Name: ____________________

Teacher: ____________________

Class: ____________________

FORT STREET HIGH SCHOOL

2017

HIGHER SCHOOL CERTIFICATE COURSE

ASSESSMENT TASK 3: TRIAL HSC

Mathematics Extension 2

Time allowed: 3 hours
(plus 5 minutes reading time)

Syllabus Outcomes	Assessment Area Description and Marking Guidelines	Questions
E1	Chooses and applies appropriate mathematical techniques in order to solve a broad range of problems effectively	1-10
E3	Uses the relationship between algebraic and geometric representations of complex numbers	11
E6	Combines the ideas of algebra and calculus to determine the important features of the graphs of a wide variety of functions	12
E4	Uses efficient techniques for the algebraic manipulation required in dealing with questions such as those involving conic sections and polynomials	13
E7, E8	Applies further techniques of integration, such as slicing and cylindrical shells, integration by parts and recurrence formulae, to problems	14,15
E2-E8	Synthesises mathematical processes to solve harder problems and communicates solutions in an appropriate form	16

Section I	Total 10	Marks
Q1-Q10	/10	
Section II	Total 90	Marks
Q11	/15	
Q12	/15	
Q13	/15	
Q14	/15	
Q15	/15	
Q16	/15	
	Percent	

Total Marks 100

Section I 10 marks
Multiple Choice, attempt all questions.
Allow about 15 minutes for this section.

Section II 90 Marks
Attempt Questions 11-16.
Allow about 2 hours 45 minutes for this section.

<u>General Instructions</u>:

- <u>Questions 11-16</u> are to be started in a new booklet.
- The marks allocated for each question are indicated.
- <u>In Questions 11 – 16</u>, show relevant mathematical reasoning and/or calculations.
- Marks may be deducted for careless or badly arranged work.
- Board – approved calculators may be used.

Section I (10 marks)

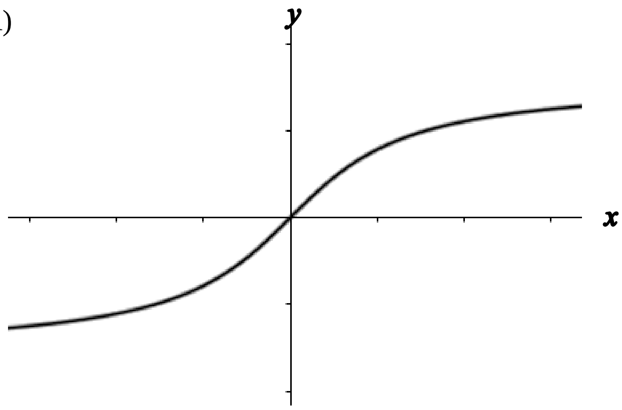
Attempt questions 1–10. Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–10.

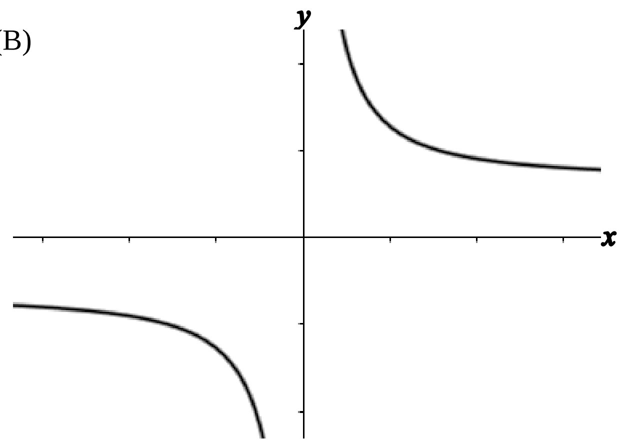
Question 1.

Which graph best describes $y = \left|\tan^{-1}(x)\right|$?

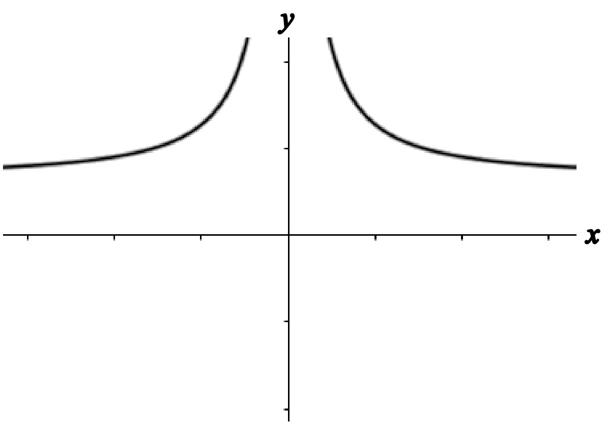
(A)



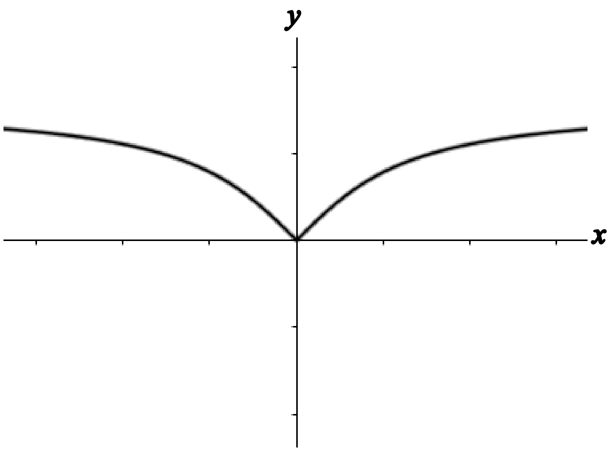
(B)



(C)



(D)



Question 2.

If $z = \sqrt{3} + i$ then $z - \dfrac{1}{z}$ is equal to

(A) $\cos\dfrac{\pi}{6} + i\sin\dfrac{\pi}{6}$

(B) $\sqrt{3}$

(C) $\dfrac{3\sqrt{3}}{4} + \dfrac{5i}{4}$

(D) 1

Question 3.

What are the coordinates of the foci if the equation of an ellipse is given by $4x^2+9y^2=36$

(A) $S\left(\pm\sqrt{5},0\right)$

(B) $S\left(\pm\sqrt{13},0\right)$

(C) $S\left(0,\pm\sqrt{5}\right)$

(D) $S\left(0,\pm\sqrt{13}\right)$

Question 4.

Find the remainder when $P(x)=x^4-3x^3+2x^2+1$ is divided by $x-i$.

(A) $-2+3i$

(B) 3

(C) $-3i$

(D) $3i$

Question 5.

What integral could be used to calculate $\int_0^{\frac{\pi}{2}}\frac{dx}{1+\sin x}$?

(A) $\int_0^1\frac{1}{(1+t)^2}dt$

(B) $\int_0^1\frac{1+t^2}{(1+t)^2}dt$

(C) $\frac{1}{2}\int_0^1\frac{1}{(1+t)^2}dt$

(D) $2\int_0^1\frac{1}{(1+t)^2}dt$

Question 6.

An object, of mass *m*, falling under gravity experiences resistance proportional to its velocity. Which expression best describes the terminal velocity of the object. Let the resistance force be given by $R = mkv$.

(A) $\dfrac{g}{k}$

(B) $\dfrac{mg}{k}$

(C) $g - k$

(D) $g + k$

Question 7.

Find $\int \sec^2\theta \tan^2\theta \ \ d\theta$.

(A) $\sec^2\theta + \dfrac{1}{2}\tan^2\theta \ + \ C$

(B) $\dfrac{1}{3}\tan^3\theta + \ C$

(C) $\tan^4\theta - \dfrac{1}{5}\tan^5\theta \ + \ C$

(D) $\tan^4\theta - \ln\left|\cos^4\theta\right| \ + \ C$

Question 8.

The polynomial $P(x) = x^3 - 5x^2 - 8x + 48$ has an integer double root at $x = \alpha$. Find the value of α.

(A) $\alpha = 0$

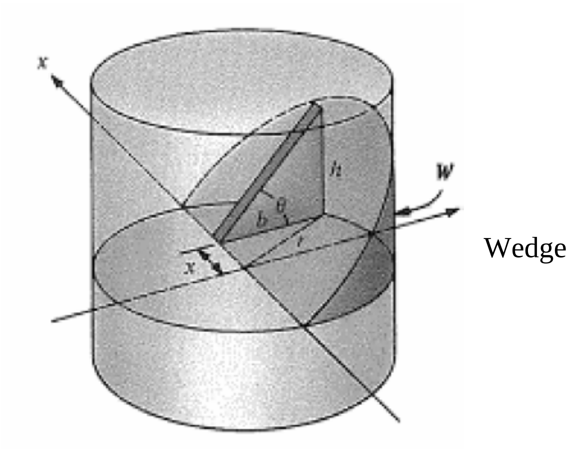
(B) $\alpha = 3$

(C) $\alpha = -3$

(D) $\alpha = 4$

Question 9.

The diagram shows a wedge cut from a cylinder of radius r. The angle from between the top and bottom of the wedge, θ , is $\frac{\pi}{6}$ radians. Triangular cross sections are taken perpendicular to the x axis.



Which expression best describes the volume of the wedge?

(A) $V = \int_{-r}^{r} \frac{1}{2\sqrt{3}}\left(r^2 - x^2\right)\, dx$

(B) $V = \int_{-r}^{r} \frac{1}{\sqrt{3}}\left(r^2 - x^2\right)\, dx$

(C) $V = \int_{-r}^{r} \left(r^2 - x^2\right)\, dx$

(D) $V = \int_{-r}^{r} \frac{\sqrt{3}}{2}\left(r^2 - x^2\right)\, dx$

Question 10.

In the Argand diagram, $ABCD$ is a square and the vertices A and B correspond to the complex numbers w and z.

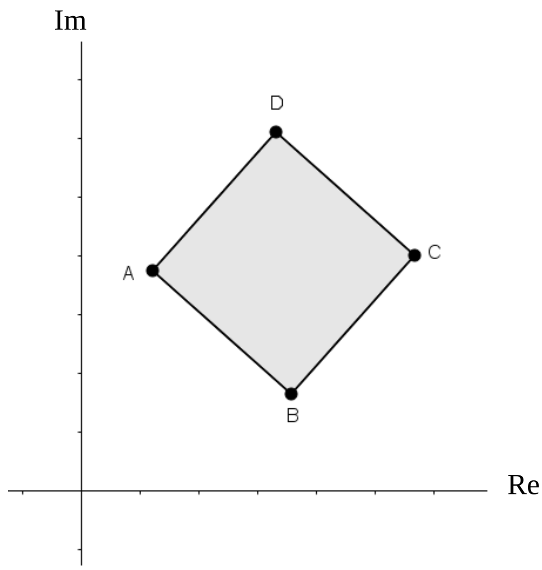
What complex number corresponds to the vector BD?

(A) $(z - w)(1 + i)$

(B) $(w - z)(1 - i)$

(C) $(w - z)(1 + i)$

(D) $(w + z)(1 - i)$



Section II (90 marks)

Attempt Questions 11–16. Allow about 2 hours and 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/ or calculations.

Question 11 (15 marks) Use a *separate* writing booklet

(a) (i) Write $z = \frac{\sqrt{2}}{2} + \frac{\sqrt{6}}{2}i$ in modulus argument form. **2**

(ii) Show that z is a solution of the equation **2**

$$z^6 + 4z^4 + 8\sqrt{3}\,i = 0$$

(b) Find two numbers whose sum is 6 and whose product is 13 **2**

(c) Describe in geometric terms the curve described by $2|z| = z + \overline{z} + 4$ **3**

(d) ω is a non-real cube root of unity.

(i) Find the value of $\frac{1}{\omega^2} + \frac{1}{\omega}$ **1**

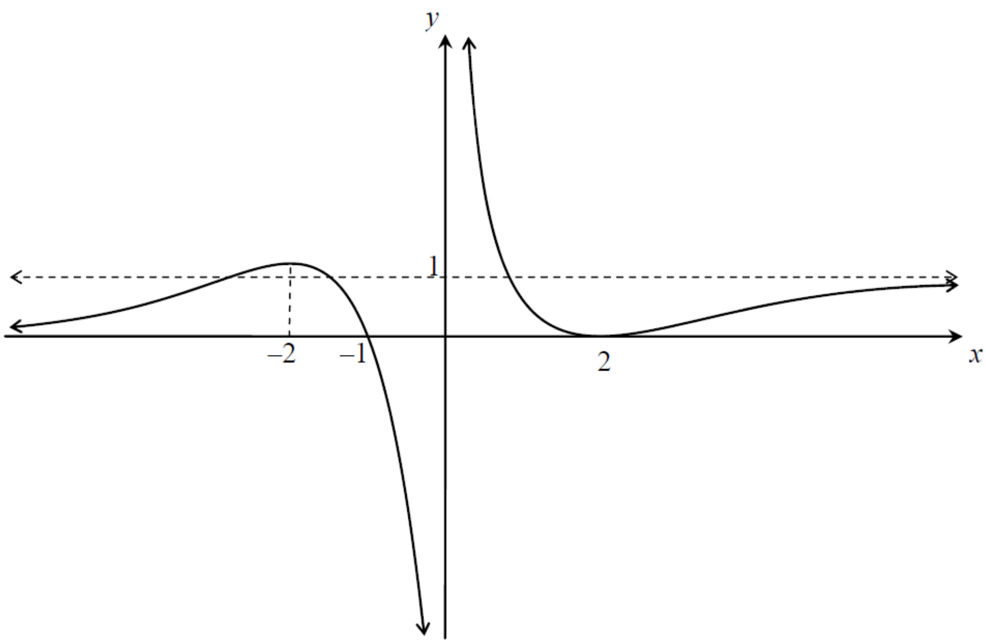
(ii) Show that $\frac{1+2\omega+3\omega^2}{2+3\omega+\omega^2} + \frac{1+2\omega+3\omega^2}{3+\omega+2\omega^2} = -1$ **2**

(e) Sketch on an Argand diagram the locus of z where the following conditions hold. **3**

$$0 \le \arg(z+1-i) \le \frac{3\pi}{4} \text{ and } |z+1-i| \le 2$$

Question 12 (15 marks) Use a *separate* writing booklet

(a) The graph of $y = f(x)$ is displayed below. The lines $y = 1, x = 0$ and $y = 0$ are asymptotes.



Sketch each of the graphs below and, without using calculus, clearly label any maxima or minima, intercepts and the equations of any asymptotes.

(i) $y = \left| f(x) \right|$ **2**

(ii) $y = e^{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = \dfrac{1}{f(x)}$ **2**

(b) State the domain and range of $f(x) = \ln\left(\cos^{-1} x\right)$ **2**

(c) Find the equation of the tangent to the curve $x^3 + y^3 - 8y + 7 = 0$ at the point $P(1,2)$ **2**

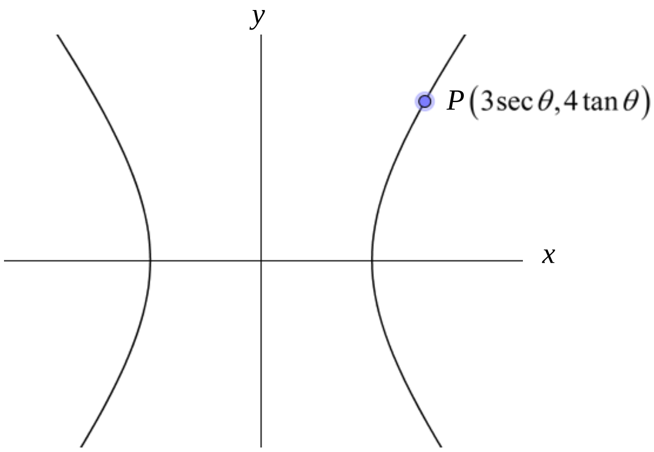
(d) Find all real roots of the polynomial **3**

$$P(x) = x^4 - x^3 - 4x^2 - 2x - 12$$

given one of the roots is $i\sqrt{2}$.

Question 13 (15 marks) Use a *separate* writing booklet

(a) A hyperbola is defined by the equation $16x^2 - 9y^2 = 144$.



(i) Find the coordinates of the foci and the equations of each directrix and asymptote. **3**

(ii) Find the gradient of the tangent to the hyperbola at point $P(3\sec\theta, 4\tan\theta)$. **2**

(iii) Show that the tangent to the hyperbola at P has the equation $4x = 3y\sin\theta + 12\cos\theta$. **2**

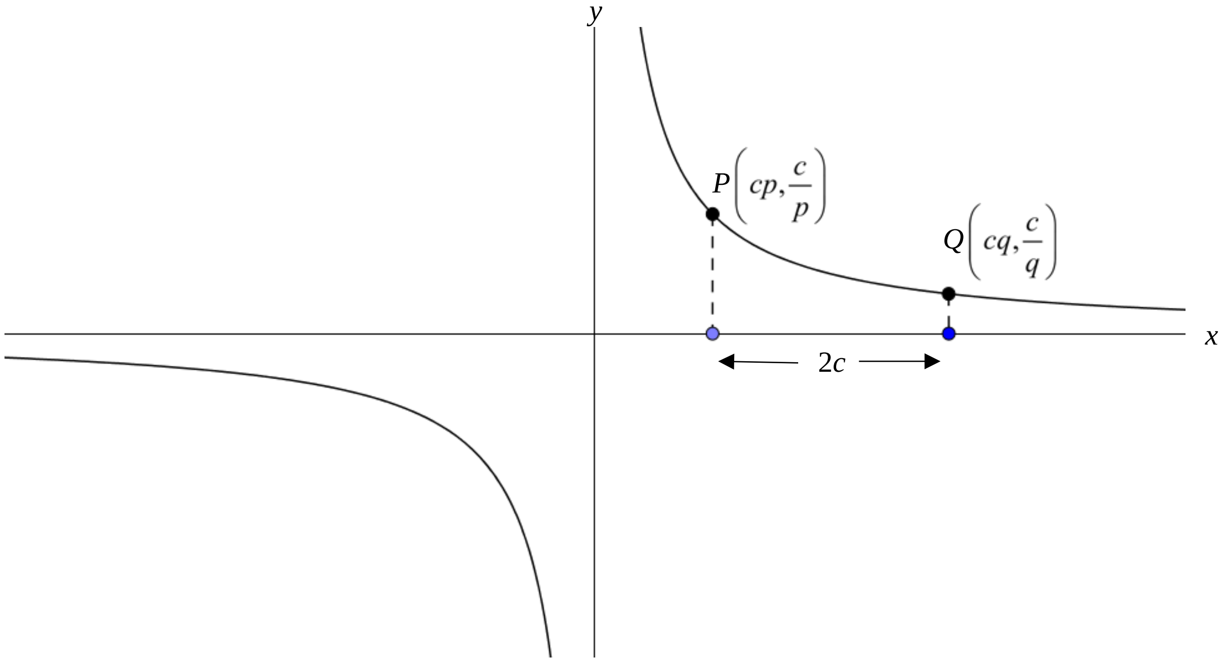
(iv) Given $0 < \theta < \frac{\pi}{2}$, show that Q, the point of intersection of the tangent to the hyperbola at P and the nearer directrix, has coordinates $Q\left(\frac{9}{5}, \frac{12 - 20\cos\theta}{5\sin\theta}\right)$. **2**

(v) Show that lines joining SP and SQ are perpendicular. **3**

(vi) Hence show the area of the triangle formed by PSQ is $\frac{2(5 - 3\cos\theta)^2}{5\sin\theta\cos\theta}$. **3**

Question 14 (15 marks) Use a *separate* writing booklet

(a) The chord PQ on the rectangular hyperbola $xy = c^2$ is constructed such that the horizontal distance between points P and Q has a constant length $2c$, where points P and Q lie in the first quadrant.



Find the locus of the midpoint of PQ in terms of x, y and c. **3**

(b) The region bounded by the parabola $y^2 = 4x$ and the line $x = 2$ is rotated about the line $x = 6$. **4**

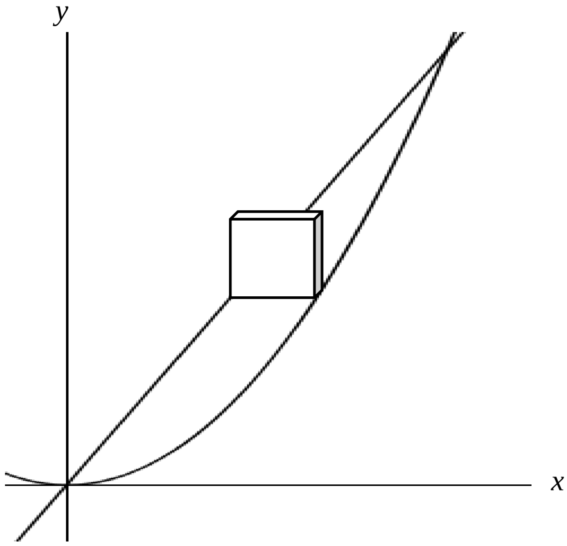
Using the method of cylindrical shells, find the volume of the solid formed.

(c) Using the substitution $u^2 = 4 - x^2$ evaluate $\int_0^2 x^3\sqrt{4-x^2}\ dx$ **4**

(d) Use the method of integration by parts to evaluate $\int_0^{\frac{\pi}{2}} e^x \cos x \quad dx$ **4**

Question 15 (15 marks) Use a *separate* writing booklet

(a)



The base of a solid is the region in the first quadrant bounded by the graphs of $y = x$ and $y = x^2$. Each cross section perpendicular to the y - axis is a square as shown in the diagram.

Find the volume of the solid formed. **4**

(b) (i) Find numbers a, b and c such that **3**

$$\frac{x^2}{4x^2 - 9} \equiv a + \frac{b}{2x-3} + \frac{c}{2x+3}$$

(ii) Hence evaluate $\int_0^1 \frac{x^2}{4x^2 - 9}\,dx$ **2**

(c) An object falls from rest, under gravity, for a time of $\frac{1}{2k}$ seconds before hitting water and experiencing an upward resistance of mkv, where m is the mass of the object, v the object's velocity and k is a positive constant.

Let g be the acceleration due to gravity and take the downwards motion to be in the positive direction.

(i) Show that when the object hits the water its velocity will be $\frac{g}{2k}$ and **2**

the distance travelled is $\frac{g}{8k^2}$

(ii) Show that the total distance travelled when the object's velocity is $\frac{3g}{4k}$ is given by **4**

$$x = \frac{g}{k^2}\ln 2 - \frac{g}{8k^2}$$

Question 16 (15 marks) Use a *separate* writing booklet

(a) The polynomial $x^4 - 5x^3 - 2x^2 + 3x + 1 = 0$ has roots α, β, γ and δ.

Find an equation with roots $\alpha^2 - 1, \beta^2 - 1, \gamma^2 - 1$ and $\delta^2 - 1$. **2**

(b) Let $I_n = \int \frac{dx}{\left(1+x^2\right)^n}$ where n is a non-negative integer.

(i) Show that $I_{n+1} = \frac{1}{2n} \frac{x}{\left(1+x^2\right)^n} + \frac{2n-1}{2n} I_n$. **3**

(ii) Hence find I_3. **2**

(c) Two stones are thrown simultaneously from the same point in the same direction and with the same angle of projection, α, but with different velocities U, V metres per second $U < V$.

The slower stone hits the ground at a point P on the same level as the point of projection. At that instant the faster stone just clears a wall of height h metres above the level of projection and its (downward) path makes an angle β with the horizontal.

(i) Express the distance from P to the foot of the wall in terms of h and α only. **3**

(ii) Show that $V\left(\tan\alpha + \tan\beta\right) = 2U \tan\alpha$. **3**

(iii) Deduce that if, $\beta = \frac{1}{2}\alpha$, then $U < \frac{3}{4}V$. **2**

End of examination.

Section I (10 marks)

Attempt questions 1–10. Allow about 15 minutes for this section.

Use the multiple-choice answer sheet for Questions 1–10.

Multiple Choice Answer Sheet

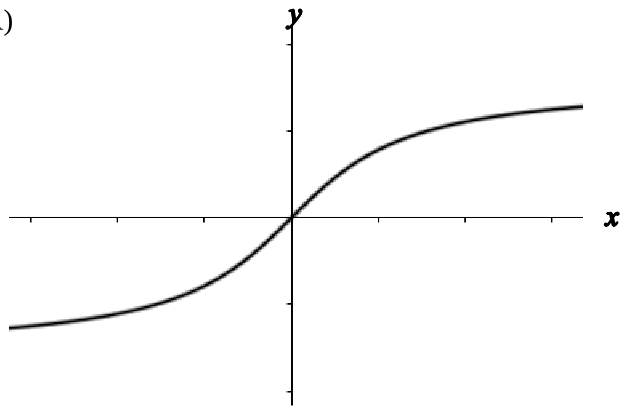
Circle the correct answer in pen

1	A	B	C	D
2	A	B	C	D
3	A	B	C	D
4	A	B	C	D
5	A	B	C	D
6	A	B	C	D
7	A	B	C	D
8	A	B	C	D
9	A	B	C	D
10	A	B	C	D

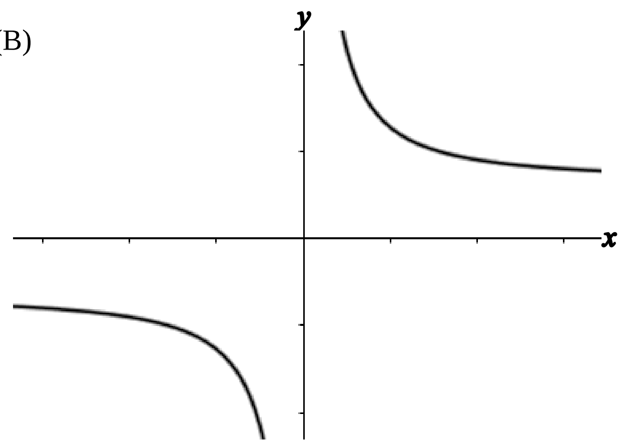
Question 1.

Which graph best describes $y = \left|\tan^{-1}(x)\right|$?

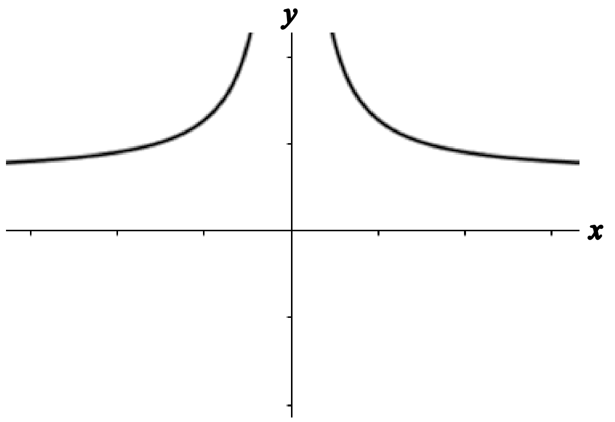
(A)



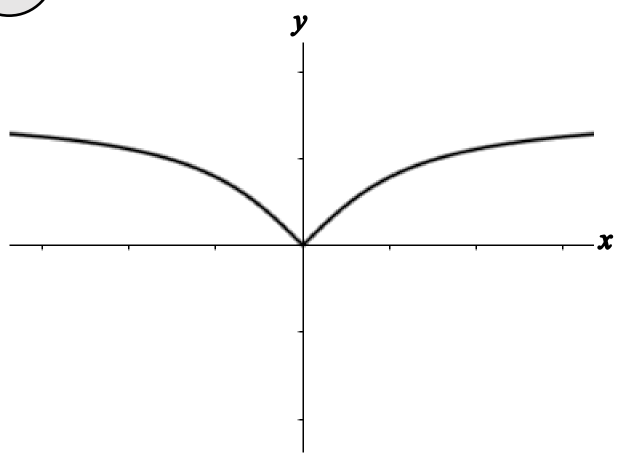
(B)



(C)



(D)



Question 2.

If $z = \sqrt{3} + i$ then $z - \frac{1}{z}$ is equal to

(A) $\cos\frac{\pi}{6} + i\sin\frac{\pi}{6}$

(B) $\sqrt{3}$

(C) $\frac{3\sqrt{3}}{4} + \frac{5i}{4}$

(D) 1

Question 3.

What are the coordinates of the foci if the equation of an ellipse is given by $4x^2 + 9y^2 = 36$

(A) $S\left(\pm\sqrt{5}, 0\right)$

(B) $S\left(\pm\sqrt{13}, 0\right)$

(C) $S\left(0, \pm\sqrt{5}\right)$

(D) $S\left(0, \pm\sqrt{13}\right)$

Question 4.

Find the remainder when $P(x) = x^4 - 3x^3 + 2x^2 + 1$ is divided by $x - i$.

(A) $-2 + 3i$

(B) 3

(C) $-3i$

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Question 5.

What integral could be used to calculate $\int_0^{\frac{\pi}{2}} \frac{dx}{1+\sin x}$?

(A) $\int_0^1 \frac{1}{(1+t)^2}\,dt$

(B) $\int_0^1 \frac{1+t^2}{(1+t)^2}\,dt$

(C) $\frac{1}{2}\int_0^1 \frac{1}{(1+t)^2}\,dt$

(D) $2\int_0^1 \frac{1}{(1+t)^2}\,dt$

Question 6.

An object, of mass *m*, falling under gravity experiences resistance proportional to its velocity. Which expression best describes the terminal velocity of the object.

(A) $\frac{g}{k}$

(B) $\frac{mg}{k}$

(C) $g-k$

(D) $g+k$

Question 7.

Find $\int \sec^2\theta \tan^2\theta \; d\theta$.

(A) $\sec^2\theta + \frac{1}{2}\tan^2\theta \; + \; C$

(B) $\frac{1}{3}\tan^3\theta + \; C$

(C) $\tan^4\theta - \frac{1}{5}\tan^5\theta \; + \; C$

(D) $\tan^4\theta - \ln\left|\cos^4\theta\right| \; + \; C$

Question 8.

The polynomial $P(x) = x^3 - 5x^2 - 8x + 48$ has an integer double root at $x = \alpha$. Find the value of α.

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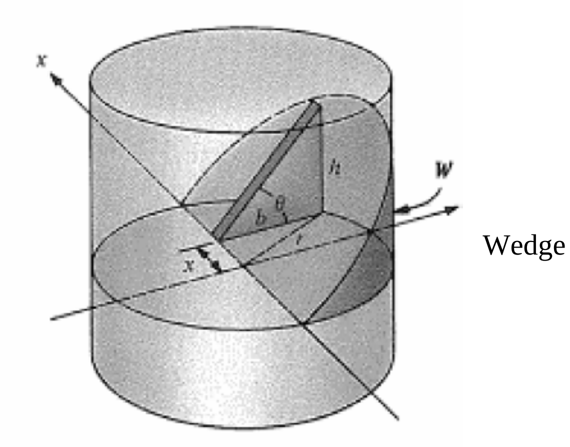
Question 9.

The diagram shows a wedge cut from a cylinder of radius r. The angle from between the top and bottom of the wedge, θ , is $\frac{\pi}{6}$ radians. Triangular cross sections are taken perpendicular to the x axis.



Which expression best describes the volume of the wedge?

(A) $V=\int_{-r}^{r}\frac{1}{2\sqrt{3}}\left(r^2-x^2\right)\,dx$

(B) $V=\int_{-r}^{r}\frac{1}{\sqrt{3}}\left(r^2-x^2\right)\,dx$

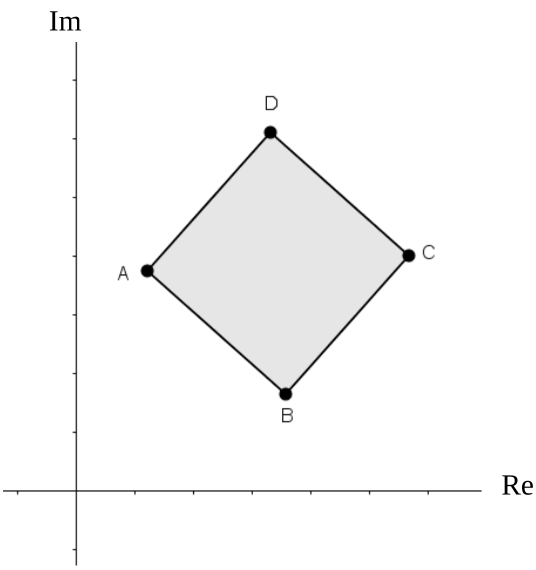
(C) $V=\int_{-r}^{r}\left(r^2-x^2\right)\,dx$

(D) $V=\int_{-r}^{r}\frac{\sqrt{3}}{2}\left(r^2-x^2\right)\,dx$

Question 10.

In the Argand diagram, $ABCD$ is a square and the vertices A and B correspond to the complex numbers w and z.

What complex number corresponds to the vector BD?



(A) $(z-w)(1+i)$

(B) $(w-z)(1-i)$

(C) $(w-z)(1+i)$

(D) $(w+z)(1-i)$

Section II (90 marks)

Question 11 (15 marks) Use a *separate* writing booklet

(a) (i) Write $z=\dfrac{\sqrt{2}}{2}+\dfrac{\sqrt{6}}{2}i$ in modulus argument form.

Solution

$$|z|=\sqrt{\left(\frac{\sqrt{2}}{2}\right)^2+\left(\frac{\sqrt{6}}{2}\right)^2} \qquad \arg z=\tan^{-1}\left(\frac{\frac{\sqrt{6}}{2}}{\frac{\sqrt{2}}{2}}\right)$$

$$=\sqrt{\frac{1}{4}+\frac{3}{2}} \qquad =\tan^{-1}\left(\sqrt{3}\right)$$

$$=\sqrt{2} \qquad =\frac{\pi}{3}$$

$$\therefore \quad z=\sqrt{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)$$

Suggested marking scheme

2 Correct response

1 Not writing in mod/arg form or
One incorrect modulus or argument

Marker's comments

Generally well done.

Students that tried to evaluate $\bmod z$ without showing working invariably got it wrong.

(ii) Show that z is a solution of the equation

$$z^6+4z^4+8\sqrt{3}\,i=0$$

Solution

$$z^6+4z^4+8\sqrt{3}\,i=\left[\sqrt{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]^6+4\left[\sqrt{2}\left(\cos\frac{\pi}{3}+i\sin\frac{\pi}{3}\right)\right]^4+8\sqrt{3}\,i$$

$$=\left[8\left(\cos 2\pi+i\sin 2\pi\right)\right]+4\left[4\left(\cos\frac{4\pi}{3}+i\sin\frac{4\pi}{3}\right)\right]+8\sqrt{3}\,i$$

$$=[8]+4\left[4\left(-\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)\right]+8\sqrt{3}\,i$$

$$=8+\left[-8-8\sqrt{3}\,i\right]+8\sqrt{3}\,i$$

$$=0$$

Therefore z is a solution as $P(z)=0$ by the remainder theorem.

Suggested marking scheme

2 Correct response

1 With one error or
Incomplete setting out

Marker's comments

Students should be mindful to make a concluding statement.

(b) Find two numbers whose sum is 6 and whose product is 13

Solution

This problem is akin to solving $z^2-6z+13=0$

$$z=\frac{6\pm\sqrt{36-4\times1\times13}}{2}$$

$$=\frac{6\pm\sqrt{-16}}{2}$$

$$=3\pm 2i$$

Suggested marking scheme

2 Correct response

1 Recognizing equation to solve

Marker's comments

Mostly well done.

Students who got the wrong answer should have checked the product or sum of their numbers.

(c) Describe in geometric terms the curve described by $2|z| = z + \overline{z} + 4$

Solution

Let $z = x + iy$

$$2|z| = z + \overline{z} + 4$$

$$2\sqrt{x^2 + y^2} = x + iy + x - iy + 4$$

$$\sqrt{x^2 + y^2} = x + 2$$

$$x^2 + y^2 = x^2 + 4x + 4$$

$$y^2 = 4(x+1)$$

Therefore the locus is a parabola with the vertex $(-1,0)$, directrix $x = -2$ and focus $S(0,0)$.

Suggested marking scheme

3 Correct response

2 Finding the equation of the locus but not describing the locus

1 Partial solution

Marker's comments

Students should leave the equation in locus form i.e. $y^2 = 4(x+1)$.

Describe means to explain the equation in words referencing the key features of the parabola. Students should avoid using *"sideways"*. Graphs while helpful do not attract marks.

There were many careless error with this question.

(d) ω is a non-real cube root of unity.

(i) Find the value of $\frac{1}{\omega^2} + \frac{1}{\omega}$

Solution

ω is a non-real cube root of unity implies

$$\omega^3 = 1$$
$$\omega^3 - 1 = 0$$
$$(\omega - 1)(\omega^2 + \omega + 1) = 0$$

$$\omega \neq 1, \quad and$$
$$\omega^2 + \omega + 1 = 0$$

$$\frac{1}{\omega^2} + \frac{1}{\omega} = \frac{\omega + \omega^2}{\omega^3}$$
$$= \frac{-1}{1}$$
$$= -1$$

Suggested marking scheme

1 Correct response

Marker's comments

Generally well done.

(ii) Show that $\dfrac{1+2\omega+3\omega^2}{2+3\omega+\omega^2}+\dfrac{1+2\omega+3\omega^2}{3+\omega+2\omega^2}=-1$

Solution

Trying to rewrite the *LHS* in terms of part (i)

$$\frac{1+2\omega+3\omega^2}{2+3\omega+\omega^2}+\frac{1+2\omega+3\omega^2}{3+\omega+2\omega^2}=\frac{1+2\omega+3\omega^2}{2+3\omega+\omega^2}\times\frac{\omega^2}{\omega^2}+\frac{1+2\omega+3\omega^2}{3+\omega+2\omega^2}\times\frac{\omega}{\omega}$$

$$=\frac{\omega^2+2\omega^3+3\omega^4}{\left(2+3\omega+\omega^2\right)\omega^2}+\frac{\omega+2\omega^2+3\omega^3}{\left(3+\omega+2\omega^2\right)}$$

$$=\frac{\omega^2+2+3\omega}{\left(2+3\omega+\omega^2\right)\omega^2}+\frac{\omega+2\omega^2+3}{\left(3+\omega+2\omega^2\right)\omega}$$

$$=\frac{1}{\omega^2}+\frac{1}{\omega}$$

$$=-1$$

Suggested marking scheme

2 Correct response

1 partial solution

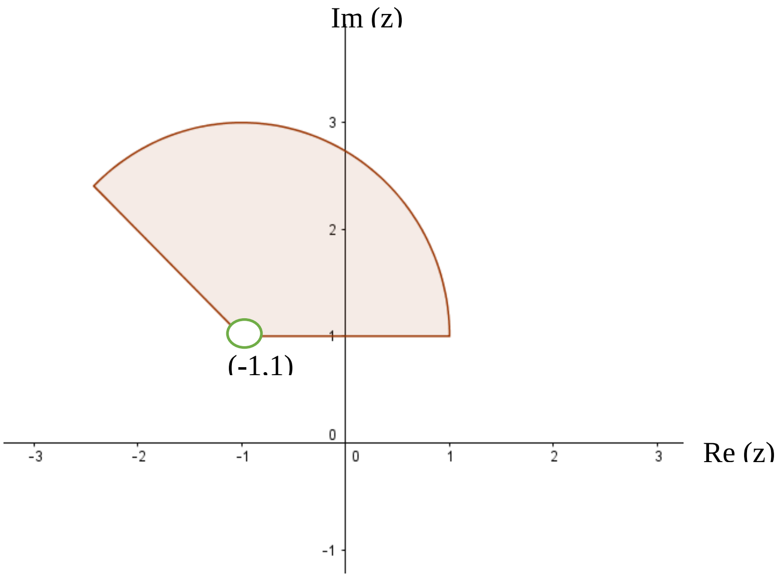
Marker's comments

This was the most difficult part of the question and many students did **not** make the connection between part (i). There were some very lengthy answers that made poor use of the time.

(e) Sketch on an Argand diagram the locus of z where the following conditions hold.

$$0\le\arg\left(z+1-i\right)\le\frac{3\pi}{4}\text{ and }\left|z+1-i\right|\le 2$$

Solution



Suggested marking scheme

3 Correct response

2 One error e.g. including the full circle or unclear radius or closed circle etc

1 Two errors.

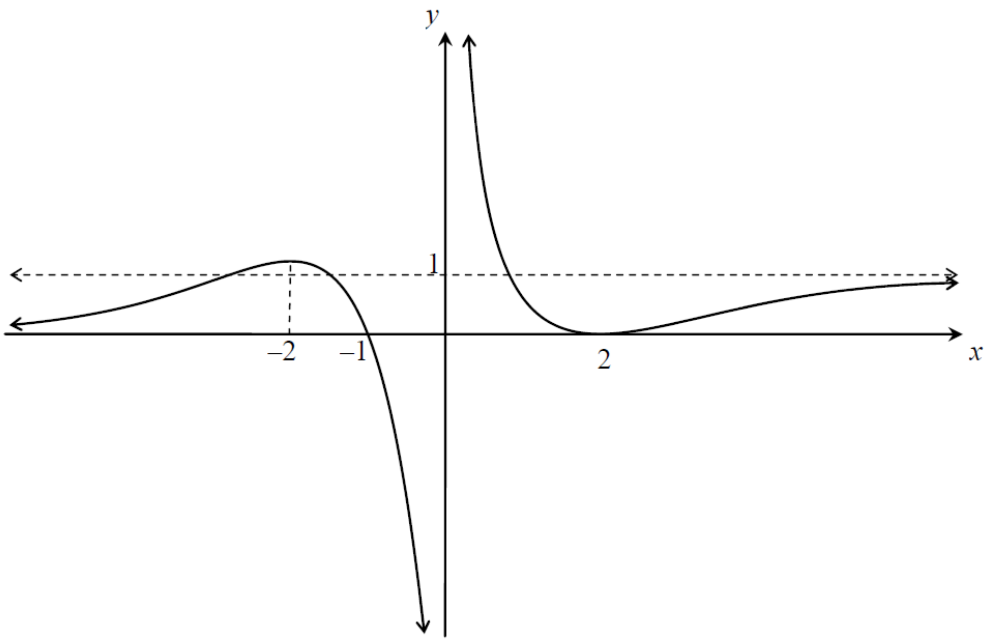
Marker's comments

Very poorly answered. Many students

- confused *AND* with *OR* and included the full circle.
- left out an open circle around $(-1,1)$ on the Argand diagram where the argument does not exist.
- did **not** clearly show the radius of the circle.

Question 12 (15 marks) Use a *separate* writing booklet

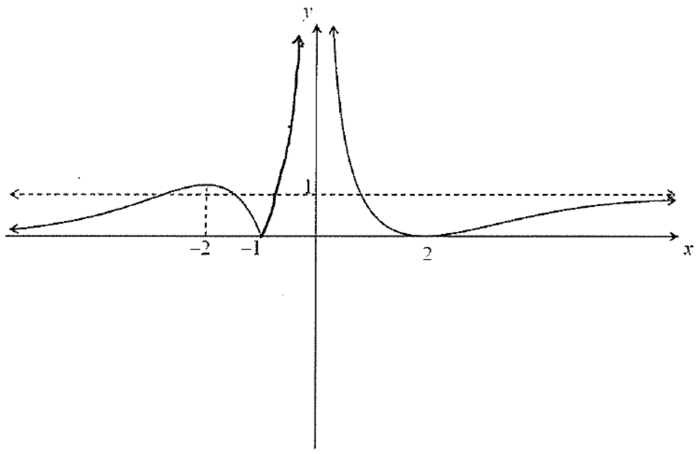
(a) The graph of $y = f(x)$ is displayed below. The lines $y = 1, x = 0$ and $y = 0$ are asymptotes.



Sketch each of the graphs below and, without using calculus, clearly label any maxima or minima, intercepts and the equations of any asymptotes.

(i) $y = |f(x)|$

Solution



Suggested marking scheme

2 Correct response including intercepts, max/min, shape i.e. how it approaches asymptotes.

1 partial solution with correct intercepts, max/min points or shape including how it approaches asymptotes.

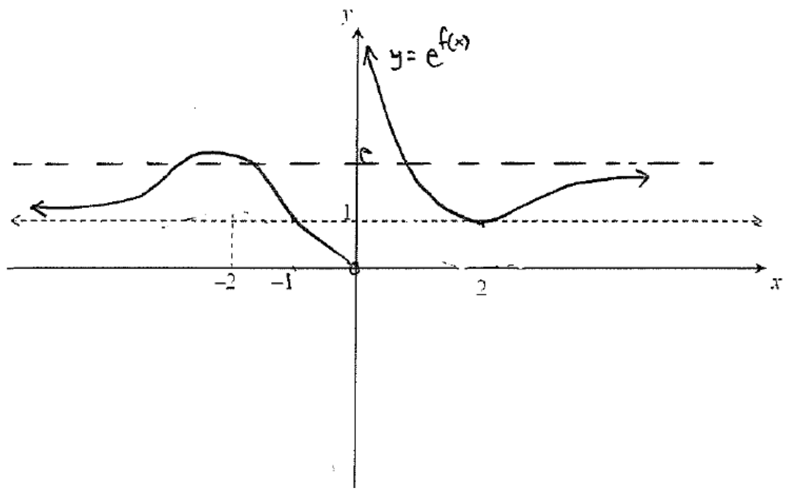
Marker's comments

Mostly well done.

Some students incorrectly found $y = f(|x|)$ instead.

(ii) $y = e^{f(x)}$

Solution



Note: At $x = -2$, $y > e$

Suggested marking scheme

2 Correct response including intercepts, max/min, shape i.e. how it approaches asymptotes.

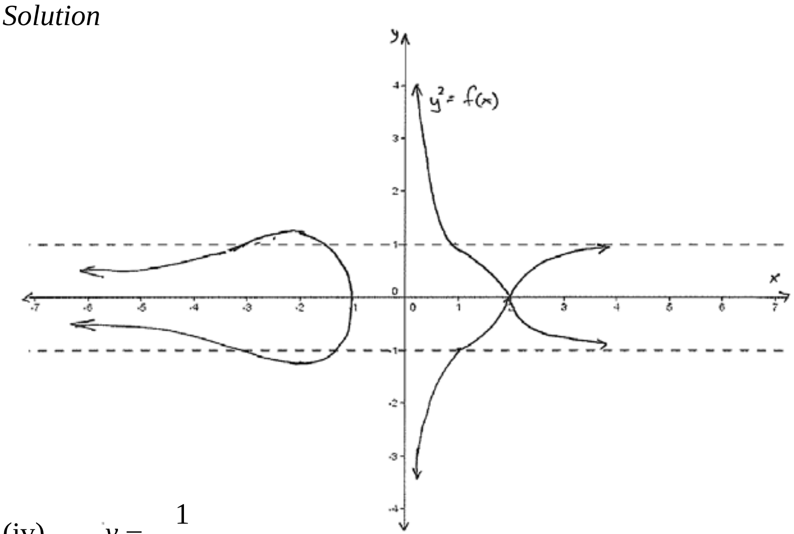
1 partial solution with correct intercepts, max/min points or shape including how it approaches asymptotes.

Marker's comments

Some students did not mark asymptotes.

(iii) $y^2 = f(x)$

Solution



Suggested marking scheme

2 Correct response including intercepts, max/min, shape i.e. how it approaches asymptotes.

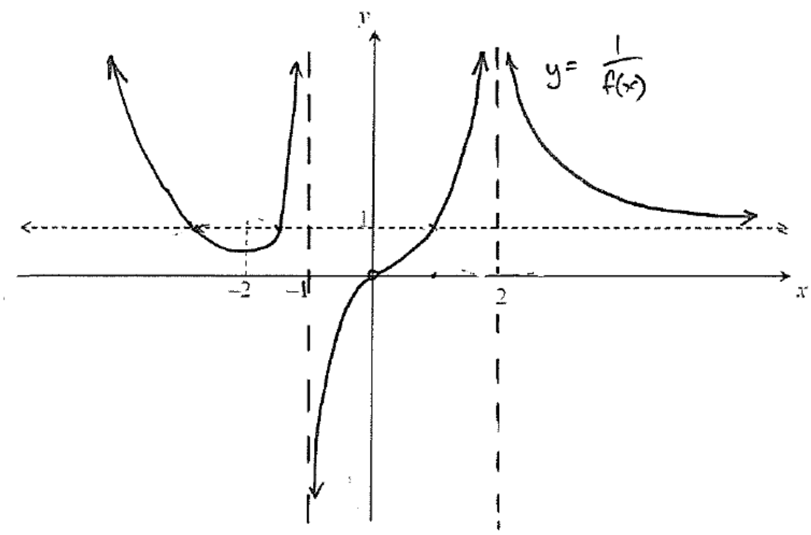
1 partial solution with correct intercepts, max/min points or shape including how it approaches asymptotes.

Marker's comments

$x = -1$ *and* $x = 2$ are critical points and students needed to indicate an undefined gradient and in the case of $x = 2$, a cusp.

(iv) $y = \dfrac{1}{f(x)}$

Solution



Suggested marking scheme

2 Correct response including intercepts, max/min, shape i.e. how it approaches asymptotes.

1 partial solution with correct intercepts, max/min points or shape including how it approaches asymptotes.

Marker's comments

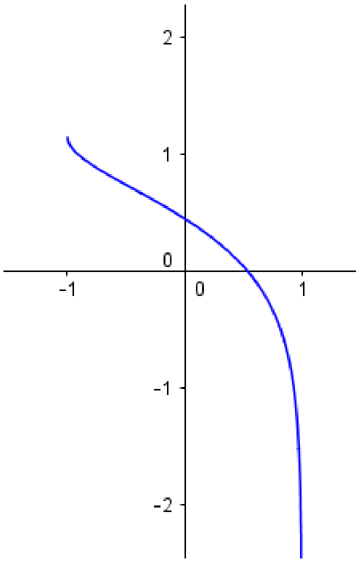
Mostly well done.

Many students did not mark $x = 0$ with an open circle.

(b) State the domain and range of $f(x)=\ln\left(\cos^{-1}x\right)$

Solution

Consider the graph of $f(x)=\ln\left(\cos^{-1}x\right)$



Domain: $-1 \le x < 1$

Range: $y \le \ln \pi$

Suggested marking scheme

2 Correct response
1 partial solution

Marker's comments

Poorly answered.

Students would have benefitted from drawing a diagram.

(c) Find the equation of the tangent to the curve $x^3+y^3-8y+7=0$ at the point $P(1,2)$

Solution

Find the equation of the gradient

$$3x^2+3y^2\times\frac{dy}{dx}-8\times\frac{dy}{dx}=0$$

$$\frac{dy}{dx}\left(3y^2-8\right)=-3x^2$$

$$\frac{dy}{dx}=\frac{-3x^2}{3y^2-8}$$

At $P(1,2)$ $\quad \frac{dy}{dx}=\frac{-3}{4}$

Therefore the equation of the tangent is

$$y-2=-\frac{3}{4}(x-1)$$

$$3x+4y-11=0$$

Suggested marking scheme

2 Correct response
1 partial solution

Marker's comments

Generally well done.

(d) Find all real roots of the polynomial

$$P(x) = x^4 - x^3 - 4x^2 - 2x - 12$$

given one of the roots is $i\sqrt{2}$.

Solution

Given $i\sqrt{2}$ is a root then by complex conjugate theorem $-i\sqrt{2}$ is a root.

$\Rightarrow (x - i\sqrt{2})(x + i\sqrt{2})$ are factors of $P(x)$

$= (x^2 + 2)$

By polynomial division $P(x) = (x^2 + 2)(x^2 - x - 6)$

Therefore the real roots of $P(x)$ are $x = -2, 3$

Suggested marking scheme

3 Correct response

2 One error or factorising $P(x)$

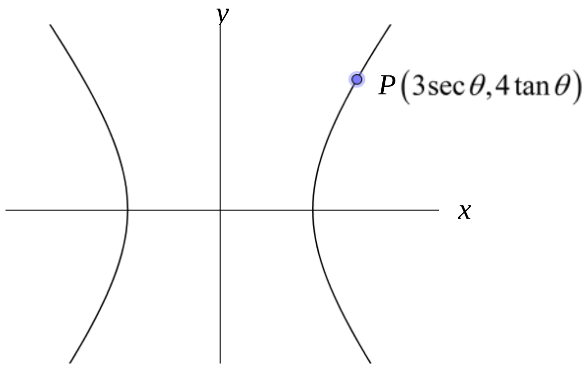
1 recognising $-i\sqrt{2}$ is a root

Marker's comments

Generally well done.

Question 13 (15 marks)

(a) A hyperbola is defined by the equation $16x^2 - 9y^2 = 144$.



(i) Find the coordinates of the foci and the equations of each directrix and asymptote.

Solution

Finding the values of *a* and *b*.

$$16x^2 - 9y^2 = 144$$
$$\frac{x^2}{9} - \frac{y^2}{16} = 1$$
$$\Rightarrow a = 3, b = 4$$

Finding the eccentricity

$$e = \sqrt{1 + \frac{b^2}{a^2}}$$
$$= \sqrt{1 + \frac{16}{9}}$$
$$= \frac{5}{3}$$

Foci: $S(\pm ae, 0)$

$$S\left(\pm 3 \times \frac{5}{3}, 0\right)$$
$$S(\pm 5, 0)$$

Directrix: $x = \pm \frac{a}{e}$

$$= \pm \frac{3}{\frac{5}{3}}$$
$$= \pm \frac{9}{5}$$

Asymptotes: $y = \pm \frac{b}{a}x$

$$= \pm \frac{4}{3}x$$

Suggested marking scheme

3 Correct response
2 One error
1 finding the values of *a* and *b*

Marker's comments

Well done.

(ii) Find the gradient of the tangent to the hyperbola at point $P\left(3\sec\theta, 4\tan\theta\right)$.

Solution

At P $\quad \dfrac{dy}{dx} = \dfrac{4\sec^2 x}{3\sec x \tan x}$

$$= \frac{4}{3}\operatorname{cosec} x \quad or$$

$$= \frac{4\sec x}{3\tan x} \quad or \quad \frac{4}{3\sin x}$$

Suggested marking scheme

2 Correct response
1 partial solution
Marker's comments

Well done.

(iii) Show that the tangent to the hyperbola at P has the equation $4x = 3y\sin\theta + 12\cos\theta$.

Solution

$$y - 4\tan\theta = \frac{4\sec\theta}{3\tan\theta}\left(x - 3\sec\theta\right)$$

$$3y\tan\theta - 12\tan^2\theta = 4x\sec\theta - 12\sec^2\theta$$

$$4x\sec\theta - 3y\tan\theta = 12\left(\sec^2\theta - \tan^2\theta\right)$$

$$\frac{4x}{\cos\theta} - \frac{3y\sin\theta}{\cos\theta} = 12\left(\tan^2\theta + 1 - \tan^2\theta\right)$$

$$4x - 3y\sin\theta = 12\cos\theta$$

$$4x = 3y\sin\theta + 12\cos\theta \quad \#$$

Suggested marking scheme

2 Correct response
1 partial solution
Marker's comments

Well done.

(iv) Given $0 < \theta < \dfrac{\pi}{2}$, show that Q, the point of intersection of the tangent to the hyperbola at P and the nearer directrix, has coordinates $Q\left(\dfrac{9}{5}, \dfrac{12 - 20\cos\theta}{5\sin\theta}\right)$. **2**

Solution

Solving $x = \dfrac{9}{5}$ with $4x = 3y\sin\theta + 12\cos\theta$

$$4\left(\frac{9}{5}\right) = 3y\sin\theta + 12\cos\theta$$

$$\frac{36}{5} - 12\cos\theta = 3y\sin\theta$$

$$y = \frac{12}{5\sin\theta} - \frac{4\cos\theta}{\sin\theta}$$

$$= \frac{12 - 20\cos\theta}{5\sin\theta} \quad \#$$

Suggested marking scheme

2 Correct response
1 partial solution
Marker's comments

Well done.

(v) Show that lines joining *SP* and *SQ* are perpendicular.

Solution

$S(5,0)\quad P(3\sec\theta, 4\tan\theta)\quad Q\left(\frac{9}{5}, \frac{12-20\cos\theta}{5\sin\theta}\right)$

$$m_{SP} = \frac{4\tan\theta - 0}{3\sec\theta - 5} = \frac{\frac{4\sin\theta}{\cos\theta}}{\frac{3}{\cos\theta} - \frac{5\cos\theta}{\cos\theta}} = \frac{4\sin\theta}{3-5\cos\theta}$$

$$m_{SQ} = \frac{\frac{12-20\cos\theta}{5\sin\theta} - 0}{\frac{9}{5} - 5} = \frac{12-20\cos\theta}{-16\sin\theta}$$

Suggested marking scheme

3 Correct response

2 One error or
Did not simplify m_{SP} *or* m_{SQ}

1 Finding the m_{SP} *or* m_{SQ}

Marker's comments

Many did not simplify m_{SP} *or* m_{SQ} (i.e. were missing steps).

$$m_{SP} \times m_{SQ} = \frac{4\sin\theta}{3-5\cos\theta} \times \frac{12-20\cos\theta}{-16\sin\theta} = \frac{1}{3-5\cos\theta} \times \frac{4(3-5\cos\theta)}{-4} = -1$$

Therefore *SP* and *SQ* are perpendicular

(vi) Hence show the area of the triangle formed by *PSQ* is $\frac{2(5-3\cos\theta)^2}{5\sin\theta\cos\theta}$.

Solution

Since *SP* and *SQ* are perpendicular are can be found by $\frac{1}{2}bh$

$$SQ^2 = \left(\frac{9}{5} - 5\right)^2 + \left(\frac{12-20\cos\theta}{5\sin\theta} - 0\right)^2$$
$$= \frac{256}{25} + \left(\frac{4(3-5\cos\theta)}{5\sin\theta}\right)^2$$
$$= \frac{256}{25} + \frac{16(9-30\cos\theta+25\cos^2\theta)}{25\sin^2\theta}$$
$$= \frac{256\sin^2\theta + 144 - 480\cos\theta + 400\cos^2\theta}{25\sin^2\theta}$$
$$= \frac{256 - 256\cos^2\theta + 144 - 480\cos\theta + 400\cos^2\theta}{25\sin^2\theta}$$
$$= \frac{400 - 480\cos\theta + 144\cos^2\theta}{25\sin^2\theta}$$
$$= \frac{(20-12\cos\theta)^2}{25\sin^2\theta}$$

$$SP^2 = (3\sec\theta - 5)^2 + (4\tan\theta - 0)^2$$
$$= 9\sec^2\theta - 30\sec\theta + 25 + 16\tan^2\theta$$
$$= 9\sec^2\theta - 30\sec\theta + 25 + 16\sec^2\theta - 16$$
$$= 25\sec^2\theta - 30\sec\theta + 9$$
$$= (5\sec\theta - 3)^2$$

$$A = \frac{1}{2} \times SP \times SQ$$
$$= \frac{1}{2} \times (5\sec\theta - 3) \times \frac{(20 - 12\cos\theta)}{5\sin\theta}$$
$$= \frac{1}{2} \times \left(\frac{5 - 3\cos\theta}{\cos\theta}\right) \times \frac{4(5 - 3\cos\theta)}{5\sin\theta}$$
$$= \frac{2(5 - 3\cos\theta)^2}{5\sin\theta\cos\theta}$$

Suggested marking scheme

3 Correct response
2 One error or
Did not simplify SP or SQ
1 finding SP^2

Marker's comments

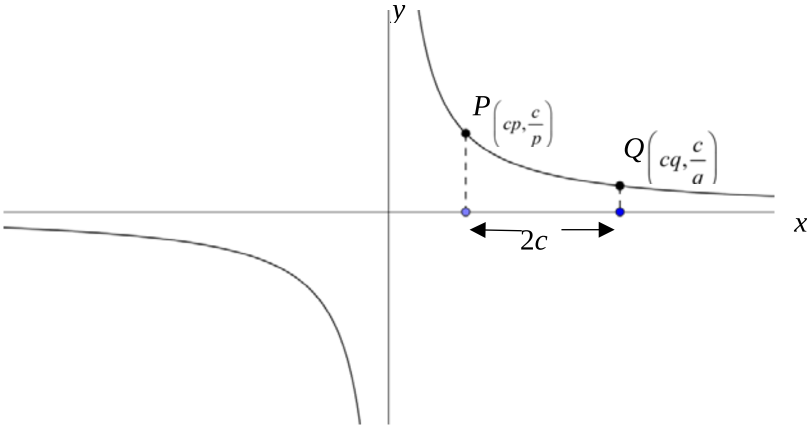
Generally well done.

Some did not simplify SP or SQ (i.e. were missing steps).

Question 14 (15 marks)

(a) The chord PQ on the rectangular hyperbola $xy = c^2$ is constructed such that the horizontal distance between points P and Q has a constant length $2c$, where points P and Q lie in the first quadrant.

Find the locus of the midpoint of PQ.



Solution

Note: $cq = cp + 2c$

$q = p + 2$

Finding the midpoint

$$x = \frac{cp + cq}{2}$$
$$= \frac{cp + (cp + 2c)}{2}$$
$$= \frac{2cp + 2c}{2}$$
$$= cp + c \qquad \Rightarrow \quad p = \frac{x - c}{c}$$

$$y = \frac{\frac{c}{p} + \frac{c}{q}}{2}$$
$$= \frac{cq + cp}{2pq}$$
$$= \frac{cp + 2c + cp}{2p(p+2)}$$
$$= \frac{cp + c}{p(p+2)}$$

Finding the locus in terms of x, y and c.

$$y = \frac{cp + c}{p(p+2)}$$
$$= \frac{c\left(\frac{x-c}{c}\right) + c}{\left(\frac{x-c}{c}\right)\left(\left(\frac{x-c}{c}\right) + 2\right)}$$
$$= \frac{x}{\left(\frac{x-c}{c}\right)\left(\frac{x-c+2c}{c}\right)}$$
$$y = \frac{c^2 x}{x^2 - c^2}$$

Suggested marking scheme

3	Correct response
2	Cartesian equation with error
1	For finding p in terms of x or eliminating q or Finding $p + q$ and progressing with working to substitute into an equation for $p + q$

Marker's comments

Many students experienced trouble with this question.

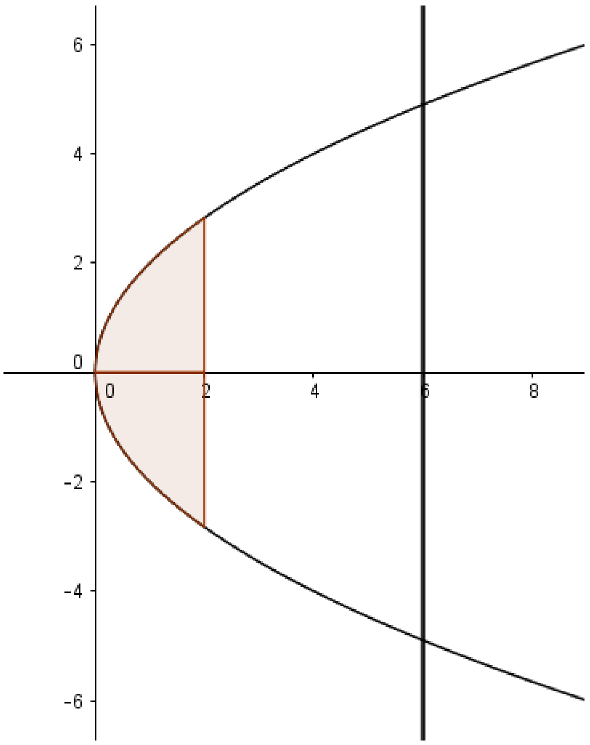
Common errors include:

- not using the identity $cq = cp + 2c$
- not trying to eliminate p or q

(b) The region bounded by the parabola $y^2 = 4x$ and the line $x = 2$ is rotated about the line $x = 6$.

Using the method of cylindrical shells, find the volume of the solid formed.

Solution



Suggested marking scheme

4 Correct response
3 One error
2 Finding δV
1 Partial solution

Marker's comments

Mostly well done.

Many silly errors including

- Incorrect radius
- Incorrect limits
- Arithmetic mistakes, particularly involving $\pm$
- Omission of π *or* $\sqrt{2}$ in answer

Students need to take greater care with their solution!

$$\delta V = 2\pi(6-x)4\sqrt{x}\ \delta x$$

$$V = 8\pi \lim_{\delta x \to 0} \sum_{0}^{2} (6-x)\sqrt{x}\ \delta x$$

$$V = 8\pi \int_0^2 6x^{\frac{1}{2}} - x^{\frac{3}{2}}\ dx$$

$$= 8\pi \left[4x^{\frac{3}{2}} - \frac{2}{5}x^{\frac{5}{2}} \right]_0^2$$

$$= 8\pi \left[\left(4(2)^{\frac{3}{2}} - \frac{2}{5}(2)^{\frac{5}{2}} \right) - (0) \right]$$

$$= 8\pi \left(4 \times 2\sqrt{2} - \frac{2}{5} \times 4\sqrt{2} \right)$$

$$= \frac{256\sqrt{2}}{5}\pi \qquad units^3$$

(c) Using the substitution $u^2 = 4 - x^2$ evaluate $\int_0^2 x^3\sqrt{4-x^2}\ dx$

Solution

$u^2 = 4 - x^2$

$x = 0,\ u = 2$

$x = 2,\ u = 0$

$$2u\frac{du}{dx} = -2x$$

$$\frac{du}{dx} = -\frac{x}{u}$$

$$dx = -\frac{udu}{x}$$

$$\int_0^2 x^3\sqrt{4-x^2}\ dx = \int_2^0 x^3\sqrt{u^2} \times \frac{udu}{-x}$$

$$= \int_0^2 x^2u^2\ \ du$$

$$= \int_0^2 \left(4-u^2\right)u^2\ \ du$$

$$= \int_0^2 4u^2 - u^4\ \ du$$

$$= \left[\frac{4u^3}{3} - \frac{u^5}{5}\right]_0^2$$

$$= \left[\left(\frac{4(2)^3}{3} - \frac{(2)^5}{5}\right) - (0)\right]$$

$$= \frac{64}{15}$$

Suggested marking scheme

4 Correct response

3 One error

2 Finding correct first line of integral

1 Partial solution i.e. limits

Marker's comments

Many silly errors including

- Incorrect limits
- Incorrect substitution with both the *dx* and x^2
- Arithmetic mistakes, particularly involving $\pm$
- Poor handwriting e.g. $4-u^2 \to 4-u$
  $4-u^2 \to 4-16$

Students need to take greater care with their solution!

(d) Use the method of integration by parts to evaluate $\int_0^{\frac{\pi}{2}} e^x \cos x\ \ dx$

Solution

$$\int_0^{\frac{\pi}{2}} e^x \cos x\ \ dx = \left[e^x \cos x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x\left(-\sin x\right)\ \ dx$$

$$= \left[(0) - 1\right] + \int_0^{\frac{\pi}{2}} e^x \sin x\ \ dx$$

$$= -1 + \left[e^x \sin x\right]_0^{\frac{\pi}{2}} - \int_0^{\frac{\pi}{2}} e^x \cos x\ \ dx$$

$$2\int_0^{\frac{\pi}{2}} e^x \cos x\ \ dx = -1 + \left[e^{\frac{\pi}{2}} - 0\right]$$

$$\int_0^{\frac{\pi}{2}} e^x \cos x\ \ dx = \frac{e^{\frac{\pi}{2}} - 1}{2}$$

Suggested marking scheme

4 Correct response

3 One error

2 Integrating by parts once

1 Partial solution

Marker's comments

Mostly well done.

Many silly errors including

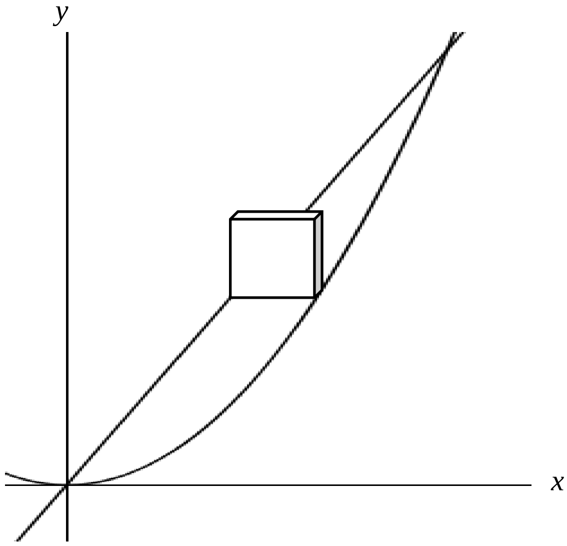
- Evaluating e^0, note $e^0 \neq e$
- Evaluating $\left[e^x \cos x\right]_0^{\frac{\pi}{2}}$

  Note: $\left[e^x \cos x\right]_0^{\frac{\pi}{2}} \neq 0$ *and* $\neq 1$

  $\left[e^x \cos x\right]_0^{\frac{\pi}{2}} = -1$

Question 15 (15 marks)

(a)



The base of a solid is the region in the first quadrant bounded by the graphs of $y = x$ and $y = x^2$. Each cross section perpendicular to the y - axis is a square as shown in the diagram.

Find the volume of the solid formed.

Solution

$$\delta V = \left(\sqrt{y} - y\right)^2 \delta y$$

$$V = \lim_{\delta y \to 0} \sum_0^1 \left(\sqrt{y} - y\right)^2 \delta y$$

$$= \int_0^1 y - 2y^{\frac{3}{2}} + y^2 \; dy$$

$$= \left[\frac{y^2}{2} - \frac{4y^{\frac{5}{2}}}{5} + \frac{y^3}{3} \right]_0^1$$

$$= \frac{1}{30} \; units^3$$

Suggested marking scheme

4	Correct response
3	One error
2	Finding an integral
1	Finding δV

Marker's comments

Mostly well done.

Some students did not show calculations of an individual slice. A mark was deducted for these students.

Some students incorrectly read the question and calculated the slice parallel to the y axis. Students lost a mark if they did not calculate it perpendicular to the y axis.

(b) (i) Find numbers a, b and c such that $\dfrac{x^2}{4x^2-9} \equiv a + \dfrac{b}{2x-3} + \dfrac{c}{2x+3}$

Solution

$$\frac{x^2}{4x^2-9} \equiv a + \frac{b}{2x-3} + \frac{c}{2x+3}$$

$$x^2 \equiv a\left(4x^2-9\right) + b\left(2x+3\right) + c\left(2x-3\right)$$
$$x^2 \equiv 4ax^2 + x\left(2b+2c\right) - 9a + 3b - 3c$$

Comparing coefficients

x^2 : $4a = 1$

$a = \dfrac{1}{4}$

x : $2b + 2c = 0$

$b = -c$

Constant: $0 = -9a + 3c - 3c$

$= -9\left(\dfrac{1}{4}\right) + 3(-c) - 3c$

$6c = -\dfrac{9}{4}$

$c = -\dfrac{9}{24}$

$= -\dfrac{3}{8}$

Therefore $a = \dfrac{1}{4},\ b = \dfrac{3}{8},\ c = -\dfrac{3}{8}$

Suggested marking scheme

3 Correct response
2 One error
1 Finding one value or partial solution

Marker's comments

Mostly well done.

Some careless errors.

(ii) Hence evaluate $\displaystyle\int_0^1 \frac{x^2}{4x^2-9}\,dx$

Solution

$$\int_0^1 \frac{x^2}{4x^2-9}\,dx = \int_0^1 \frac{1}{4} + \frac{3}{8(2x-3)} - \frac{3}{8(2x+3)}\,dx$$
$$= \int_0^1 \frac{1}{4} + \frac{3}{16} \times \frac{2}{(2x-3)} - \frac{3}{16} \times \frac{2}{(2x+3)}\,dx$$
$$= \left[\frac{1}{4}x + \frac{3}{16}\ln\left|2x-3\right| - \frac{3}{16}\ln\left|2x+3\right|\right]_0^1$$
$$= \left(\frac{1}{4} + \frac{3}{16}\ln\left|2-3\right| - \frac{3}{16}\ln\left|2+3\right|\right) - \left(0 + \frac{3}{16}\ln\left|-3\right| - \frac{3}{16}\ln\left|+3\right|\right)$$
$$= \frac{1}{4} + \frac{3}{16}\ln\left|\frac{-1}{5}\right|$$
$$= \frac{1}{4} + \frac{3}{16}\ln\left(\frac{1}{5}\right)$$
$$\approx -0\cdot 05$$

Suggested marking scheme

2 Correct response
1 Partial solution

Marker's comments

Mostly well done.

Some students did not take the absolute value when integrating and incorrectly concluded that there is no solution.

(c) An object falls from rest, under gravity, for a time of $\frac{1}{2k}$ seconds before hitting water and experiencing an upward resistance of mkv, where m is the mass of the object, v the object's velocity and k is a positive constant.

Let g be the acceleration due to gravity and take the downwards motion to be in the positive direction.

(i) Show that when the object hits the water its velocity will be $\frac{g}{2k}$ and the distance travelled is $\frac{g}{8k^2}$

Solution

$$\ddot{x} = g$$
$$\frac{d\dot{x}}{dt} = g$$
$$\dot{x} = gt + C_1, \quad when\ \dot{x} = 0, t = 0 \Rightarrow C_1 = 0$$
$$\dot{x} = gt$$
$$x = \frac{1}{2}gt^2 + C_2, \quad when\ x = 0, t = 0 \Rightarrow C_2 = 0$$
$$x = \frac{1}{2}gt^2$$

After $t = \frac{1}{2k}$

$$\dot{x} = g\left(\frac{1}{2k}\right) = \frac{g}{2k}$$

$$x = \frac{1}{2}g\left(\frac{1}{2k}\right)^2 = \frac{g}{8k^2}$$

Suggested marking scheme

2 Correct response
1 Partial solution

Marker's comments

Some students did not realise the object falls from rest under gravity only (until it hits the water), consequently they incorrectly integrated from $\ddot{x} = g - kv$.

(ii) Show that the total distance travelled when the object's velocity is $\frac{3g}{4k}$ is given by $x=\frac{g}{k^2}\ln 2-\frac{g}{8k^2}$

Solution

$$\begin{aligned}F&=mg-mkv\\&=m(g-kv)\end{aligned}$$

$$\Rightarrow a=g-kv$$

$$\begin{aligned}a&=g-kv\\v\frac{dv}{dx}&=g-kv\\\frac{dv}{dx}&=\frac{g-kv}{v}\\\frac{dx}{dv}&=\frac{v}{g-kv}\\-k\frac{dx}{dv}&=\frac{-kv}{g-kv}\\-k\frac{dx}{dv}&=\frac{g-kv-g}{g-kv}\\-k\frac{dx}{dv}&=1+\frac{-g}{g-kv}\\-\frac{k}{g}\frac{dx}{dv}&=\frac{1}{g}+\frac{-1}{g-kv}\\-\frac{k^2}{g}\frac{dx}{dv}&=\frac{k}{g}+\frac{-k}{g-kv}\end{aligned}$$

$$-\frac{k^2}{g}x=\frac{k}{g}v+\ln\left|g-kv\right|+C$$

When $x=\frac{g}{8k^2}$, $v=\frac{g}{2k}$

$$-\frac{k^2}{g}\left(\frac{g}{8k^2}\right)=\frac{k}{g}\left(\frac{g}{2k}\right)+\ln\left|g-k\left(\frac{g}{2k}\right)\right|+C$$

$$\begin{aligned}C&=-\frac{1}{8}-\frac{1}{2}-\ln\left|\frac{g}{2}\right|\\&=-\frac{5}{8}-\ln\left|\frac{g}{2}\right|\end{aligned}$$

$$-\frac{k^2}{g}x=\frac{k}{g}v+\ln\left|g-kv\right|-\frac{5}{8}-\ln\left|\frac{g}{2}\right|$$

Now when $v=\frac{3g}{4k}$

$$\begin{aligned}-\frac{k^2}{g}x&=\frac{k}{g}\left(\frac{3g}{4k}\right)+\ln\left|g-k\left(\frac{3g}{4k}\right)\right|-\frac{5}{8}-\ln\left|\frac{g}{2}\right|\\-\frac{k^2}{g}x&=\frac{3}{4}-\frac{5}{8}+\ln\left|\frac{g}{4}\right|-\ln\left|\frac{g}{2}\right|\\-\frac{k^2}{g}x&=\frac{1}{8}+\ln\left|\frac{1}{2}\right|\\x&=-\frac{g}{8k^2}-\frac{g}{k^2}\ln\left|\frac{1}{2}\right|\\&=\frac{g}{k^2}\ln 2-\frac{g}{8k^2}\end{aligned}$$

Suggested marking scheme

4	Correct response
3	One error
2	Finding an expression for x
1	Partial solution

Marker's comments

Many students had the correct acceleration equation. Some replaced a with $\frac{dv}{dt}$ and were successful; other did not know how to proceed beyond this point.

Many used incorrect initial conditions i.e. $x=0, v=0$ instead of $x=\frac{g}{8k^2}$ and $v=\frac{g}{2k}$, failing to realise the motion equation they were calculating was from the point of contact with the water.

Alternate solution

Taking $x=0$ to be the moment the object hits the water.

$$\int_{\frac{g}{2k}}^{v}\frac{v}{g-kv}dv=\int_{0}^{x}dx$$

$$-\frac{1}{k}\int_{\frac{g}{2k}}^{v}\frac{g-kv}{g-kv}-\frac{g}{g-kv}dv=x$$

$$-\frac{1}{k}\int_{\frac{g}{2k}}^{v}\left(1-\frac{g}{g-kv}\right)dv=x$$

$$-\frac{1}{k}\left[v+\frac{g}{k}\ln\left|g-kv\right|\right]_{\frac{g}{2k}}^{v}=x$$

$$-\frac{1}{k}\left[v+\frac{g}{k}\ln\left|g-kv\right|-\frac{g}{2k}-\frac{g}{k}\ln\left|\frac{g}{2}\right|\right]=x$$

At $v=\frac{3g}{4k}$

$$\begin{aligned}x&=-\frac{1}{k}\left[\frac{3g}{4k}+\frac{g}{k}\ln\left|g-k\times\frac{3g}{4k}\right|-\frac{g}{2k}-\frac{g}{k}\ln\left|\frac{g}{2}\right|\right]\\&=-\frac{1}{k}\left[\frac{g}{4k}+\frac{g}{k}\ln\left|\frac{g}{4}\right|-\frac{g}{k}\ln\left|\frac{g}{2}\right|\right]\\&=-\frac{1}{k}\left[\frac{g}{4k}+\frac{g}{k}\ln\left|\frac{\frac{g}{4}}{\frac{g}{2}}\right|\right]\\&=-\frac{1}{k}\left[\frac{g}{4k}+\frac{g}{k}\ln\left|\frac{1}{2}\right|\right]\\&=-\frac{g}{4k^2}-\frac{g}{k^2}\left[\ln 1-\ln 2\right]\\&=-\frac{g}{4k^2}+\ln 2\times\frac{g}{k^2}\end{aligned}$$

So the total distance is $-\frac{g}{4k^2}+\ln 2\times\frac{g}{k^2}+\frac{g}{8k^2}=-\frac{g}{8k^2}+\frac{g}{k^2}\ln 2$

Question 16 (15 marks) Use a *separate* writing booklet

(a) The polynomial $x^4 - 5x^3 - 2x^2 + 3x + 1 = 0$ has roots α, β, γ and δ.

Find an equation with roots $\alpha^2 - 1, \beta^2 - 1, \gamma^2 - 1$ and $\delta^2 - 1$.

Suggested marking scheme

2 Correct response
1 Partial solution

Marker's comments

Mostly well done.

Some students did not know the method to find equations with roots involving α, β, γ & δ.

Solution

$$x = \alpha^2 - 1$$
$$\alpha^2 = x + 1$$
$$\alpha = \sqrt{x+1}$$

$$\left(\sqrt{x+1}\right)^4 - 5\left(\sqrt{x+1}\right)^3 - 2\left(\sqrt{x+1}\right)^2 + 3\sqrt{x+1} + 1 = 0$$
$$(x+1)^2 - 5(x+1)\sqrt{x+1} - 2(x+1) + 3\sqrt{x+1} + 1 = 0$$
$$(x+1)^2 - 2(x+1) + 1 = 5(x+1)\sqrt{x+1} - 3\sqrt{x+1}$$
$$x^2 + 2x + 1 - 2x - 2 + 1 = \sqrt{x+1}(5x + 5 - 3)$$
$$x^2 = \sqrt{x+1}(5x+2)$$
$$x^4 = (x+1)\left(25x^2 + 20x + 4\right)$$
$$x^4 = 25x^3 + 20x^2 + 4x + 25x^2 + 20x + 4$$
$$x^4 - 25x^3 - 45x^2 - 24x - 4 = 0$$

(b) Let $I_n = \int \frac{dx}{\left(1+x^2\right)^n}$ where n is a non-negative integer.

(i) Show that $I_{n+1} = \frac{1}{2n}\frac{x}{\left(1+x^2\right)^n} + \frac{2n-1}{2n} I_n$.

Solution

PTO

$$I_n = \int \frac{dx}{\left(1+x^2\right)^n}$$

$$= \int 1.\left(1+x^2\right)^{-n}\ dx$$

$$= x.\frac{1}{\left(1+x^2\right)^n} - \int x(-n)\left(1+x^2\right)^{-n-1}(2x)\ dx$$

$$= \frac{x}{\left(1+x^2\right)^n} + 2n\int \frac{x^2}{\left(1+x^2\right)^{n+1}}\ dx$$

$$= \frac{x}{\left(1+x^2\right)^n} + 2n\int \frac{1+x^2-1}{\left(1+x^2\right)^{n+1}}\ dx$$

$$= \frac{x}{\left(1+x^2\right)^n} + 2n\int \frac{1+x^2}{\left(1+x^2\right)^{n+1}} - 2n\int \frac{1}{\left(1+x^2\right)^{n+1}}\ dx$$

$$= \frac{x}{\left(1+x^2\right)^n} + 2n\int \frac{1}{\left(1+x^2\right)^{n}} - 2n\int \frac{1}{\left(1+x^2\right)^{n+1}}\ dx$$

$$= \frac{x}{\left(1+x^2\right)^n} + 2nI_n - 2nI_{n+1}$$

$$2nI_{n+1} = \frac{x}{\left(1+x^2\right)^n} + 2nI_n - I_n$$

$$2nI_{n+1} = \frac{x}{\left(1+x^2\right)^n} + (2n-1)I_n$$

$$I_{n+1} = \frac{1}{2n}\frac{x}{\left(1+x^2\right)^n} + \frac{(2n-1)}{2n}I_n$$

Suggested marking scheme

3 Correct response
2 One error
1 Finding partial solution

Marker's comments

Some students chose the wrong initial *u* and *v* when integrating by parts.

Some students did not know the correct procedure for integrating by parts and incorrectly added the integral.

(ii) Hence find I_3.

Solution

$$I_{2+1} = \frac{1}{2(2)}\frac{x}{\left(1+x^2\right)^2} + \frac{(2(2)-1)}{2(2)}I_2$$

$$= \frac{1}{4}\frac{x}{\left(1+x^2\right)^2} + \frac{3}{4}I_2$$

$$= \frac{1}{4}\frac{x}{\left(1+x^2\right)^2} + \frac{3}{4}\left[\frac{1}{2}\frac{x}{\left(1+x^2\right)} + \frac{1}{2}I_1\right]$$

$$= \frac{1}{4}\frac{x}{\left(1+x^2\right)^2} + \frac{3}{8}\frac{x}{\left(1+x^2\right)} + \frac{3}{8}I_1$$

$$= \frac{1}{4}\frac{x}{\left(1+x^2\right)^2} + \frac{3}{8}\frac{x}{\left(1+x^2\right)} + \frac{3}{8}\int\frac{dx}{1+x^2}$$

$$= \frac{1}{4}\frac{x}{\left(1+x^2\right)^2} + \frac{3}{8}\frac{x}{\left(1+x^2\right)} + \frac{3}{8}\tan^{-1}x + C$$

Suggested marking scheme

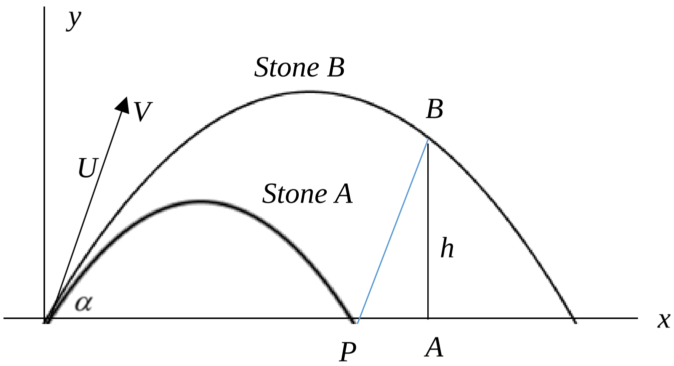
2 Correct response
1 Partial solution

Marker's comments

Mostly well done.

Some student chose the incorrect value for *n*.

(c) Two stones are thrown simultaneously from the same point in the same direction and with the same angl



The slower stone hits the ground at a point P on the same level as the point of projection. At that instant the faster stone just clears a wall of height h metres above the level of projection and its (downward) path makes an angle β with the horizontal.

(i) Express the distance from P to the foot of the wall in terms of h and α only.

Solution

The equations of motion

Stone A

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$
$$\dot{x} = U\cos\alpha \qquad \dot{y} = -gt + U\sin\alpha$$
$$x = Ut\cos\alpha \qquad y = -\frac{1}{2}gt^2 + Ut\sin\alpha$$

Stone B

$$\ddot{x} = 0 \qquad \ddot{y} = -g$$
$$\dot{x} = V\cos\alpha \qquad \dot{y} = -gt + V\sin\alpha$$
$$x = Vt\cos\alpha \qquad y = -\frac{1}{2}gt^2 + Vt\sin\alpha$$

Finding angle APB

$$\tan APB = \frac{-\frac{1}{2}gt^2 + Vt\sin\alpha - \left(-\frac{1}{2}gt^2 + Ut\sin\alpha\right)}{Vt\cos\alpha - Ut\cos\alpha}$$
$$= \frac{Vt\sin\alpha - Ut\sin\alpha}{t\cos\alpha(V-U)}$$
$$= \frac{t\sin\alpha(V-U)}{t\cos\alpha(V-U)}$$
$$= \tan\alpha$$
$$\angle APB = \alpha$$

In triangle APB

$$\tan\alpha = \frac{h}{PA}$$
$$PA = h\cot\alpha$$

Suggested marking scheme

3 Correct response
2 One error
1 Finding equations of motion

Marker's comments

Students should draw a diagram.

Many students did not attempt this question.

Some students assumed $\tan\alpha = \frac{h}{PA}$ which is not sufficient.

(ii) Show that $V\left(\tan\alpha + \tan\beta\right) = 2U\tan\alpha$.

Solution

Stone *A* hits the ground at *P*.

$$0 = Ut\sin\alpha - \frac{1}{2}gt^2$$

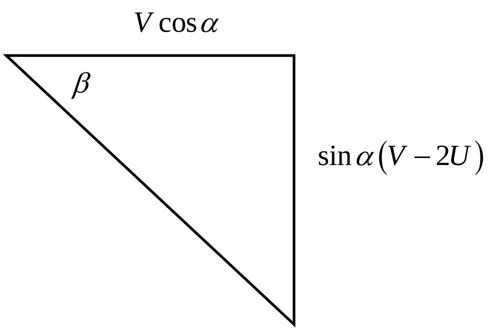
$$t\left(U\sin\alpha - \frac{g}{2}t\right) = 0$$

$$t = \frac{2U\sin\alpha}{g} \quad \text{since } t = 0 \text{ is the origin}$$

When $t = \frac{2U\sin\alpha}{g}$, $\dot{y} = -g\left(\frac{2U\sin\alpha}{g}\right) + V\sin\alpha$

$$= V\sin\alpha - 2U\sin\alpha$$
$$= \sin\alpha\left(V - 2U\right)$$

Consider the velocity vector of stone *B* when it clears the wall.



Since β is angled downwards

$$\tan\beta = \frac{-\left[\sin\alpha\left(V-2U\right)\right]}{V\cos\alpha}$$
$$= \frac{2U - V}{V}\tan\alpha$$

$$V\left(\tan\alpha + \tan\beta\right) = V\left(\tan\alpha + \frac{2U-V}{V}\tan\alpha\right)$$
$$= V\tan\alpha + \left(2U - V\right)\tan\alpha$$
$$= V\tan\alpha + 2U\tan\alpha - V\tan\alpha$$
$$= 2u\tan\alpha \qquad \#$$

Suggested marking scheme

3 Correct response
2 One error
1 Finding equations of motion

Marker's comments

Most students found this question challenging.

(iii) Deduce that if, $\beta = \frac{1}{2}\alpha$, then $U < \frac{3}{4}V$.

Solution

$$V\left(\tan\alpha + \tan\beta\right) = 2U\tan\alpha$$

$$V\left(\tan 2\beta + \tan\beta\right) = 2U\tan 2\beta \quad \text{since } \beta = \frac{1}{2}\alpha$$

$$\frac{V}{U} = \frac{2\tan 2\beta}{\tan\beta + \tan 2\beta}$$

$$= \frac{2\left(\frac{\tan\beta + \tan\beta}{1 - \tan^2\beta}\right)}{\tan\beta + \frac{\tan\beta + \tan\beta}{1 - \tan^2\beta}}$$

$$= \frac{\frac{4\tan\beta}{1 - \tan^2\beta}}{\frac{\left(1 - \tan^2\beta\right)\tan\beta + 2\tan\beta}{1 - \tan^2\beta}}$$

$$= \frac{4}{3 - \tan^2\beta}$$

Now since $\tan^2\beta > 0$

$$\frac{V}{U} > \frac{4}{3}$$

$$U < \frac{3V}{4} \qquad \#$$

Suggested marking scheme

2 Correct response

1 Partial solution

Marker's comments

Many students did not attempt this question.

Students should be aware they could attempt this question without completing parts (i) and (ii).

Alternate method

$$V(\tan\alpha + \tan\beta) = 2U\tan\alpha$$

$$V\left(\tan\alpha + \tan\frac{\alpha}{2}\right) = 2U\tan\alpha \qquad \text{since } \beta = \frac{1}{2}\alpha$$

$$\frac{V}{U} = \frac{2\tan\alpha}{\tan\alpha + \tan\frac{\alpha}{2}}$$

$$= \frac{2\left(\frac{2t}{1-t^2}\right)}{\frac{2t}{1-t^2} + t}$$

$$= \frac{\frac{4t}{1-t^2}}{\frac{2t + t(1-t^2)}{1-t^2}}$$

$$= \frac{4}{2+1-t^2}$$

$$= \frac{4}{3-t^2}$$

$$= \frac{4}{3-\tan^2\frac{\alpha}{2}}$$

Now since $\tan^2\frac{\alpha}{2} > 0$

$$\frac{V}{U} > \frac{4}{3}$$

$$U < \frac{3V}{4} \qquad \#$$