = 3$ or $x = 1$

So tangents at $(3, 1)$ $(1, 3)$ are $\parallel$ to $y = 2x$

c) Diameter $= x_2 - x_1 = 7y^{\frac{1}{4}} - y^{\frac{1}{4}} = 6y^{\frac{1}{4}}$ so $r = 3y^{\frac{1}{4}}$

Area $= \pi\left(3y^{\frac{1}{4}}\right)^2 = 9\pi y^{\frac{1}{2}}$

So $\delta V = 9\pi y^{\frac{1}{2}} \delta y$

$\therefore V = \lim_{\delta y \to 0} \sum_{y=0}^{16} 9\pi y^{\frac{1}{2}} \delta y \therefore V = 9\pi \int_0^{16} y^{\frac{1}{2}} dy$

$= 6\pi y^{\frac{3}{2}} \Big]_0^{16} = 384\pi \text{ cm}^3.$

Caringbah High School

Student Number

Examination:..................................

Q14 i) $x = \cos\theta$

$8\cos^3\theta - 6\cos\theta + 1 = 0$

$4\cos^3\theta - 3\cos\theta = -\frac{1}{2}$

$\cos 3\theta = -\frac{1}{2}$

$3\theta = \frac{2\pi}{3}, \frac{4\pi}{3}, \frac{8\pi}{3}, \frac{10\pi}{3}, \frac{14\pi}{3}, \frac{16\pi}{3}$

$\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}, \frac{10\pi}{9}, \frac{14\pi}{9}, \frac{16\pi}{9}$

So $x = \cos\theta = \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}, \cos\frac{10\pi}{9}, \cos\frac{14\pi}{9}, \cos\frac{16\pi}{9}$

$= -\cos\frac{\pi}{9}, = -\cos\frac{\pi}{9} \quad = \cos\frac{4\pi}{9} \quad = \cos\frac{2\pi}{9}$

$\therefore$ Roots are $\cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, -\cos\frac{\pi}{9}$

ii) $\alpha+\beta+\gamma = 0 \quad \therefore \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} - \cos\frac{\pi}{9} = 0$

$\therefore \cos\frac{2\pi}{9} + \cos\frac{4\pi}{9} = \cos\frac{\pi}{9}$

iii) $\alpha\beta\gamma = \frac{-1}{8} \quad \cos\frac{2\pi}{9} \times \cos\frac{4\pi}{9} \times -\cos\frac{\pi}{9} = -\frac{1}{8}$

$\therefore \sec\frac{2\pi}{9} \times \sec\frac{4\pi}{9} \times \sec\frac{\pi}{9} = 8$

Start here for Question Number: 11

$\alpha+\beta+\gamma+\delta=-p$

$\alpha\beta\gamma\delta=r$

Q14b) i) $-p = r+6$

$(\sqrt{2}\operatorname{cis}\frac{\pi}{4})^4$

$(1+i)^4 = -4$

$(1+i)^3 = (\sqrt{2}\operatorname{cis}\frac{\pi}{4})^3$

$= -2+2i$

$(1-i)^4 = (\sqrt{2}\operatorname{cis}\frac{-\pi}{4})^4$

$= -4$

$(1-i)^3 = (\sqrt{2}\operatorname{cis}\frac{-\pi}{4})^3$

$= -2-2i$

Using $z = 1+i$

$-4+p(-2+2i)+q(1+i)-p-6=0$

$-10-3p+q+2ip+iq=0 \quad -①$

Using $z = 1-i$

$-4+p(-2-2i)+q(1-i)-p-6=0$

$-10-3p+q-2ip-iq=0 \quad -②$

① + ②

$-20-6p+2q=0$

$\therefore -3p+q=10.$

① − ②

$4ip+2iq=0$

$\therefore 2p+q=0$

ii) $-3p+q=10$

$2p+q=0$

$-5p=10$

$p=-2$

$q=4$

$r=+2-6=-4$

/2

iii) Equation is

$z^4-2z^3+4z-4=0$

$(z-(1+i))(z-(1-i))=z^2-2z+2$ is a factor

$$\begin{array}{r}
z^2-2 \\
z^2-2z+2\overline{)\,z^4-2z^3+4z-4} \\
\underline{z^4-2z^3+2z^2} \\
-2z^2+4z-4
\end{array}$$

/2.

$\therefore$ Roots are $(1+i)$ $(1-i)$ $\sqrt{2}$, $-\sqrt{2}$

Additional writing space on back page.

c)

i) $\angle ATC = 90^\circ$ (Δ in semi-circle)

1 $\angle PTC = \angle TAC = y$ (angle in alt segment)

In ΔAPT

$2x + 2y + 90 = 180$

$\therefore 2x + 2y = 90$

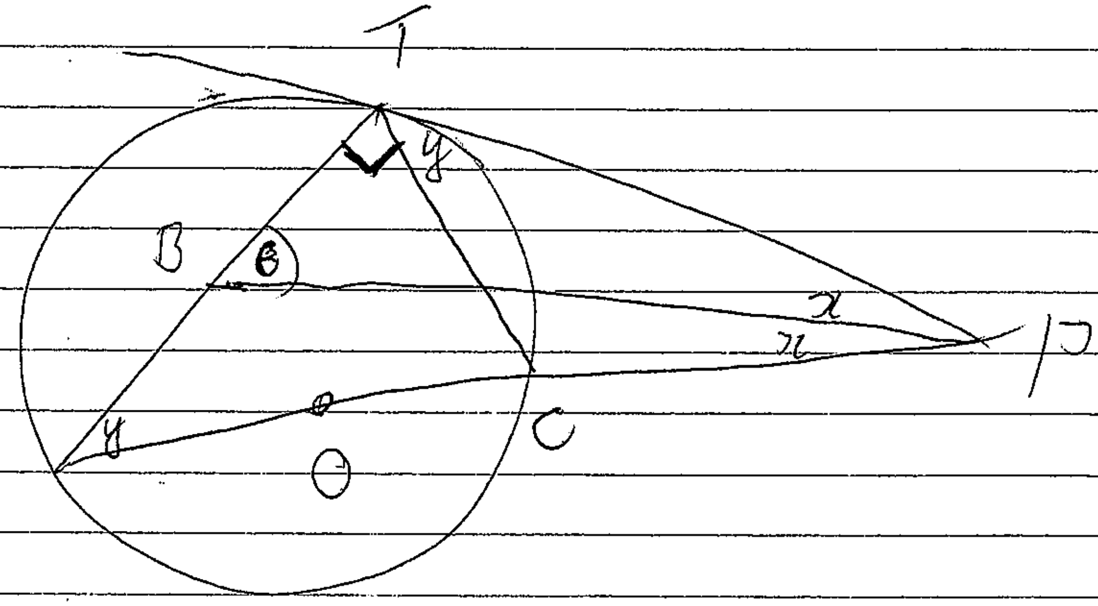
1 $\therefore x + y = 45$

$\therefore$ In ΔTBP

$90 + x + y + \angle TBP = 180$

1 $\therefore \angle TBP = 180 - 135$

$= 45^\circ$



Q15 a) i) $L(a\cos\theta, b\sin\theta)$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1 \quad (a>b)$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\frac{dy}{dx}=0$$

$$\frac{dy}{dx}=\frac{-b^2x}{a^2y}$$

$$=\frac{-b^{\cancel{2}}\cancel{a}\cos\theta}{a^{\cancel{2}}\cancel{b}\sin\theta}$$

$$=\frac{-b}{a}\frac{\cos\theta}{\sin\theta}$$

1

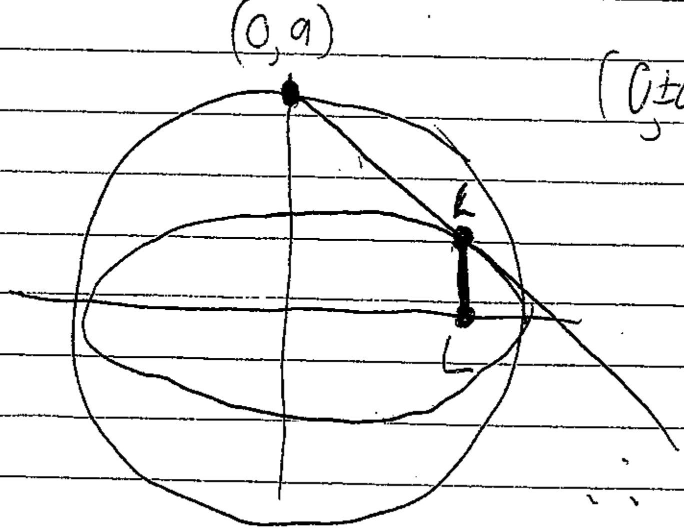
$$y-b\sin\theta=\frac{-b\cos\theta}{a\sin\theta}(x-a\cos\theta)$$

$$ay\sin\theta-ab\sin^2\theta=-bx\cos\theta+ab\cos^2\theta$$

$$bx\cos\theta+ay\sin\theta=ab(\sin^2\theta+\cos^2\theta)$$
$$=ab$$

1

ii)



$(0,\pm a)$ satisfies $bx\cos\theta+ay\sin\theta=ab$

$\pm a^2\sin\theta=ab$

$\cancel{a}\sin\theta=\pm\frac{b}{a}$ 1

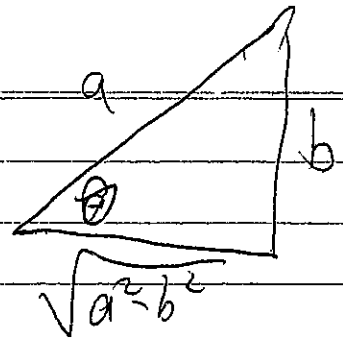
$\therefore$ ~~$L(a\cos\theta, a\sin^2\theta)$~~

$\therefore L$ is $\left(a\cos\theta, \pm\frac{b^2}{a}\right)$ 1

~~[illegible] to [illegible]~~

$\therefore$ ~~[illegible]~~

since $\sin\theta = \pm\frac{b}{a}$



$\therefore \cos\theta = \frac{\sqrt{a^2-b^2}}{a}$ 1

So $x = a\cos\theta$

$= \pm a\frac{\sqrt{a^2-b^2}}{a} = \pm\sqrt{a^2-b^2}$ 1

for ellipse $b^2 = a^2(1-e^2)$

$\therefore a^2-b^2 = a^2e^2$

$\therefore x = \pm\sqrt{a^2e^2}$

$= \pm ae$

which is the foci 1

Q15 b) i) $xy = 4$ $\quad P\left(2p, \frac{2}{p}\right)$

$$y = \frac{4}{x}$$

$$y' = -\frac{4}{x^2} = -\frac{1}{p^2}$$ 1

Normal: p^2

$$y - \frac{2}{p} = p^2(x - 2p)$$

$$py - 2 = p^3x - 2p^4$$

$$py - p^3x = 2(1-p^4)$$ 1

ii) $Q\left(2q, \frac{2}{q}\right)$

$$\frac{2p}{q} - 2p^3q = 2(1-p^4)$$ 1

$$\cancel{2}p - \cancel{2}p^3q^2 = \cancel{2}q(1-p^4)$$

$$p - p^3q^2 = q - qp^4$$

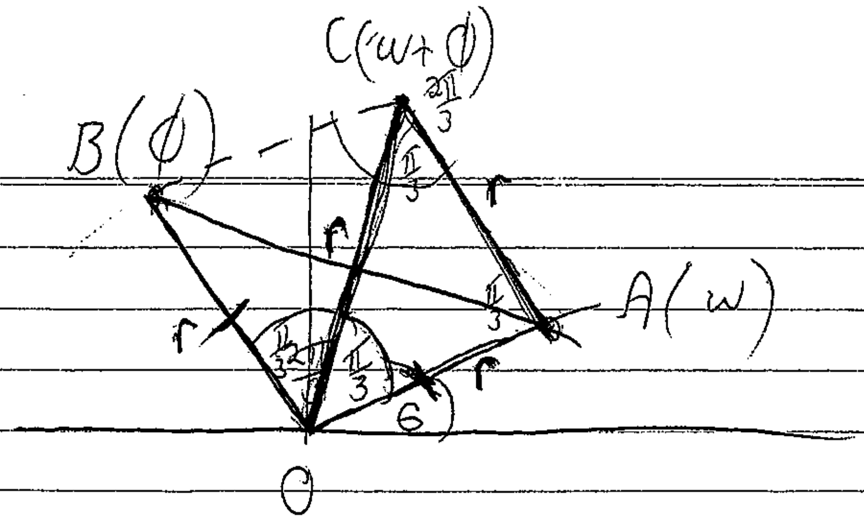
$$p - q = p^3q^2 - qp^4$$

$$-(q-p) = p^3q(q-p)$$

$$\therefore p^3q = -1.$$ 1

$\boxed{4}$

b)



OACB is a rhombus $\therefore$ CO bisects $\angle BOA$

~~let Arg $w = \theta$~~

$\therefore \text{Arg}(w+\phi) = \left(\frac{\pi}{3}+\theta\right)$

$\text{Arg}(w+\phi)^2 = \left(2\frac{\pi}{3}+2\theta\right)$

$|w+\phi| = r$

$\therefore |(w+\phi)^2| = r^2$

$\therefore (w+\phi)^2 = r^2 \text{cis}\left(\frac{2\pi}{3}+2\theta\right)$

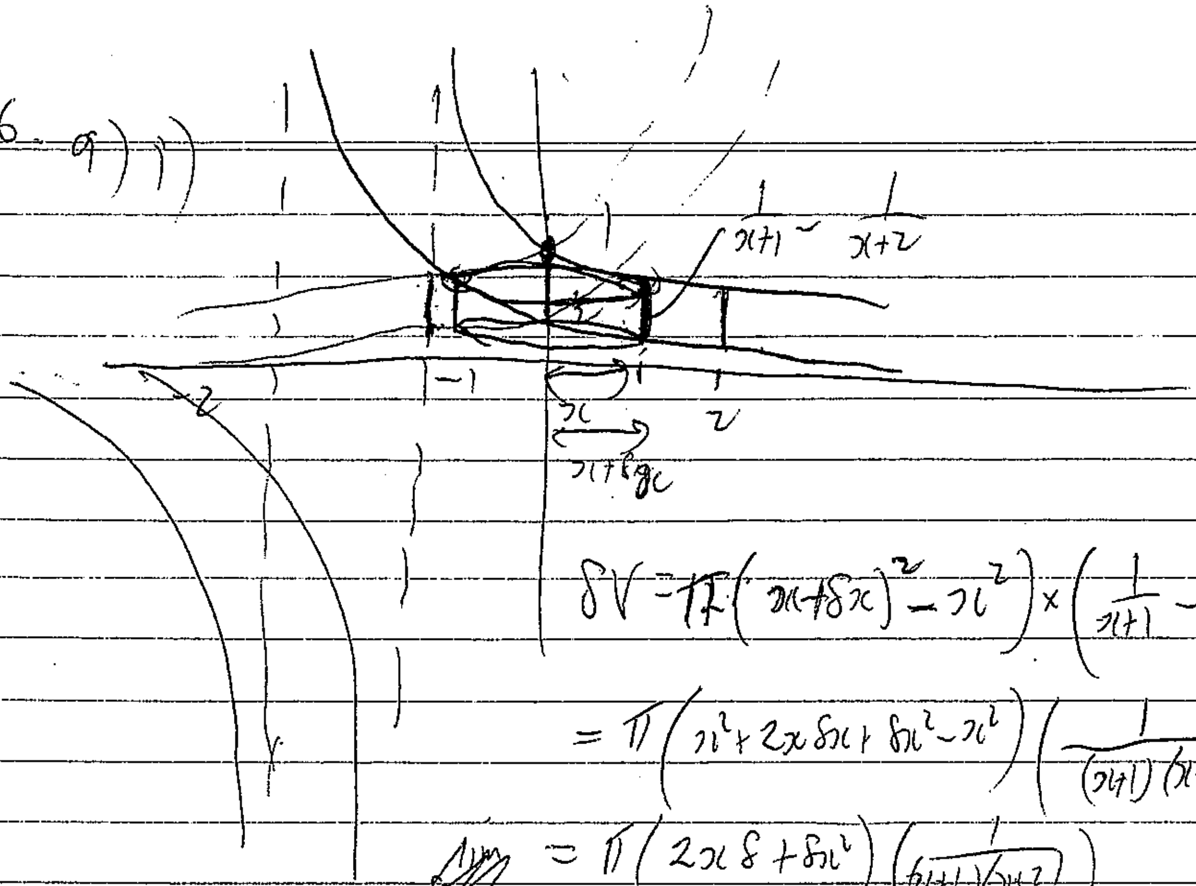
$w\phi = r\,\text{cis}\,\theta \times r\,\text{cis}\left(\theta+\frac{2\pi}{3}\right)$

$= r^2 \text{cis}\left(\theta+\theta+\frac{2\pi}{3}\right)$

$= r^2 \text{cis}\left(2\theta+\frac{2\pi}{3}\right)$

$= (w+\phi)^2$.

16. a) i)



$$\delta V = \pi\left((x+\delta x)^2 - x^2\right) \times \left(\frac{1}{x+1} - \frac{1}{x+2}\right)$$

$$= \pi\left(x^2 + 2x\delta x + \delta x^2 - x^2\right)\left(\frac{1}{(x+1)(x+2)}\right)$$

$$= \pi\left(2x\delta + \delta x^2\right)\left(\frac{1}{(x+1)(x+2)}\right)$$

$$\frac{\delta V}{\cancel{\delta x}} = \pi\left(2x\delta x + \overset{\to 0}{\delta x^2}\right)\left(\frac{1}{(x+1)(x+2)}\right)$$

$\lim_{\delta x \to 0}$

$$\frac{dV}{\cancel{dx}} = \pi \sum 2x\left(\frac{1}{(x+1)(x+2)}\right) dx$$

$$V = \pi \int_0^2 2x \times \frac{1}{(x+1)(x+2)}\, dx$$

$$= 2\pi \int_0^2 \frac{x}{(x+1)(x+2)}\, dx$$

$$\frac{a}{(x+1)} + \frac{b}{x+2} = a(x+2) + b(x+1) = x$$

$$2a + b = 0 \qquad a + b = 1$$

$$\therefore\ a = -1 \qquad b = 2$$

$$= 2\pi \int_0^2 \frac{-1}{x+1} + \frac{2}{x+2}\, dx = 2\pi\left(2\ln(x+2) - \ln(x+1)\right)$$

$$= 2\pi\left(2\ln 4 - \ln 3 - (2\ln 2 - \ln 1)\right)$$

$$= 2\pi\left(\ln \frac{16}{3} - \ln 4\right)$$

$$= 2\pi \ln\left(\frac{4}{3}\right)$$

b) $$I_n = \int_0^{\frac{1}{2}} \frac{1}{(1+4x^2)^n}\,dx$$

$$I_1 = \int_0^{\frac{1}{2}} \frac{1}{1+4x^2}\,dx$$

$$= \frac{1}{4}\int_0^{\frac{1}{2}} \frac{1}{\left(\frac{1}{2}\right)^2 + x^2}\,dx$$

$$= \frac{1}{2}\tan^{-1} 2x \Big]_0^{\frac{1}{2}}$$

$$= \frac{1}{2}\left(\tan^{-1}(1) - \tan^{-1} 0\right)$$

$$= \frac{1}{2} \times \frac{\pi}{4}$$

$$= \frac{\pi}{8}$$

ii) $$\int_0^{\frac{1}{2}} \frac{1}{(1+4x^2)^n}\,dx = x\cdot(1+4x^2)^{-n}\Big]_0^{\frac{1}{2}} - -n\int_0^{\frac{1}{2}} 8x^2 / (1+4x^2)^{n+1}\,dx$$

$$= \frac{1}{2\times 2^n} + 2n\int_0^{\frac{1}{2}} \frac{4x^2}{(1+4x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^{n+1}} + 2n\int_0^{1} \frac{1+4x^2}{(1+4x^2)^{n+1}} - \frac{1}{(1+4x^2)^{n+1}}\,dx$$

$$= \frac{1}{2^{n+1}} + 2n\int_0^{1} \frac{1}{(1+4x^2)^n} - \frac{1}{(1+4x^2)^{n+1}}\,dx$$

$$I_n = \frac{1}{2^{n+1}} + 2n\left(I_n - I_{n+1}\right)$$

$$I_n = \frac{1}{2^{n+1}} + 2nI_n - 2nI_{n+1}$$

$$\therefore (1-2n)\,I_n = \frac{1}{2^{n+1}} - 2n\,I_{n+1}$$

$$\therefore I_n = \frac{1}{2^{n+1}(1-2n)} - \frac{2n\,I_{n+1}}{1-2n}$$

$$= \frac{1}{2^{n+1}(1-2n)} + \frac{2n\,I_{n+1}}{2n-1}$$

$$\therefore I_1 = \frac{1}{2^2 \times (1-2)} + \frac{2I_2}{1} = \frac{\pi}{8}$$

$$2I_2 = \frac{\pi}{8} + \frac{1}{4}$$

$$I_2 = \frac{\pi+2}{16}$$

$$I_2 = \frac{1}{2^3(1-4)} + \frac{4I_3}{3} = \frac{\pi+2}{16}$$

$$\frac{4I_3}{3} = \frac{\pi+2}{16} + \frac{1}{24}$$

$$\therefore I_3 = \frac{3}{4}\left(\frac{\pi+2}{16} + \frac{1}{24}\right)$$

$$= \frac{3}{4}\left(\frac{3\pi+6+2}{48}\right)$$

$$= \frac{9\pi+24}{4\times 48} = \frac{3\pi+8}{64}$$