Complex Numbers $i^{2018} = -1$

The more straight forward questions on Complex Numbers appear in multiple choice or Question 11 of the HSC.

Longer (and usually more difficult) problems will appear in the latter half of the paper and are often combined with concepts from other topics such as Polynomials, Induction and Binomial Expansions.

(i) De Moivre's Theorem

$$\begin{aligned}(\cos\theta + i\sin\theta)^n &= \cos n\theta + i\sin n\theta \\ z^n &= r^n(\cos n\theta + i\sin n\theta) \\ &= r^n cisn\theta\end{aligned}$$

Examples

1. Express the complex number $z=\frac{6}{\sqrt{3}-i}$ in modulus argument form.

$$z=\frac{6cis0}{2cis\left(-\frac{\pi}{6}\right)}$$

$$=3cis\frac{\pi}{6}$$

Hence express z^3 in the form *a* + *ib* where *a* and *b* are real.

$$z^3=3^3cis\frac{3\pi}{6}$$

$$=27cis\frac{\pi}{2}$$

$$=27i$$

2. Express $\cos 3\theta$ in the terms of $\cos\theta$

$$\cos 3\theta+i\sin 3\theta=(cis\theta)^3$$

$$=c^3+3c^2si-3cs^2-s^3i$$

$$\therefore \cos 3\theta=\cos^3\theta-3\cos\theta\sin^2\theta$$

$$=\cos^3\theta-3\cos\theta(1-\cos^2\theta)$$

$$=4\cos^3\theta-3\cos\theta$$

(v) Roots of Complex Numbers

To find the square root of a complex number we can;

a) Let the solution be $a + ib$ and solve simultaneous equations in a and b

$$\text{If } \sqrt{x+iy} = a+ib$$

$$\text{then } a^2 - b^2 = x$$

$$2ab = y$$

b) Convert into mod arg form and use De Moivre's Theorem

$$\text{If } z^2 = x+iy$$

$$z^2 = rcis\theta$$

$$z = \sqrt{r}cis\left[\frac{2\pi k+\theta}{2}\right] \qquad k = 0,1$$

c) Use the *"alternative"* formula

If $\sqrt{x+iy} = a+ib$

then $a = \sqrt{\dfrac{x+|x+iy|}{2}}$ $\qquad b = \dfrac{y}{2a}$

Example

Find the square root of 8 + 6*i*, giving your answer in the form *x* + *iy*

$$a^2 - b^2 = 8 \qquad 2ab = 6$$

$$a^2 - \frac{9}{a^2} = 8 \qquad b = \frac{3}{a}$$

$$a^4 - 8a^2 - 9 = 0$$

$$(a^2 - 9)(a^2 + 1) = 0$$

$$a^2 = 9 \quad \text{or} \quad a^2 = -1$$

$$a = \pm 3 \qquad \text{no real solutions}$$

$$\therefore b = \pm 1$$

$$\sqrt{8+6i} = \pm(3+i)$$

$$z^2 = 8+6i$$

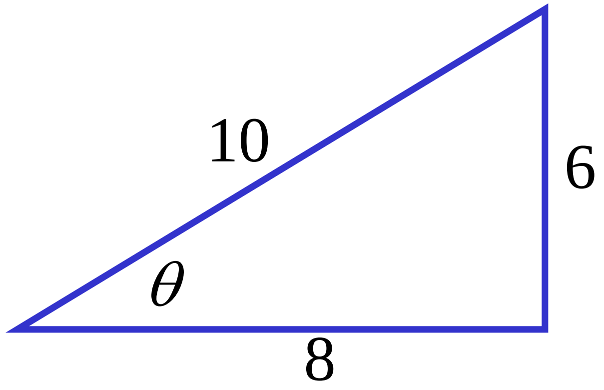
$$z^2 = 10cis\left(\tan^{-1}\frac{3}{4}\right)$$

$$z = \sqrt{10}cis\left[\frac{2\pi k + \tan^{-1}\frac{3}{4}}{2}\right] \quad k = 0,1$$

$$z = \sqrt{10}cis\frac{\tan^{-1}\frac{3}{4}}{2}, \sqrt{10}cis\left(\pi + \frac{\tan^{-1}\frac{3}{4}}{2}\right)$$

$$z = \sqrt{10}\left(\frac{3}{\sqrt{10}} + \frac{1}{\sqrt{10}}i\right), \sqrt{10}\left(-\frac{3}{\sqrt{10}} - \frac{1}{\sqrt{10}}i\right)$$

$$z = 3+i, -3-i$$



$$\cos^2\frac{\tan^{-1}\frac{3}{4}}{2} = \frac{1}{2}\left(1+\cos\tan^{-1}\frac{3}{4}\right)$$

$$= \frac{1}{2}\left(1+\frac{4}{5}\right)$$

$$= \frac{9}{10}$$

$$\therefore \sin^2\frac{\tan^{-1}\frac{3}{4}}{2} = \frac{1}{10}$$

$$|8+6i| = \sqrt{8^2+6^2}$$

$$= 10$$

$$a = \sqrt{\frac{8+10}{2}}$$

$$= \sqrt{\frac{18}{2}}$$

$$= 3$$

$$b = \frac{6}{6}$$

$$= 1$$

$$\therefore \sqrt{8+6i} = \pm(3+i)$$

For cube roots (and higher) we use De Moivre's Theorem (unless of course the polynomial in the equation is easily factorised)

If $z^n = x + iy$

$$z^n = r\,cis\,\theta$$

$$z = \sqrt[n]{r}\,cis\left[\frac{2\pi k + \theta}{n}\right] \qquad k = 0,1,\ldots,n-1$$

Equations of the form $z^n = a + ib$ have n roots which are equally spaced on the circle of radius r, where r is the modulus of any root.

Example

Solve $z^5 = -4 - 4i$

$$z^5 = 4\sqrt{2}\,cis\left(-\frac{3\pi}{4}\right)$$

$$z = \sqrt{2}\,cis\left[\frac{2\pi k - \frac{3\pi}{4}}{5}\right] \qquad k = 0,1,2,3,4$$

$$z = \sqrt{2}\,cis\frac{-3\pi}{20}, \sqrt{2}\,cis\frac{\pi}{4}, \sqrt{2}\,cis\frac{13\pi}{20},$$

$$\sqrt{2}\,cis\frac{-19\pi}{20}, \sqrt{2}\,cis\frac{-11\pi}{20}$$

The equation $z^n - 1 = 0$ has n roots which lie on the unit circle, centre (0,0), and if ω is a complex root of this equation, then the roots are;

$$1, \omega, \omega^2, \omega^3, \ldots, \omega^{n-1}$$

and we can show that;

$$1 + \omega + \omega^2 + \omega^3 + \cdots + \omega^{n-1} = 0 \qquad (\text{sum of the roots})$$

$$\omega^n = 1$$

Examples

1. Solve $z^3 - 1 = 0$

$$z = cis\left[\frac{2\pi k}{3}\right] \qquad k = 0, \pm 1$$

$$z = 1, cis\frac{2\pi}{3}, cis\frac{-2\pi}{3}$$

2. Solve $z^4 - 1 = 0$

$$z = cis\left[\frac{2\pi k}{4}\right] \quad k = 0, \pm 1, 2$$

$$z = 1, cis\frac{\pi}{2}, cis\frac{-\pi}{2}, cis\pi$$

$$\underline{z = 1, i, -i, -1}$$

3. If ω is a complex cube root of unity, use the fact that $1 + \omega + \omega^2 = 0$ to;

a) Evaluate $(1+\omega^2)^3$

$$= (-\omega)^3$$

$$= -\omega^3$$

$$\underline{= -1}$$

b) Evaluate $\frac{1}{1+\omega} + \frac{1}{1+\omega^2}$

$$= -\frac{1}{\omega^2} - \frac{1}{\omega}$$

$$= \frac{-1-\omega}{\omega^2}$$

$$= \frac{\omega^2}{\omega^2}$$

$$\underline{= 1}$$

c) Form the cubic equation with roots $1,\ 1+\omega,\ 1+\omega^2$

$$(z-1)\{z^2 - (2+\omega+\omega^2)z + (1+\omega+\omega^2+\omega^3)\} = 0$$

$$(z-1)(z^2 - z + 1) = 0$$

$$z^3 - z^2 + z - z^2 + z - 1 = 0$$

$$\underline{z^3 - 2z^2 + 2z - 1 = 0}$$

Some important results

$$z^n + \frac{1}{z^n} = 2\cos n\theta \qquad\qquad z^n - \frac{1}{z^n} = 2i\sin n\theta$$

$$z^n - 1 = (z-1)\left(z^{n-1} + \ldots + z^2 + z + 1\right) \Rightarrow \frac{z^n - 1}{z-1} = z^{n-1} + \ldots + z^2 + z + 1$$

$$z^n + 1 = (z+1)\left(z^{n-1} - \ldots + z^2 - z + 1\right) \Rightarrow \frac{z^n + 1}{z+1} = z^{n-1} - \ldots + z^2 - z + 1$$

Examples

1. Resolve $z^7 - 1 = 0$ into real linear and quadratic factors. Hence prove

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2} \qquad z = cis\left[\frac{2\pi k}{7}\right] \qquad k = 0, \pm 1, \pm 2, \pm 3$$

$$z = 1, cis\frac{2\pi}{7}, cis\frac{-2\pi}{7}, cis\frac{4\pi}{7}, cis\frac{-4\pi}{7}, cis\frac{6\pi}{7}, cis\frac{-6\pi}{7}$$

$$z^7 - 1 = (z-1)\left(z^2 - 2z\cos\frac{2\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{4\pi}{7} + 1\right)\left(z^2 - 2z\cos\frac{6\pi}{7} + 1\right)$$

sum of roots = 0

$$1 + 2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = 0$$

$$2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = -1$$

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$$

2. $z^6+z^5+z^4+z^3+z^2+z+1=0$

$$\frac{(z-1)\left(z^6+z^5+z^4+z^3+z^2+z+1\right)}{z-1}=0$$

$$\frac{z^7-1}{z-1}=0$$

$$z^7-1=0, \qquad z\neq 1$$

$$z=cis\frac{2\pi}{7},cis\frac{-2\pi}{7},cis\frac{4\pi}{7},cis\frac{-4\pi}{7},cis\frac{6\pi}{7},cis\frac{-6\pi}{7}$$

3. Solve $6z^4-7z^3+9z^2-7z+6=0$

$$6z^2-7z+9-\frac{7}{z}+\frac{6}{z^2}=0$$

$$6\left(z^2+\frac{1}{z^2}\right)-7\left(z+\frac{1}{z}\right)+9=0$$

$$12\cos 2\theta-14\cos\theta+9=0$$

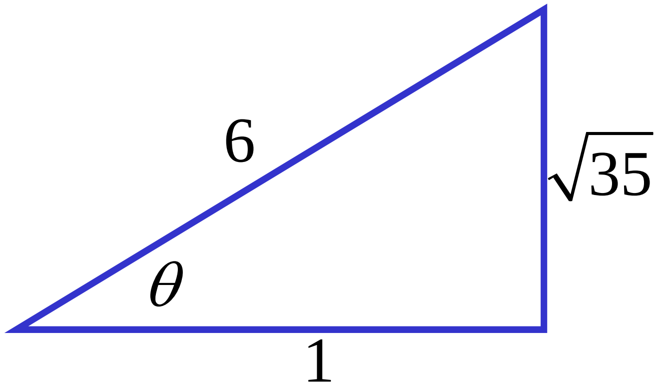
$$24\cos^2\theta-12-14\cos\theta+9=0$$

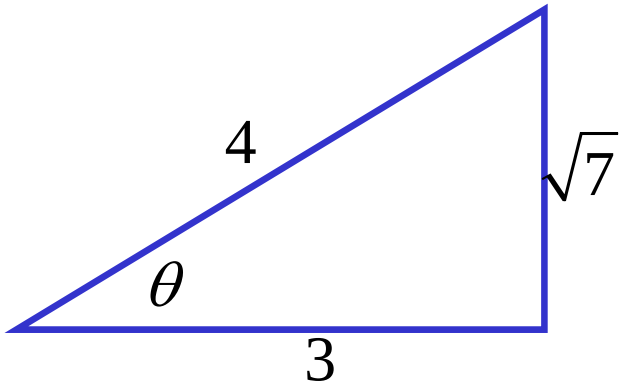
$$24\cos^2\theta-14\cos\theta-3=0$$

$$24\cos^2\theta - 14\cos\theta - 3 = 0$$

$$(6\cos\theta + 1)(4\cos\theta - 3) = 0$$

$$\cos\theta = -\frac{1}{6} \quad \text{or} \quad \cos\theta = \frac{3}{4}$$





$$z = -\frac{1}{6} \pm \frac{\sqrt{35}}{6} i \qquad \text{OR} \qquad z = \frac{3}{4} \pm \frac{\sqrt{7}}{4} i$$

4. $(z-1)^6+(z+1)^6=0$

$$\left(\frac{z-1}{z+1}\right)^6=-1$$

$$\frac{z-1}{z+1}=cis\left(\frac{2\pi k+\pi}{6}\right),\ k=0,\pm1,\pm2,3$$

$$z-1=zcis\left(\frac{2\pi k+\pi}{6}\right)+cis\left(\frac{2\pi k+\pi}{6}\right)$$

$$z\left[1-cis\left(\frac{(2k+1)\pi}{6}\right)\right]=1+cis\left(\frac{(2k+1)\pi}{6}\right)$$

$$z=\frac{1+cis\left(\frac{(2k+1)\pi}{6}\right)}{1-cis\left(\frac{(2k+1)\pi}{6}\right)}$$

$$z=\frac{1+\cos\left(\frac{(2k+1)\pi}{6}\right)+i\sin\left(\frac{(2k+1)\pi}{6}\right)}{1-\cos\left(\frac{(2k+1)\pi}{6}\right)-i\sin\left(\frac{(2k+1)\pi}{6}\right)}$$

$$z=\frac{1+2\cos^2\left(\frac{(2k+1)\pi}{12}\right)-1+2i\sin\left(\frac{(2k+1)\pi}{12}\right)\cos\left(\frac{(2k+1)\pi}{12}\right)}{1-2\cos^2\left(\frac{(2k+1)\pi}{12}\right)+1-2i\sin\left(\frac{(2k+1)\pi}{12}\right)\cos\left(\frac{(2k+1)\pi}{12}\right)}$$

$$z=\frac{2\cos^2\left(\frac{(2k+1)\pi}{12}\right)+2i\sin\left(\frac{(2k+1)\pi}{12}\right)\cos\left(\frac{(2k+1)\pi}{12}\right)}{2\sin^2\left(\frac{(2k+1)\pi}{12}\right)-2i\sin\left(\frac{(2k+1)\pi}{12}\right)\cos\left(\frac{(2k+1)\pi}{12}\right)}$$

$$z=\frac{2\cos^{2}\left(\frac{(2k+1)\pi}{12}\right)+2i\sin\left(\frac{(2k+1)\pi}{12}\right)\cos\left(\frac{(2k+1)\pi}{12}\right)}{2\sin^{2}\left(\frac{(2k+1)\pi}{12}\right)-2i\sin\left(\frac{(2k+1)\pi}{12}\right)\cos\left(\frac{(2k+1)\pi}{12}\right)}$$

$$z=\frac{2\cos\left(\frac{(2k+1)\pi}{12}\right)\left[\cos\left(\frac{(2k+1)\pi}{12}\right)+i\sin\left(\frac{(2k+1)\pi}{12}\right)\right]}{2\sin\left(\frac{(2k+1)\pi}{12}\right)\left[\sin\left(\frac{(2k+1)\pi}{12}\right)-i\cos\left(\frac{(2k+1)\pi}{12}\right)\right]}$$

$$z=\frac{\cos\left(\frac{(2k+1)\pi}{12}\right)\left[\cos\left(\frac{(2k+1)\pi}{12}\right)+i\sin\left(\frac{(2k+1)\pi}{12}\right)\right]}{-i\sin\left(\frac{(2k+1)\pi}{12}\right)\left[i\sin\left(\frac{(2k+1)\pi}{12}\right)+\cos\left(\frac{(2k+1)\pi}{12}\right)\right]}$$

$$z=\frac{\cos\left(\frac{(2k+1)\pi}{12}\right)\left[\cos\left(\frac{(2k+1)\pi}{12}\right)+i\sin\left(\frac{(2k+1)\pi}{12}\right)\right]}{-i\sin\left(\frac{(2k+1)\pi}{12}\right)\left[i\sin\left(\frac{(2k+1)\pi}{12}\right)+\cos\left(\frac{(2k+1)\pi}{12}\right)\right]}$$

$$z=i\cot\left(\frac{(2k+1)\pi}{12}\right)\ ,\ k=0,\pm1,\pm2,3$$

5. Solve $8x^3 - 6x + 1 = 0$

let $x = \cos\theta$

$$8\cos^3\theta - 6\cos\theta + 1 = 0$$

$$2\left(4\cos^3\theta - 3\cos\theta\right) = -1$$

$$2\cos 3\theta = -1$$

$$\cos 3\theta = -\frac{1}{2}$$

$$3\theta = 2\pi k \pm \frac{2\pi}{3}$$, where k is an integer

$$\theta = \frac{6\pi k \pm 2\pi}{9}$$

$$\theta = \frac{2\pi}{9}, \frac{4\pi}{9}, \frac{8\pi}{9}$$

$$x = \cos\frac{2\pi}{9}, \cos\frac{4\pi}{9}, \cos\frac{8\pi}{9}$$

6. Find the exact value of $\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}$

$$\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\cos\frac{8\pi}{9}=-\frac{1}{8}$$ (product of the roots)

$$\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}\left(-\cos\frac{\pi}{9}\right)=-\frac{1}{8}$$

$$\underline{\cos\frac{\pi}{9}\cos\frac{2\pi}{9}\cos\frac{4\pi}{9}=\frac{1}{8}}$$

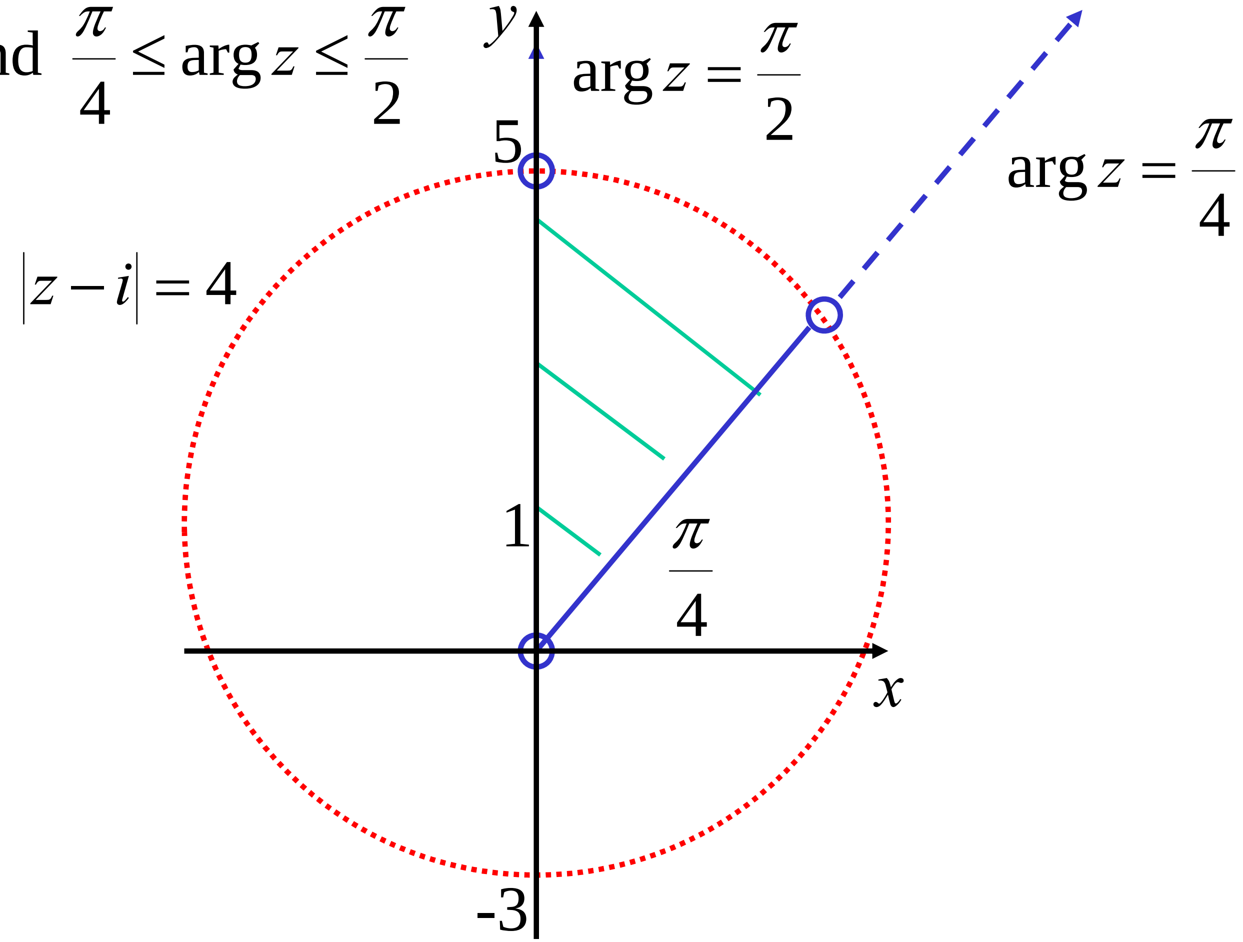
(ii) Locus

Locus problems can be done by an intuitive method or by letting $z = x + iy$ and using methods in Coordinate Geometry

Examples

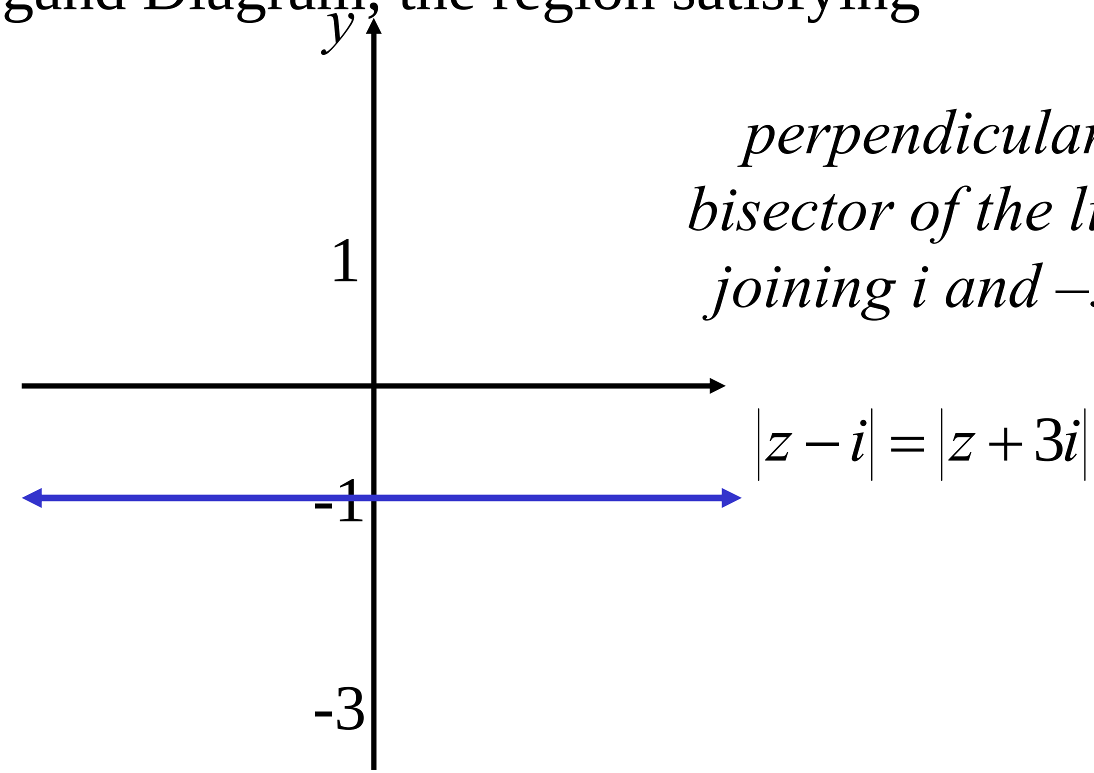
1. Sketch on an Argand Diagram, the region satisfying

$$|z-i| < 4 \text{ and } \frac{\pi}{4} \le \arg z \le \frac{\pi}{2}$$



2. Sketch on an Argand Diagram, the region satisfying

$|z-i|=|z+3i|$



perpendicular bisector of the line joining i and –3i

3. Describe the locus described by $|2z-3|=1$

$$|2z-3|=1$$

$$2\left|z-\frac{3}{2}\right|=1$$

$$\left|z-\frac{3}{2}\right|=\frac{1}{2}$$

$\therefore$ locus is a circle

$\text{centre}=\left(\frac{3}{2},0\right)$

$\text{radius}=\frac{1}{2}$

4. Describe the locus described by $|z-1|+|z+1|=4$

locus is ellipse

$$2a=4 \qquad ae=\pm 1 \qquad b^2=a^2\left(1-e^2\right)$$

$$a=2 \qquad e=\frac{1}{2} \qquad =4\left(1-\frac{1}{4}\right)$$

$$=3$$

$$\therefore \text{ locus is } \frac{x^2}{4}+\frac{y^2}{3}=1$$

5. Find the locus of z if $w=\dfrac{z+1}{z-1}$ and w is purely imaginary

If w is purely imaginary then $\arg w=\pm\dfrac{\pi}{2}$

$$i.e. \arg\left(\frac{z+1}{z-1}\right)=\pm\frac{\pi}{2}$$

$\therefore$ locus is the circle $x^2+y^2=1$, excluding $(\pm 1,0)$

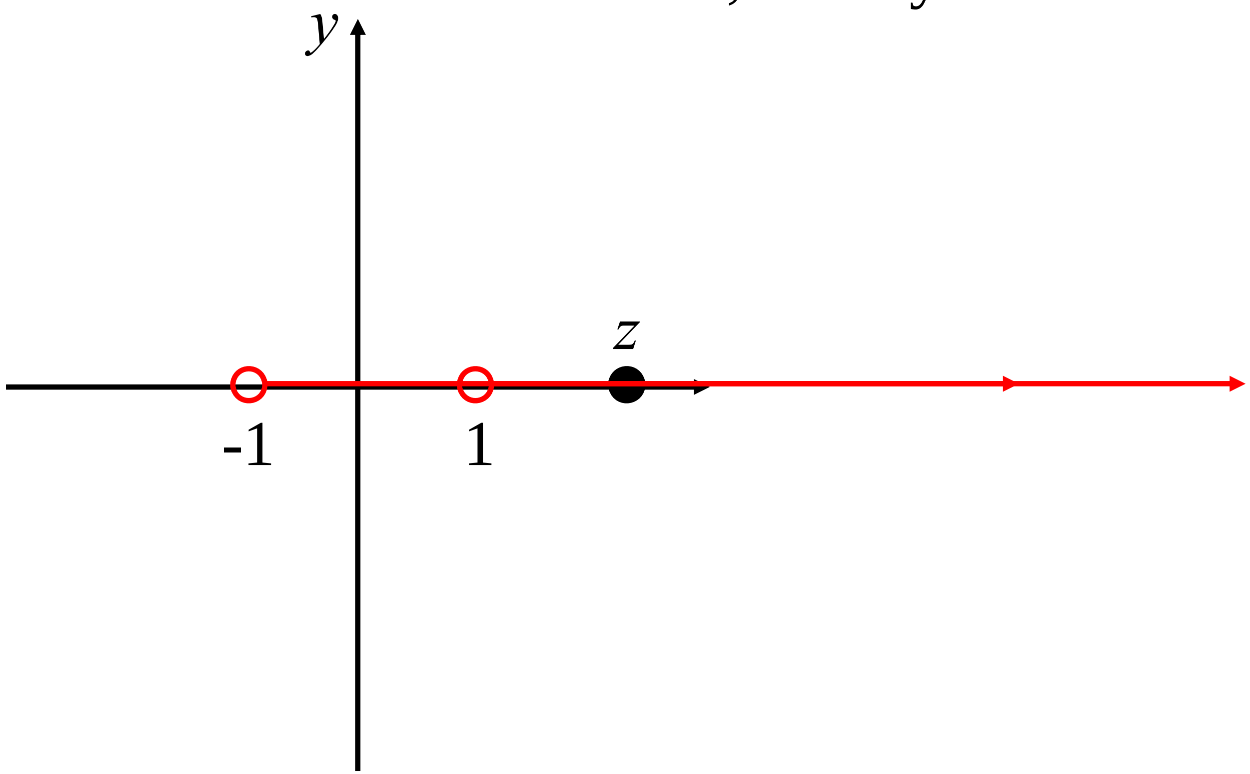
6. Find the locus of z if $w = \dfrac{z+1}{z-1}$ and w is purely real

If w is purely real then $\arg w = 0 \text{ or } \pi$

$$\text{i.e. } \arg\left(\frac{z+1}{z-1}\right) = 0 \text{ or } \pi$$

So the vectors $z+1$ and $z-1$ are either parallel or are in opposite directions

But z is on both vectors, so they must lie on the same line



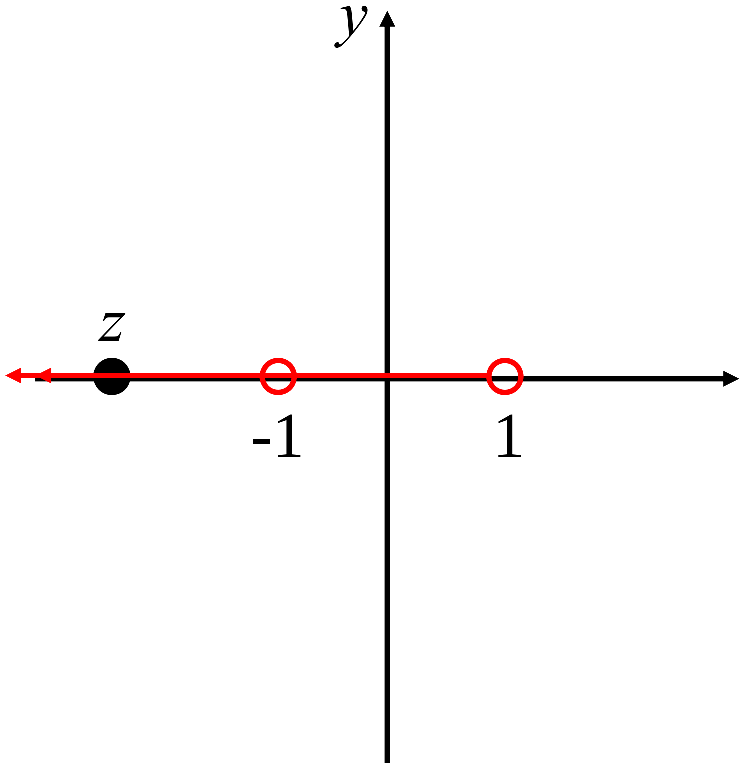
6. Find the locus of z if $w = \dfrac{z+1}{z-1}$ and w is purely real

If w is purely real then $\arg w = 0$ or π

$$\text{i.e. } \arg\left(\frac{z+1}{z-1}\right) = 0 \text{ or } \pi$$

So the vectors $z+1$ and $z-1$ are either parallel or are in opposite directions

But z is on both vectors, so they must lie on the same line



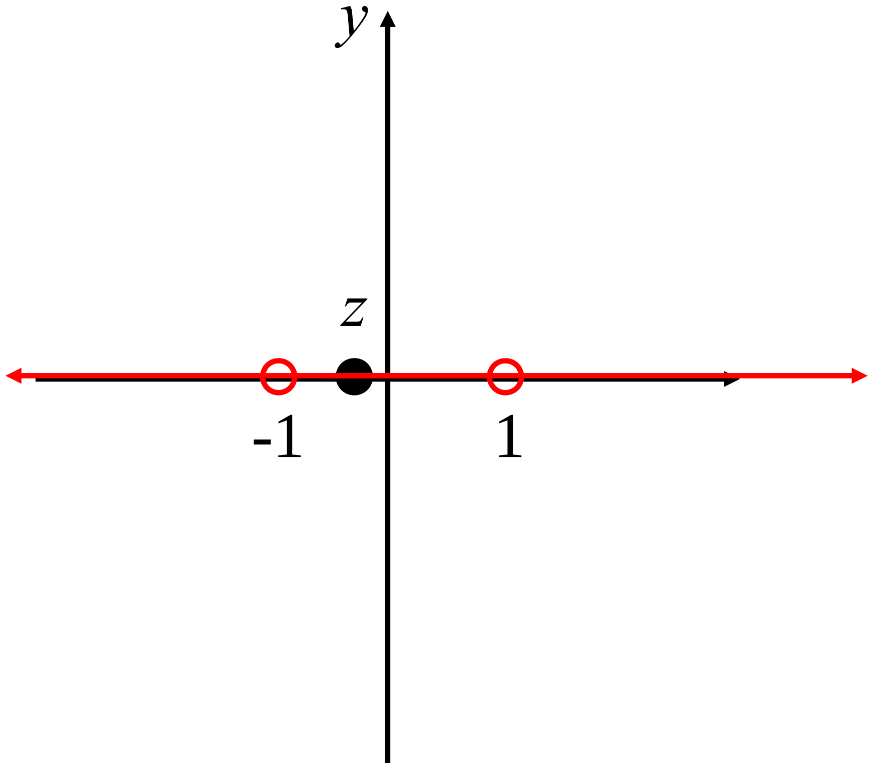
6. Find the locus of z if $w = \dfrac{z+1}{z-1}$ and w is purely real

If w is purely real then $\arg w = 0$ or π

$$\text{i.e. } \arg\left(\frac{z+1}{z-1}\right) = 0 \text{ or } \pi$$

So the vectors $z+1$ and $z-1$ are either parallel or are in opposite directions

But z is on both vectors, so they must lie on the same line



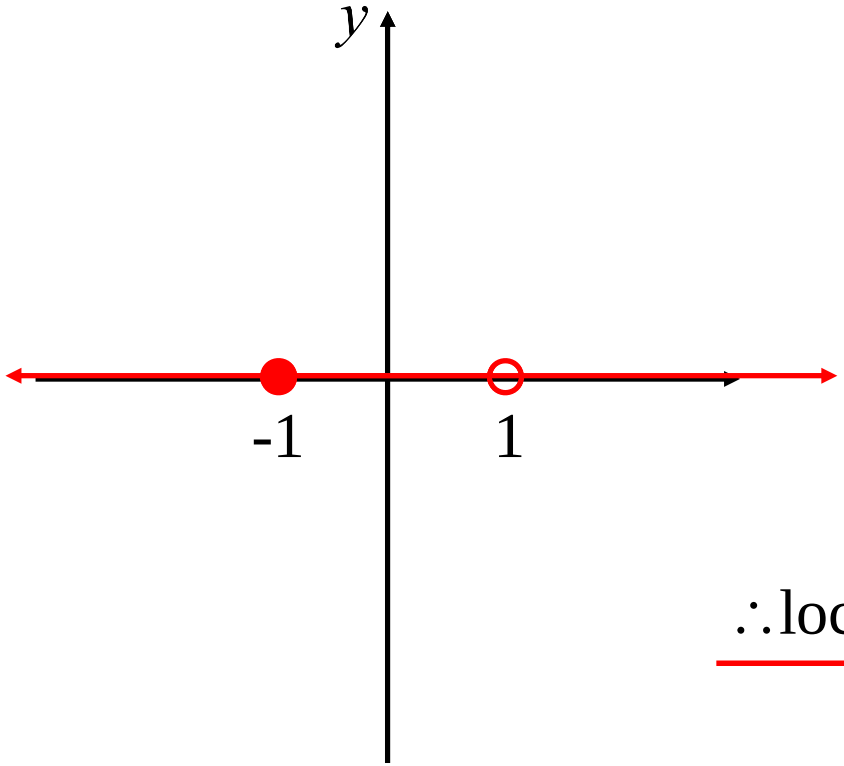
6. Find the locus of z if $w = \dfrac{z+1}{z-1}$ and w is purely real

If w is purely real then $\arg w = 0$ or π

$$\text{i.e. } \arg\left(\frac{z+1}{z-1}\right) = 0 \text{ or } \pi$$

So the vectors $z+1$ and $z-1$ are either parallel or are in opposite directions

But z is on both vectors, so they must lie on the same line

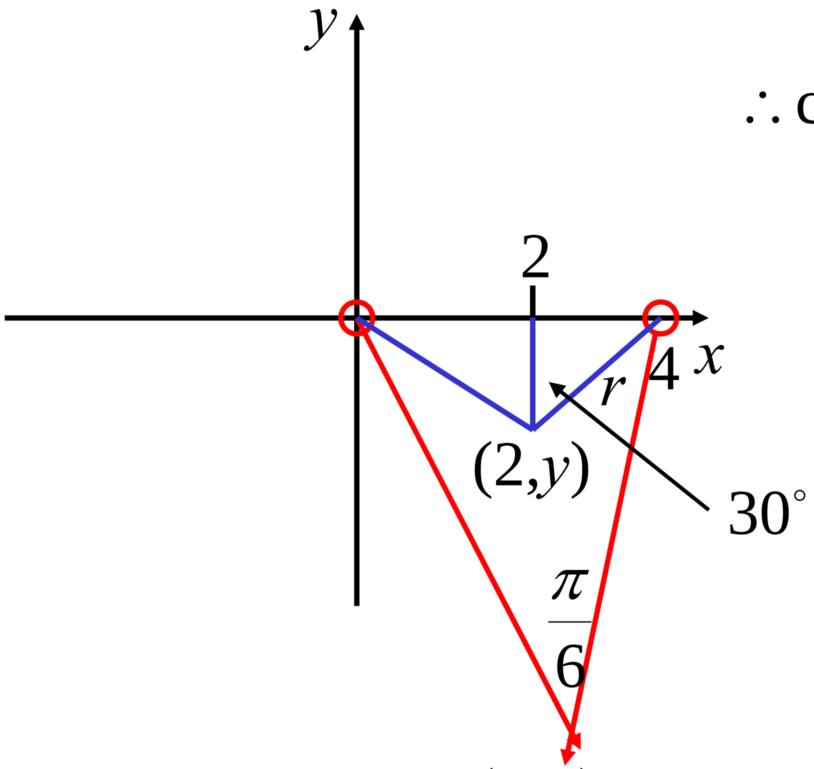


$\therefore$ locus is the line $y = 0$, excluding $(1, 0)$

7. Find the locus of z if $\arg\left(\frac{z}{z-4}\right)=\frac{\pi}{6}$

$$\arg\left(\frac{z}{z-4}\right)=\frac{\pi}{6}$$

$$\arg z-\arg(z-4)=\frac{\pi}{6}$$



NOTE: $\arg z>\arg(z\text{-}4)$

$\therefore$ below axis

$$\frac{y}{2}=\tan 60$$

$$y=2\tan 60$$

$$=2\sqrt{3}$$

$$r^2=2^2+\left(2\sqrt{3}\right)^2$$

$$r^2=16$$

$$r=4$$

$\therefore$ centre is $\left(2,-2\sqrt{3}\right)$

$\therefore$ locus is the major arc of the circle $(x-2)^2+\left(y+2\sqrt{3}\right)^2=16$ formed by the chord joining $(0,0)$ and $(4,0)$ but not including these points.

8. Find the locus of $w = z^2$ if $z^2 + (\overline{z})^2 = 4$

$$z^2 + (\overline{z})^2 = 4$$

$$(x+iy)^2 + (x-iy)^2 = 4$$

$$2x^2 - 2y^2 = 4$$

$$x^2 - y^2 = 2$$

$$w = X + iY \quad = z^2$$

$$= x^2 + 2ixy - y^2$$

$$X = x^2 - y^2 \qquad Y = 2xy$$

$$= 2$$

$\therefore$ the locus of w is the line $x = 2$