Name..........................................................................

Number..........................................................................

# Gosford High School

HIGHER SCHOOL CERTIFICATE

**2017** TRIAL EXAMINATION

# Mathematics Extension 2

- **General Instructions**
- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided at the back of this paper
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

**Total Marks – 100**

**Section I** Pages 2 – 6

**10 marks**

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

**Section II** Pages 7 – 13

**90 marks**

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section

**Section I** **10 marks** **Allow approximately. 15 minutes for section 1**

Use the multiple-choice answer sheet for Questions 1 – 10.

1. The expression for $\frac{dy}{dx}$ for the curve $x^2 - y^2 + x^3 \cos y - 6 = 0$ is

(A) $\frac{-2x - 3x^2 \cos y}{2y}$

(B) $\frac{2x + 3x^2 \cos y}{2y}$

(C) $\frac{-2x - 3x^2 \cos y}{2y + x^3 \sin y}$

(D) $\frac{2x + 3x^2 \cos y}{2y + x^3 \sin y}$

2. An ellipse has equation $9x^2 + 25y^2 = 225$. The eccentricity and equation of the directrices for this ellipse are:

(A) $e = \frac{4}{5}$ *and* $x = \pm\frac{25}{4}$

(B) $e = \frac{4}{5}$ *and* $x = \pm 4$

(C) $e = \frac{3}{5}$ *and* $x = \pm 4$

(D) $e = \frac{3}{5}$ *and* $x = \pm\frac{25}{4}$

3. What is the double root of the equation $x^3 - 5x^2 + 8x - 4 = 0$ ?

(A) $x = -2$

(B) $x = -1$

(C) $x = 1$

(D) $x = 2$

4 Given $z = 2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$, which expression is equal to $\left(\overline{z}\right)^{-1}$?

(A) $\frac{1}{2}\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

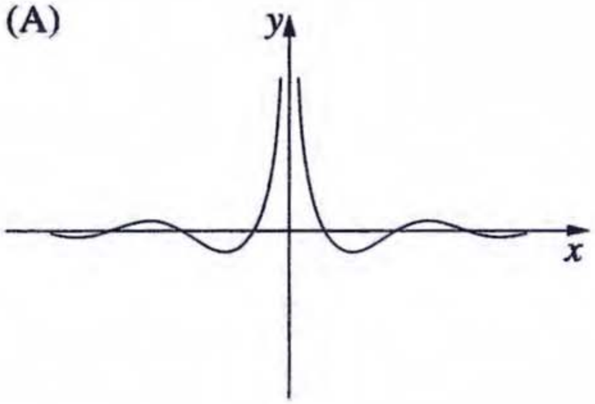
(B) $2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right)$

(C) $\frac{1}{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

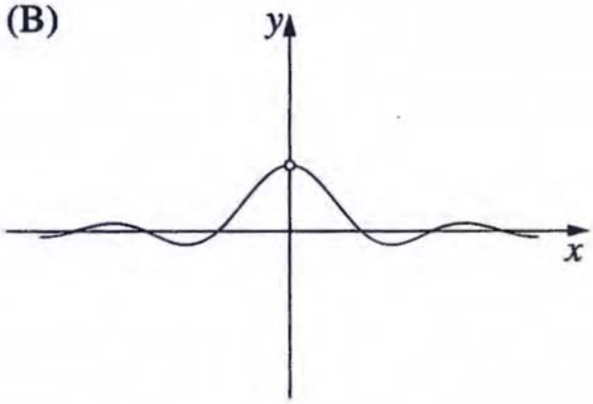
(D) $2\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$

5 Which diagram best represents the graph $y = \frac{\sin x}{x}$ ?

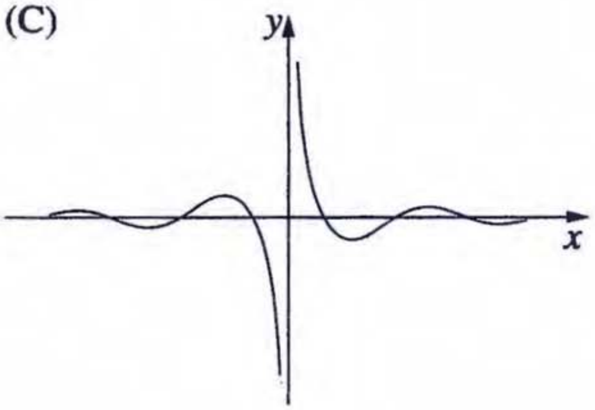
(A)



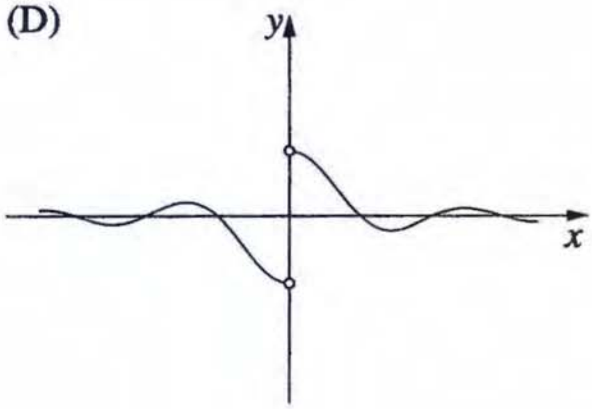

(B)



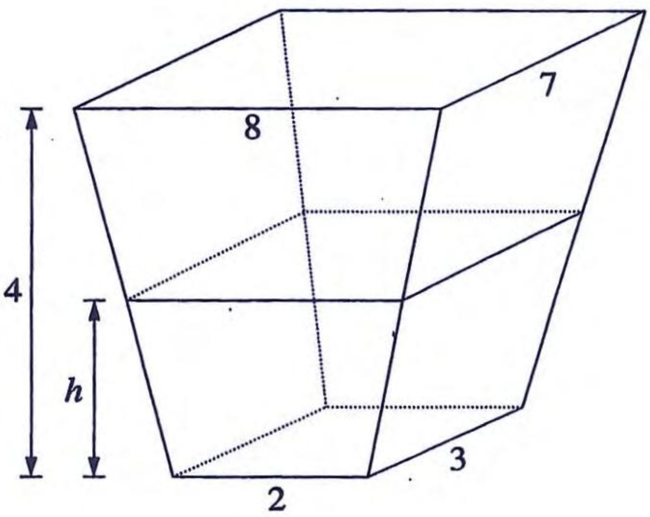

(C)




(D)



**6** The diagram shows the dimensions of a polyhedron with parallel base and top. A slice taken at height $h$ parallel to the base is a rectangle.




NOT TO SCALE

What is a correct expression for the volume of the polyhedron?

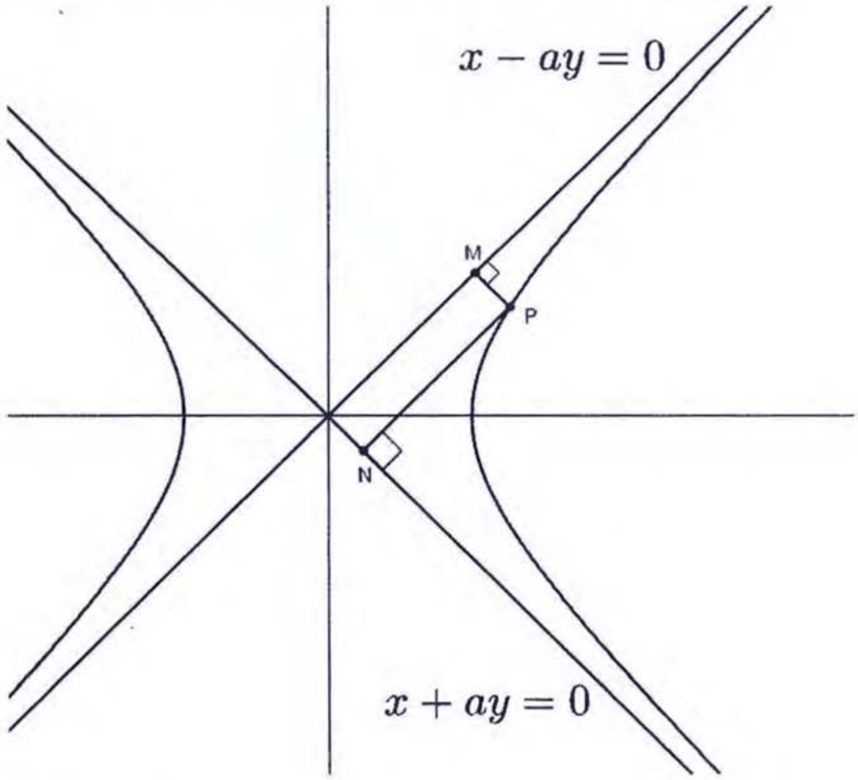
(A) $\displaystyle\int_0^4 (h+3)\left(\frac{3h}{2}+2\right) dh$

(B) $\displaystyle\int_0^4 \left(\frac{5h}{4}+3\right)\left(\frac{3h}{2}+2\right) dh$

(C) $\displaystyle\int_0^4 (h+3)\left(\frac{5h}{4}+2\right) dh$

(D) $\displaystyle\int_0^4 \left(\frac{5h}{4}+3\right)\left(\frac{5h}{4}+2\right) dh$

7) $P(a\,sec\theta, tan\theta)$ is a point on the hyperbola $\frac{x^2}{a^2} - y^2 = 1, a > 1$, with eccentricity $e$ and asymptotes $x - ay = 0$ and $x + ay = 0$. $M$ and $N$ are the feet of the perpendiculars from $P$ to the asymptotes as shown.




Which expression is $PM \times PN$?

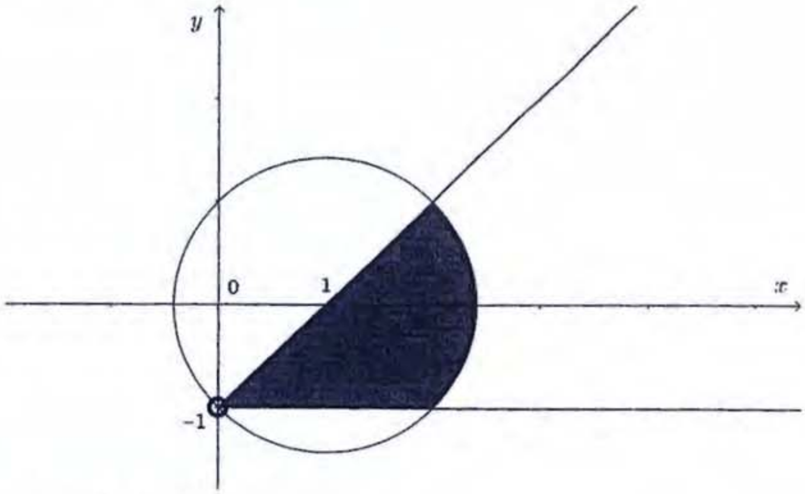
(A) $\dfrac{1}{e^2}$

(B) $\dfrac{e^2 - 1}{e^2}$

(C) $\dfrac{1}{2 - e^2}$

(D) $\dfrac{1 - e^2}{2 - e^2}$

8. Consider the Argand diagram below.




Which inequality could define the shaded area?

(A) $|z-1| \leq \sqrt{2}$ and $0 \leq \arg(z-i) \leq \dfrac{\pi}{4}$

(B) $|z-1| \leq \sqrt{2}$ and $0 \leq \arg(z+i) \leq \dfrac{\pi}{4}$

(C) $|z-1| \leq 1$ and $0 \leq \arg(z-i) \leq \dfrac{\pi}{4}$

(D) $|z-1| \leq 1$ and $0 \leq \arg(z+i) \leq \dfrac{\pi}{4}$

9. Consider the region bounded by the $y$–axis, the line $y = 4$ and the curve $y = x^2$.

If this region is rotated about the line $y = 4$, which expression gives the volume of the solid of revolution?

(A) $V = \pi \int_0^4 x^2 \, dy$

(B) $V = 2\pi \int_0^2 (4-y)x \, dy$

(C) $V = \pi \int_0^2 (4-y)^2 \, dx$

(D) $V = \pi \int_0^4 (4-y)^2 \, dx$

10. A committee of 5 people is to be chosen from a group of 6 girls and 4 boys. How many different committees could be formed that have at least one boy.

(A) $\binom{10}{5} - 1$

(B) $\binom{4}{1} + \binom{6}{4}$

(C) $\binom{4}{1} \times \binom{6}{4}$

(D) $\binom{10}{5} - 6$

**Section II 90 marks Attempt Questions 11 – 16.**

**Allow about 2 hours and 45 minutes for this section.**

Answer each question in a SEPARATE writing booklet. Extra booklets are available. In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

**Question 11 (15 marks) Use a SEPARATE writing booklet.**

(a) Given $z = -3\sqrt{3} + 3i$

(i) Express $z$ in modulus-argument form **2**

(ii) Evaluate $z^3$ **1**

(b) Use the substitution $x = 3\tan\theta$ to find $\int \frac{dx}{x^2\sqrt{9+x^2}}$ **4**

(c) Find (i) $\int \frac{dx}{\sqrt{4+2x-x^2}}$ **3**

(ii) $\int \frac{2x\,dx}{x^2+4x+8}$ **3**

(d) Multiplying a non-zero complex number by $\frac{1-i}{1+i}$ results in a rotation about the origin on an Argand diagram. Find the angle and direction of this rotation? **2**

**End of Question 11**

**Question 12 (15 marks) Use a SEPARATE writing booklet.**

(a) The normals to the ellipse $16x^2 + 25y^2 = 400$ at the points $P(5\cos\alpha, 4\sin\alpha)$ and $Q(5\cos\beta, 4\sin\beta)$ are at right angles to each other.

(i) Show that the gradient of the normal at $P$ is $\dfrac{5\sin\alpha}{4\cos\alpha}$. **2**

(ii) Show that $25\tan\alpha\tan\beta = -16$. **1**

(b) Evaluate: $\displaystyle\int_0^{\frac{\pi}{4}} e^x \cos 2x \, dx$ **4**

(c) (i) Find values of $a$, $b$ and $c$ such that : **3**

$$\frac{5x^2 - 3x - 8}{(x-1)(x-2)(x+2)} = \frac{a}{x-1} + \frac{b}{x-2} + \frac{c}{x+2}$$

(ii) Hence find: $\displaystyle\int \frac{5x^2 - 3x - 8}{(x-1)(x-2)(x+2)} dx.$ **1**

(d) Consider the complex numbers $\omega = -5 - 12i$ and $Z = 3 + 4i$.

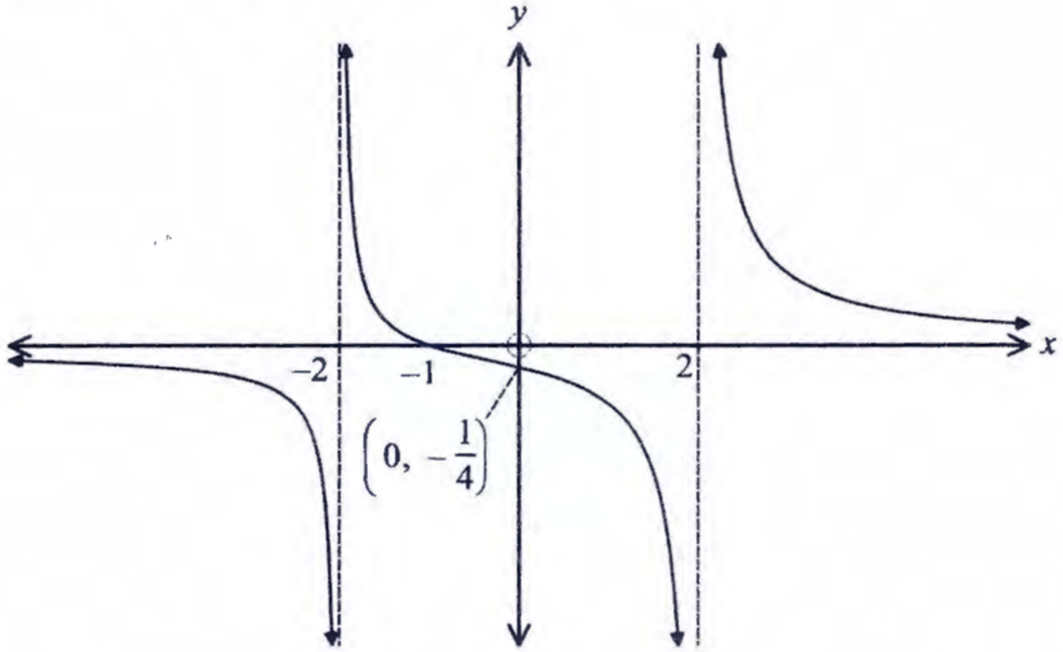
(i) Evaluate $\sqrt{\omega}$. **2**

(ii) Evaluate $\dfrac{\overline{\omega}}{Z}$ **2**

**End of Question 12**

**Question 13 (15 Marks) Use a SEPARATE writing booklet**

(a) The graph of $y = f(x)$ is drawn below.




Draw separate half page graphs for each of the following functions, showing all asymptotes and intercepts. **Templates are provided at the end of the paper.**

(i) $y = |f(x)|$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y^2 = f(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

(b) (i) Differentiate $x f(x) - \int x\, f'(x)\, dx$. **1**

(ii) Hence, or otherwise, find $\int \tan^{-1} x\, dx$. **2**

(c) The roots of the equation $x^3 - 9x^2 + 31x + m = 0$ are in an arithmetic sequence. Find the roots of the equation and the value of $m$. **4**

**Question 14 (15 marks) Use a SEPARATE writing booklet.**

(a) (i) Show that a reduction formula for $I_n = \int x^n \cos x \; dx$ is **2**

$$I_n = x^n sinx + nx^{n-1}cosx - n(n-1)I_{n-2}$$

(ii) Hence evaluate $\int x^4 \cos x \; dx$ **2**

(b) The area bounded by the curve $y = e^{x^2}$, the lines $x = 1$ and $y = 1$ is rotated about the $y$–axis. **3**

Use the method of cylindrical shells to calculate the volume of the solid of revolution formed.

(c) The polynomial $P(x) = x^3 - 3x^2 - 4x - 5$ has roots $\alpha, \beta,$ and $\gamma$.

(i) Find the equation with roots $\alpha^2$, $\beta^2$ and $\gamma^2$ **2**

(ii) Find the value of $\alpha^3 + \beta^3 + \gamma^3$ for $P(x)$ **3**

(d) Solve $2sin^3\theta + 1 = 2sin^2\theta + sin\,\theta$ for $0 \leq \theta \leq 2\pi$. **3**

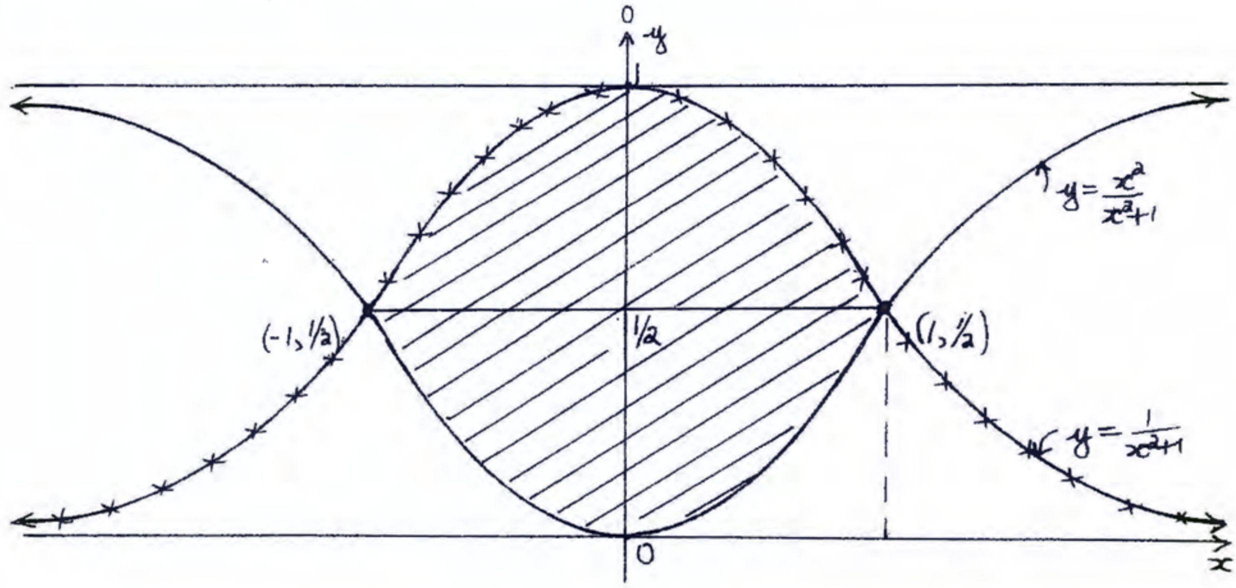
**End of Question 14**

## Question 15 (15 Marks) Use a SEPARATE writing booklet

(a) Clearly sketch on an Argand diagram the locus given by **2**

$$\arg(z-3) - \arg(z+3) = \frac{\pi}{4}$$

(b)




The curves $y = \frac{1}{x^2+1}$ and $y = \frac{x^2}{x^2+1}$ are sketched above

i) Find the area bounded by the curves. **2**

ii) Find the volume of the solid generated when this area is rotated about the y-axis. **3**

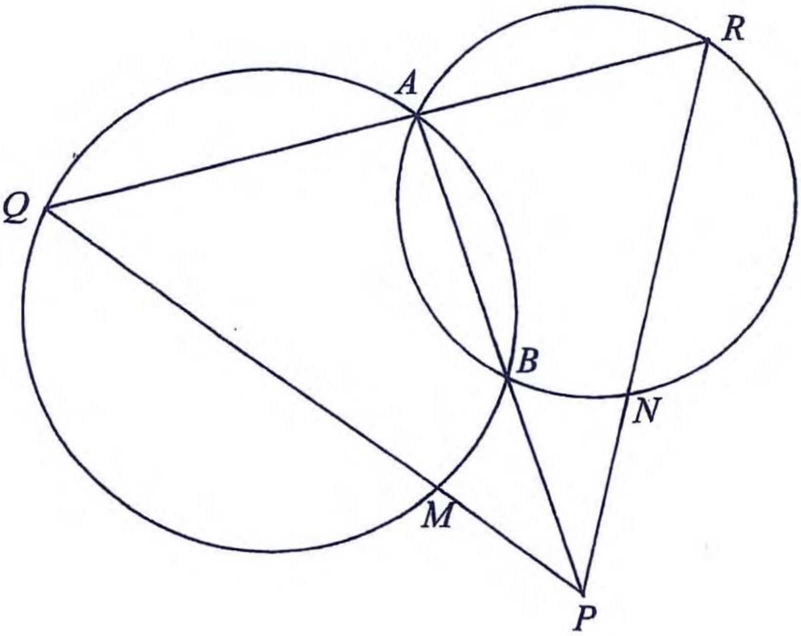
c) By means of the substitution $y = a - x$ or otherwise, prove that

i) $\int_0^a f(x)\,dx = \int_0^a f(a-x)\,dx.$ **1**

ii) Hence evaluate $\int_0^{\pi} \frac{x \sin x \, dx}{1+\cos^2 x}$. **3**

**Question 15**

**(d)**




In the diagram, two circles intersect at $A$ and $B$. Chord $QA$ on the first circle is produced to cut the second circle at $R$. From $P$ on $AB$ produced secants are drawn to $Q$ and $R$, cutting the circles at $M$ and $N$ respectively.

(i) Show that $PMBN$ is a cyclic quadrilateral. **2**

(ii) Hence show that $MQRN$ is a cyclic quadrilateral. **2**

**End of Question 15**

**Question 16 (15 Marks) Use a SEPARATE writing booklet**

a) A particle of mass $m$ kg is fired vertically upwards in a medium where the resistance to motion has magnitude $mkv^2$ newtons when the speed is $v$ ms$^{-1}$ The particle has height $x$ metres above the point of projection at time $t$ seconds. The maximum height $H$ metres is reached at time $T$ seconds. The speed of projection $U$ ms$^{-1}$ is equal to the terminal velocity of a particle falling in the medium. The acceleration due to gravity has magnitude $g$ ms$^{-2}$.

(i) Express $U^2$ in terms of $g$ and $k$, and deduce that $\ddot{x} = -\frac{g}{U^2}(U^2 + v^2)$. **2**

(ii) Show that $\frac{v}{U} = \tan\left(\frac{\pi}{4} - \frac{g}{U}t\right)$ **2**

(iii) Show that $\frac{x}{U} = \frac{U}{g}\log\left\{\sqrt{2}\cos\left(\frac{\pi}{4} - \frac{g}{U}t\right)\right\}$ **2**

(iv) Show that at time $\frac{1}{2}T$ seconds $\frac{x}{U} = \frac{U}{2g}\log\left\{1 + \frac{1}{\sqrt{2}}\right\}$ and calculate the percentage of the maximum height attained during the first half of the ascent time, giving your answer to the nearest 1%. **3**

b) The equation of the tangent to the hyperbola $xy = c^2$ at the point P $\left(cp, \frac{c}{p}\right)$ is $x + p^2y = 2cp$. **Do not prove this.**

(i) If the tangents at P $\left(cp, \frac{c}{p}\right)$ and Q $\left(cq, \frac{c}{q}\right)$ meet at R $(x_o, y_o)$, **2**

Prove that $pq = \frac{x_o}{y_0}$ and $p + q = \frac{2c}{y_o}$.

(ii) If the length of the chord PQ is $d$ units, find an expression for $d^2$ in terms of $c, p$ and $q$. (in factorised form). **2**

(iii) If d is fixed, deduce that the locus of R has equation **2**

$$4c^2(x^2 + y^2)(c^2 - xy) = x^2y^2d^2$$

**End of examination**

# Solutions to 2017 Trial HSC Extension 2

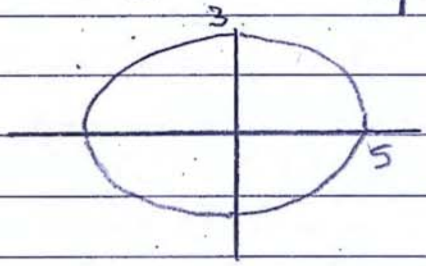
1. $x^2 - y^2 + x^3\cos y - 6 = 0$

$2x - 2y\frac{dy}{dx} + x^3\left(-\sin y\frac{dy}{dx}\right) + 3x^2\cos y = 0$

$2x + 3x^2\cos y = \frac{dy}{dx}\left(2y + x^3\sin y\right)$

$\frac{dy}{dx} = \frac{2x + 3x^2\cos y}{2y + x^3\sin y}$ D

2. $\frac{x^2}{25} + \frac{y^2}{9} = 1$  $a = 5$, $b = 3$




$b^2 = a^2(1 - e^2)$

$9 = 25(1 - e^2)$

$\frac{9}{25} = 1 - e^2$

$e^2 = \frac{16}{25} \Rightarrow e = \frac{4}{5}$

$x = \pm\frac{a}{e} = \pm\frac{5\times5}{4}$

$x = \pm\frac{25}{4}$ A

3. $3x^2 - 10x + 8 = 0$

$(3x-4)(x-2) = 0$

$x = \frac{4}{3}$ or $2$ D

$3x$ $-4$
$x$ $-2$

4. $\bar{z} = 2\left(\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}\right) = 2\left(\cos-\frac{\pi}{3} + i\sin-\frac{\pi}{3}\right)$

$\left(\bar{z}\right)^{-1} = \frac{1}{2}\left(\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}\right)$ C

5. $\frac{\text{odd}}{\text{odd}} = \text{even}$  $\lim_{x\to0}\frac{\sin x}{x} = 1$

B

DADCB | AABCD

6 $x = m_1 h + b_1$ $y = m_2 h + b_2$

$h=0$ $x=2$ $h=0$ $y=3$

$x = m_1 h + 2$ $y = m_2 h + 3$

$h=4$ $x=8$ $h=4$ $y=7$

$8 = 4m_1 + 2$ $7 = 4m_2 + 3$

$\frac{3}{2} = m_1$ $m_2 = 1$

$x = \frac{3}{2}h + 2$ $y = h + 3$

A

7. $PM.PN = \left|\frac{(a\sec\theta - a\tan\theta)(a\sec\theta + a\tan\theta)}{1+a^2}\right|$

$= \frac{a^2(\sec^2\theta - \tan^2\theta)}{1+a^2} = \frac{a^2}{1+a^2}$

$b^2 = a^2(e^2 - 1)$ $e^2 = 1 + \frac{1}{a^2} = \frac{a^2+1}{a^2}$

$1 = a^2(e^2-1)$

$\frac{1}{a^2} = e^2 - 1$ $\therefore \frac{a^2}{a^2+1}$

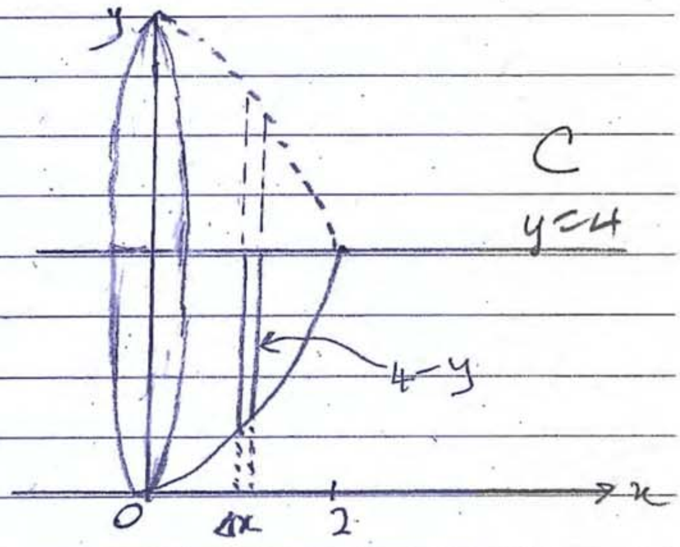
A

8 Circle $|z-1| \leq \sqrt{2}$

Ray $0 \arg(z+i) \leq \frac{\pi}{4}$

B

9 $V = \pi\int_0^2 (4-y)^2 dx$




C

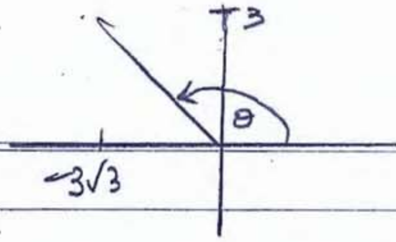
10 Total − No Boys

${}^{10}C_5 - {}^{6}C_5 = {}^{10}C_5 - 6$

D

Question 11

a) i) $z = -3\sqrt{3} + 3i$




$\tan\theta = -\frac{3}{3\sqrt{3}} = -\frac{1}{\sqrt{3}}$

1 $\theta = \pi - \frac{\pi}{6} = \frac{5\pi}{6}$

1 $|z| = \sqrt{(3\sqrt{3})^2 + 3^2} = \sqrt{27+9} = \sqrt{36} = 6$

$z = 6 \text{ cis } \frac{5\pi}{6}$

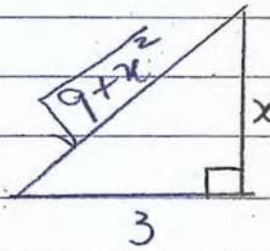
ii) $z^3 = \left(6 \text{ cis } \frac{5\pi}{6}\right)^3 = 6^3 \text{ cis } \frac{5\pi}{2}$

1 $= 6^3 \text{ cis } \frac{\pi}{2} = \underline{\underline{216i}}$

b) $\int \frac{dx}{x^2\sqrt{9+x^2}}$ $\qquad x = 3\tan\theta$

$dx = 3\sec^2\theta \, d\theta$

1 $= \int \frac{3\sec^2\theta \, d\theta}{9\tan^2\theta\sqrt{9+9\tan^2\theta}}$




$= \int \frac{3\sec^2\theta \, d\theta}{9\tan^2\theta . 3\sec\theta}$

$= \frac{1}{9}\int \frac{\sec\theta}{\tan^2\theta} d\theta$

$= \frac{1}{9}\int \frac{1}{\cos\theta} \times \frac{\cos^2\theta}{\sin^2\theta} d\theta$

1 $= \frac{1}{9}\int \frac{\cos\theta}{\sin^2\theta} d\theta$

$= \frac{1}{9}\int (\sin\theta)^{-2}\cos\theta \, d\theta$

1 $= \frac{1}{9} \frac{(\sin\theta)^{-1}}{-1} = -\frac{1}{9\sin\theta}$

1 $= -\frac{\sqrt{9+x^2}}{9x} + c$

1 1 c i) $\int \frac{dx}{\sqrt{5-(x-1)^2}}$ $\qquad 4-(x^2-2x+1)+1$

1 $= \sin^{-1}\frac{x-1}{\sqrt{5}} + c$

1 ii) $\int \frac{2x+4-4}{x^2+4x+8}\,dx$

1 $= \int \frac{2x+4}{x^2+4x+8}\,dx - \int \frac{4}{(x+2)^2+4}\,dx$

1 $= \ln(x^2+4x+8) - 4\times\frac{1}{2}\tan^{-1}\frac{x+2}{2}$

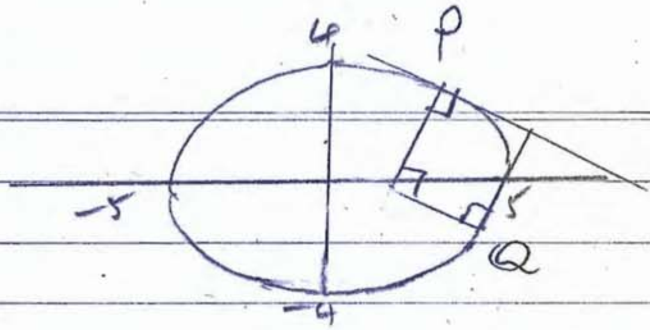
$= \ln(x^2+4x+8) - 2\tan^{-1}\frac{x+2}{2}$

d) $\frac{1-i}{1+i}\times\frac{1-i}{1-i} = \frac{1-2i+i^2}{1+1}$

1 $= \frac{-2i}{2} = -i$

1 $\frac{\pi}{2}$ clockwise

Question 12




a) $\frac{x^2}{25} + \frac{y^2}{16} = 1$

$x = 5\cos\theta$

$y = 4\sin\theta$

1 $\frac{dy}{dx} = \frac{dy}{d\theta} \Big/ \frac{dx}{d\theta}$

1 $m = \frac{4\cos\alpha}{-5\sin\alpha}$ for tangent at P

1 $m_1 = \frac{5\sin\alpha}{4\cos\alpha}$ for normal at P

as $mm_1 = -1$

ii) Normal at P $m_1 = \frac{5\sin\alpha}{4\cos\alpha}$

Normal at Q $m_2 = \frac{5\sin\beta}{4\cos\beta}$

These are at right angles

1 $\frac{5\sin\alpha}{4\cos\alpha} \times \frac{5\sin\beta}{4\cos\beta} = -1$

$25\tan\alpha\tan\beta = -1$

b) $\int e^x \cos 2x \, dx$ $\quad u = \cos 2x \quad dv = e^x dx$

1 $\quad du = -2\sin 2x \, dx \quad v = e^x$

$I = e^x \cos 2x - \int e^x . -2\sin 2x \, dx$

1 $= e^x \cos 2x + 2\int e^x \sin 2x \, dx$

$u = \sin 2x \quad dv = e^x dx$

$du = 2\cos 2x \, dx \quad v = e^x$

$I = e^x \cos 2x + 2\left\{e^x \sin 2x - 2\int \cos 2x \, e^x dx\right.$

1 $= e^x \cos 2x + 2e^x \sin 2x - 4\int \cos 2x \, e^x dx$

$5I = e^x \{\cos 2x + 2\sin 2x\}$

$$\int_0^{\frac{\pi}{4}} e^x \cos 2x \, dx = \left[\frac{e^x}{5}\{\cos 2x + 2\sin 2x\right]_0^{\pi/4}$$

$$= \frac{e^{\frac{\pi}{4}}}{5}\left\{\cos\frac{\pi}{2} + 2\sin\frac{\pi}{2}\right\} - \frac{e^0}{5}\left\{\cos 0 + 2\sin 0\right\}$$

$$= \frac{1}{5}\left\{2e^{\pi/4} - 1\right\}$$

c) $$\frac{5x^2 - 3x - 8}{(x-1)(x-2)(x+2)} \equiv \frac{a}{x-1} + \frac{b}{x-2} + \frac{c}{x+2}$$

$x=1 \quad 5 - 3 - 8 = -3a \implies a = 2$

$x=2 \quad 20 - 6 - 8 = 4b \implies b = \frac{3}{2}$

$x=-2 \quad 20 + 6 - 8 = (-3)(-4)c$

$18 = 12c \implies c = \frac{3}{2}$

$$\int \frac{5x^2 - 3x - 8}{(x-1)(x-2)(x+2)} = \int \frac{2\,dx}{x-1} + \frac{3}{2}\int \frac{dx}{x-2} + \frac{3}{2}\int \frac{dx}{x+2}$$

$$= 2\ln(x-1) + \frac{3}{2}\ln(x-2) + \frac{3}{2}\ln(x+2) + c$$

d) $w = -5 - 12i \qquad z = 3 + 4i$

i) $(a+ib)^2 = -5 - 12i$

$a + ib = \sqrt{-5 - 12i}$

$a^2 - b^2 = -5$

$2ab = -12$

$b = -\frac{6}{a}$

$a^2 - \frac{36}{a^2} = -5$

$a^4 + 5a^2 - 36 = 0$

$(a^2 + 9)(a^2 - 4) = 0$

$a = \pm 2$ as $a$ is real

$b = \mp 3$

$\therefore (2 - 3i)$ and $(-2 + 3i)$ are square roots of $w$

Q12

ii) $\dfrac{-5+12i}{3+4i} \times \dfrac{3-4i}{3-4i}$

$$\frac{-15+20i+36i+48}{9+16}$$

$$\frac{33}{25}+\frac{56i}{25}$$

<u>Question 13 b</u>

i) $\frac{d}{dx}\, x f(x) - \int x f'(x)\, dx$

$= x.f'(x) + f(x).1 - x f'(x)$

$= f(x)$

ii) $f(x) = \tan^{-1} x$

$\int \tan^{-1} x\, dx = x \tan^{-1} x - \int x \cdot \frac{1}{1+x^2}\, dx$

$= x \tan^{-1} x - \frac{1}{2} \ln(1+x^2) + C$

c) $x^3 - 9x^2 + 31x + m = 0$

Let roots be $\alpha - d,\ \alpha,\ \alpha + d$

$\Sigma\alpha = 3\alpha = 9 \implies \alpha = 3$

Sub $\alpha = 3$

$27 - 81 + 93 + m = 0$

$m = -39$

$\therefore\ x^3 - 9x^2 + 31x - 39 = 0$

$\Sigma\alpha\beta$:

$3(3-d) + 3(3+d) + (3-d)(3+d) = 31$

$9 - 3d + 9 + 3d + 9 - d^2 = 31$

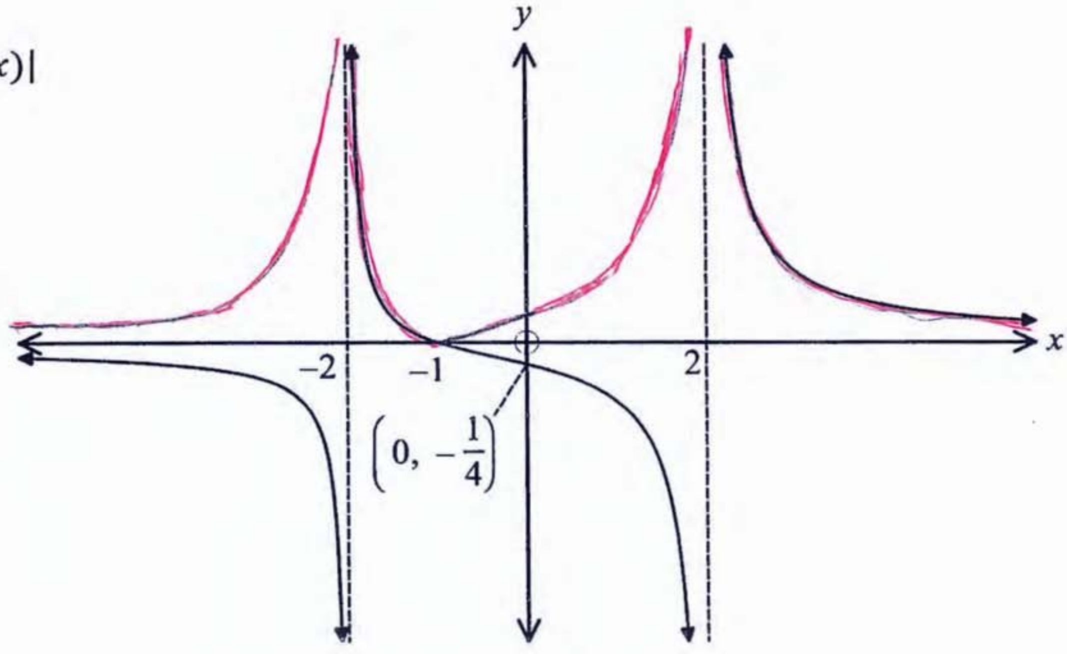
$-4 = d^2 \implies d = \pm 2i$

$\therefore$ Roots are $3-2i,\ 3,\ 3+2i$

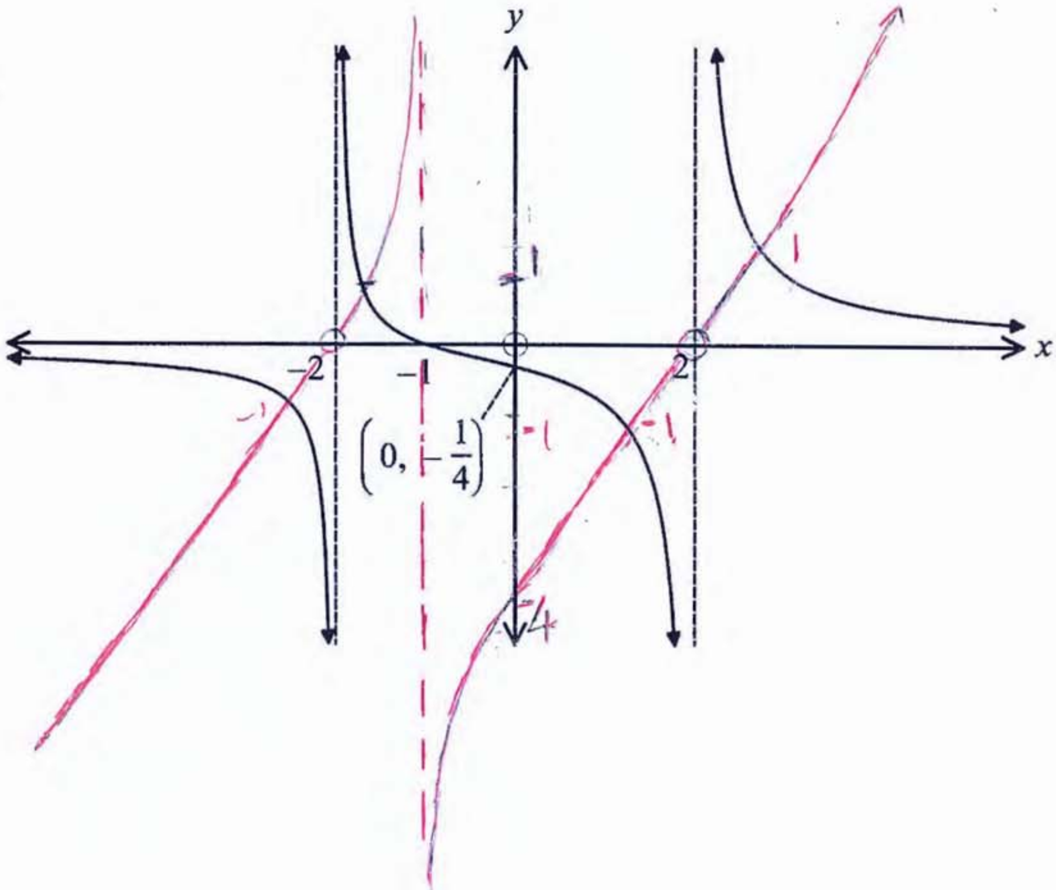
name / number ____________________

13 a)

(i) $y = |f(x)|$

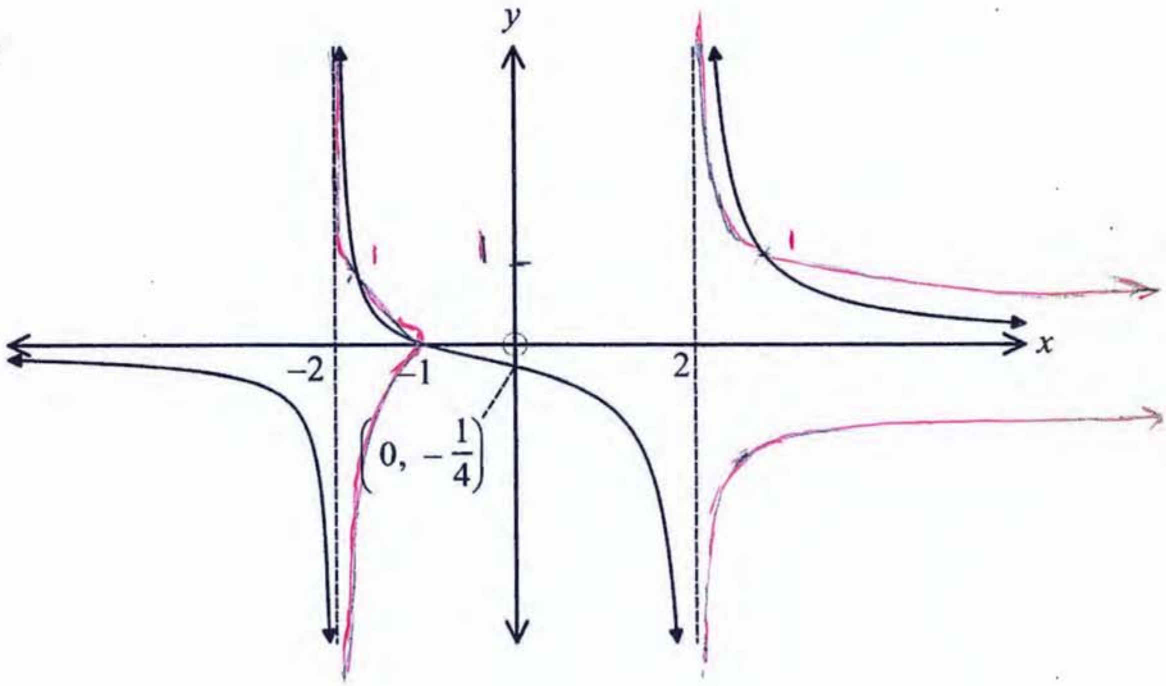



(ii) $y = \frac{1}{f(x)}$

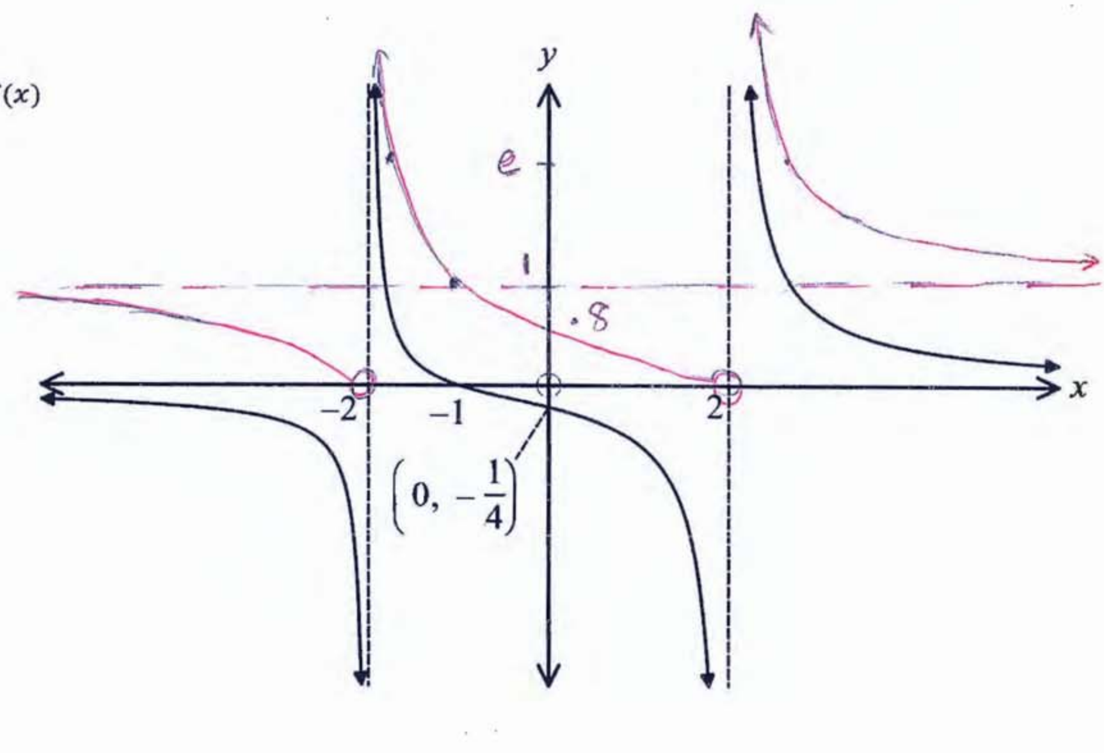


name / number ____________________

(iii) $y^2 = f(x)$




(iv) $y = e^{f(x)}$



Question 14

a) i) $I_n = \int x^n \cos x \, dx$

$u = x^n \qquad dv = \cos x \, dx$

$du = n x^{n-1} \, dx \qquad v = \sin x$

$I_n = x^n \sin x - \int \sin x . \, n x^{n-1} \, dx$

1 $= x^n \sin x - n \int x^{n-1} \sin x \, dx$

$\int x^{n-1} \sin x \, dx$

$u = x^{n-1} \qquad dv = \sin x \, dx$

$du = (n-1) x^{n-2} \, dx \qquad v = -\cos x$

$= -x^{n-1} \cos x + \int \cos x (n-1) x^{n-2} \, dx$

$\therefore I_n = x^n \sin x - n \left\{ -x^{n-1} \cos x + (n-1) \int \cos x \, x^{n-2} \, dx \right\}$

1 $= x^n \sin x + n x^{n-1} \cos x - n(n-1) \int \cos x \, x^{n-2} \, dx$

$= x^n \sin x + n x^{n-1} \cos x - n(n-1) I_{n-2}$

ii)

$I_4 = x^4 \sin x + 4x^3 \cos x - 12 I_2$

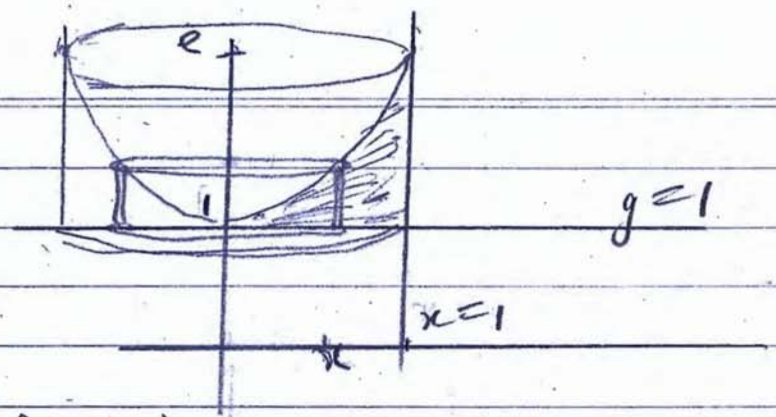
$I_2 = x^2 \sin x + 2x \cos x - 2 I_0$

1 $I_0 = \int \cos x \, dx = \sin x$

$I_2 = x^2 \sin x + 2x \cos x - 2 \sin x$

1 $I_4 = x^4 \sin x + 4x^3 \cos x - 12 x^2 \sin x - 24 x \cos x + 24 \sin x$

14 b)




$V = 2\pi \int x(y-1)\,dx$

1 $= 2\pi \int_0^1 x\left(e^{x^2}-1\right) dx$

$= 2\pi \left\{ \int_0^1 x e^{x^2}\,dx - \int_0^1 x\,dx \right\}$

1 $= 2\pi \cdot \left[ \frac{1}{2} e^{x^2} - \frac{x^2}{2} \right]_0^1$

$= \pi \left( e^1 - 1 - e^0 + 0 \right)$

1 $V = \pi (e-2)\ u^3$

c) i) Put $y = \alpha^2 \Rightarrow \alpha = \pm\sqrt{y}$

1 $(\pm\sqrt{y})^3 - 3(\pm\sqrt{y})^2 - 4(\pm\sqrt{y}) - 5 = 0$

$\pm\sqrt{y}\,y - 3y - 4(\pm\sqrt{y}) - 5 = 0$

$\pm\sqrt{y}\,(y-4) = 3y+5$

$y(y-4)^2 = (3y+5)^2$

$y(y^2-8y+16) = 9y^2+30y+25$

$y^3 - 8y^2 + 16y = 9y^2 + 30y + 25$

1 $y^3 - 17y^2 - 14y - 25 = 0$

Note $\sum \alpha^2 = 17$

14 c)

ii) $P(x) = x^3 - 3x^2 - 4x - 5$

$P(\alpha) = \alpha^3 - 3\alpha^2 - 4\alpha - 5 = 0$

1 $P(\beta) = \beta^3 - 3\beta^2 - 4\beta - 5 = 0$

$P(\gamma) = \gamma^3 - 3\gamma^2 - 4\gamma - 5 = 0$

1 $\alpha^3 + \beta^3 + \gamma^3 = 3(\alpha^2 + \beta^2 + \gamma^2) + 4(\alpha + \beta + \gamma) + 15$

But $\Sigma \alpha^2 = 17$, $\Sigma \alpha = 3$

1 $\therefore \alpha^3 + \beta^3 + \gamma^3 = 3 \times 17 + 4 \times 3 + 15$

$= 78$

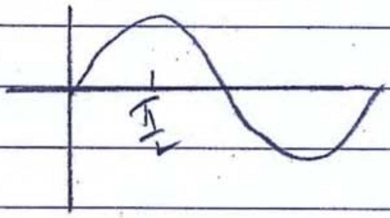
d) $2\sin^3\theta + 1 = 2\sin^2\theta + \sin\theta$

$2\sin^3\theta - 2\sin^2\theta - \sin\theta + 1 = 0$

$2\sin^2\theta(\sin\theta - 1) - (\sin\theta - 1) = 0$

1 $(\sin\theta - 1)(2\sin^2\theta - 1) = 0$

1 $\sin\theta = 1$ or $\sin\theta = \pm\frac{1}{\sqrt{2}}$




All 4 quads

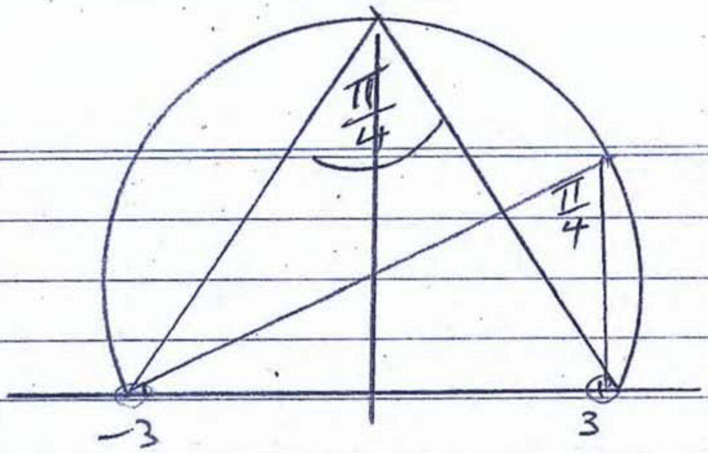
1 $\theta = \frac{\pi}{2}$ or $\frac{\pi}{4}, \frac{3\pi}{4}, \frac{5\pi}{4}, \frac{7\pi}{4}$

Question 15

a)

1

1




b) i)

$$A = \int_{-1}^{1} \frac{1}{x^2+1} - \frac{x^2}{x^2+1} \, dx$$

$$= 2\int_0^1 \frac{1-x^2}{x^2+1} \, dx$$

$$= -2\int_0^1 \frac{x^2+1-2}{x^2+1} \, dx$$

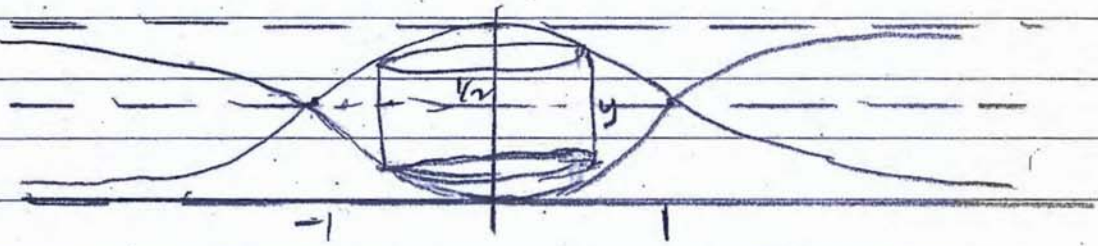
1 $$= -2\int_0^1 1 - \frac{2}{x^2+1} \, dx$$

$$= -2\left[x - 2\tan^{-1}x\right]_0^1$$

1 $$= -2\left(1 - 2\tan^{-1}1\right) - 0 + 0$$

$$= -2\left(1 - 2\times\frac{\pi}{4}\right) = \pi - 2 \ u^2$$

ii)




$$V = 2\pi \int_0^1 x\,y \, dx$$

1 where $$y = \frac{1}{x^2+1} - \frac{x^2}{x^2+1} = \frac{1-x^2}{x^2+1}$$

$$V = 2\pi \int_0^1 x \left\{ \frac{1-x^2}{x^2+1} \right\} dx$$

$$= -2\pi \int_0^1 x \left\{ \frac{x^2+1-2}{x^2+1} \right\} dx$$

1 $$= -2\pi \int_0^1 x - \frac{2x}{x^2+1} \quad dx$$

$$= -2\pi \left[ \frac{x^2}{2} - \ln(x^2+1) \right]_0^1$$

$$= -2\pi \left( \frac{1}{2} - \ln 2 - 0 + \ln 1 \right)$$

1 $$= \pi \left( 2\ln 2 - 1 \right) \quad u^3$$

c) i) $\int_0^a f(a-x)\,dx$ Put $y = a - x$

$dy = -dx$

1 $= \int_a^0 f(y) . -dy$ $x=0, \; y=a$

$x=a, \; y=0$

$$= \int_0^a f(y)\,dy = \int_0^a f(x)\,dx$$

ii) $$\int_0^\pi \frac{x \sin x}{1+\cos^2 x} dx = \int_0^\pi \frac{(\pi - x)\sin(\pi - x)}{1+\cos^2(\pi - x)} dx$$

$$= \int \frac{(\pi - x)\sin x}{1 + \{-\cos x\}^2}$$

$$= \int \frac{\pi \sin x}{1+\cos^2 x} - \frac{x \sin x}{1+\cos^2 x} \quad dx$$

$$2\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \int_0^{\pi} \frac{\pi \sin x}{1+\cos^2 x}$$

$$\int_0^{\pi} \frac{x\sin x}{1+\cos^2 x}\,dx = \left[-\frac{\pi}{2}\tan^{-1}\cos x\right]_0^{\pi}$$

$$= -\frac{\pi}{2}\tan^{-1}\cos\pi + \frac{\pi}{2}\tan^{-1}\cos 0$$

$$= -\frac{\pi}{2}\tan^{-1}(-1) + \frac{\pi}{2}\tan^{-1} 1$$

$$= \frac{\pi}{2}\cdot\frac{\pi}{4} + \frac{\pi}{2}\cdot\frac{\pi}{4}$$

$$= \frac{\pi^2}{4}$$

Alternative 15b(ii)

OR $V = \pi \int_0^{\frac{1}{2}} x_1^2 \, dy + \pi \int_{\frac{1}{2}}^1 x_2^2 \, dy$

$y = \frac{x_1^2}{x_1^2+1}$ $\qquad$ $y = \frac{1}{x_2^2+1}$

$= 1 - \frac{1}{x^2+1}$ $\qquad$ $x_2^2 + 1 = \frac{1}{y}$

$\frac{1}{x^2+1} = 1-y$ $\qquad$ $x^2 = \frac{1}{y} - 1$

$\frac{1}{1-y} = x^2+1$

$\frac{1}{1-y} - 1 = x^2$

$\therefore \; V = \pi \int_0^{\frac{1}{2}} \frac{1}{1-y} - 1 \, dy + \pi \int_{\frac{1}{2}}^1 \frac{1}{y} - 1 \, dy$

$V = \pi \left[ -\ln(1-y) - y \right]_0^{\frac{1}{2}} + \pi \left[ \ln y - y \right]_{\frac{1}{2}}^1$

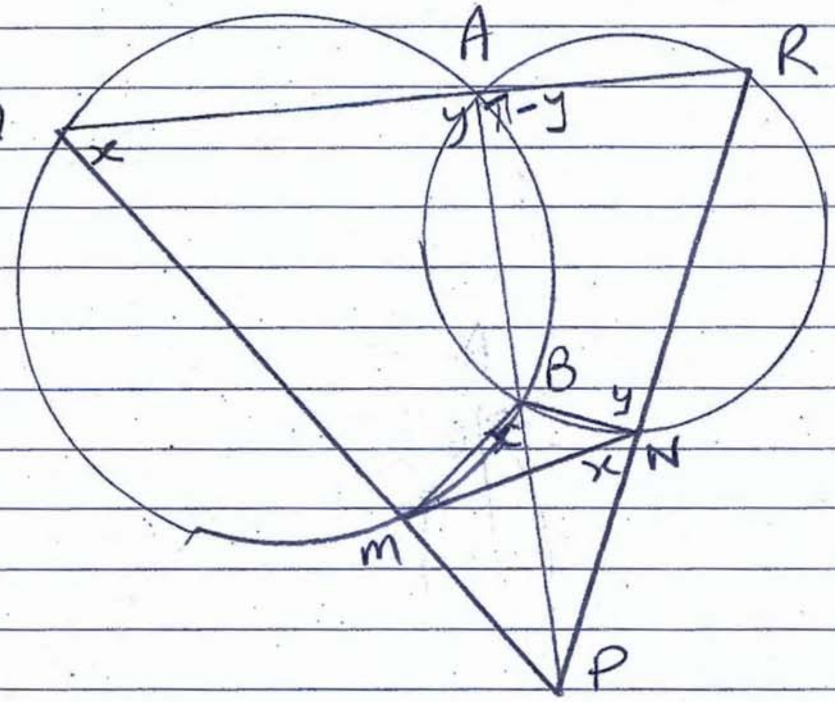
$= \pi \left( -\ln\frac{1}{2} - \frac{1}{2} + \ln 1 + 0 \right) + \pi \left( \ln 1 - 1 - \ln\frac{1}{2} + \frac{1}{2} \right)$

$= \pi \left\{ \ln 2 - \frac{1}{2} - 1 + \ln 2 + \frac{1}{2} \right\}$

$= \pi \left( 2\ln 2 - 1 \right) \quad u^3$

15

d)




i)

$x = \angle mBP = \angle AQm$ Exterior $\angle$ of cyclic quad $BMQA$

$y = \angle PmB = \angle QAB$ Exterior $\angle$ of cyclic quad $BMQA$

$\angle RAP = \pi - y$ Straight $\angle QAR$

$\angle RNB = y = \pi - (\pi - y)$ Opposite $\angle$'s cyclic quad $ARNB$

$\therefore \angle RNB = \angle BmP$

But this is exterior $\angle$ of quad $PmBN$

$\therefore PmBN$ is a cyclic quad

ii) $\angle mBP = \angle mNP$ Angles at circumference standing on arc $mP$ of cyclic quad $PmBN$

$\therefore \angle RQm = \angle mNP = x$

But $\angle mNP$ is exterior angle of quad $QRNm$

$\therefore QRNm$ is a cyclic quadrilateral.

Q 11

i) Upwards as Positive $\uparrow$ $\quad \downarrow mg \quad \downarrow mkv^2$

Terminal velocity $= U$

<u>Downwards</u> $\downarrow mg \quad \uparrow mkv^2$

$$mg - mkv^2 = 0$$
$$mkv^2 = mg$$
$$kv^2 = g$$
$$v^2 = \frac{g}{k}$$
1 $$U^2 = \frac{g}{k} \implies k = \frac{g}{U^2}$$

Upwards motion $\downarrow mg \quad \downarrow mkv^2$

$$m\ddot{x} = -mg - mkv^2$$
$$\ddot{x} = -g - kv^2$$
$$\ddot{x} = -g - \frac{gv^2}{U^2}$$
1 $$\ddot{x} = \frac{-g}{U^2}\left(U^2 + v^2\right)$$

ii) $$\frac{dv}{dt} = -\frac{g}{U^2}\left(U^2 + v^2\right)$$

$$\frac{dt}{dv} = -\frac{U^2}{g}\,\frac{1}{U^2 + v^2}$$

1 $$t = -\frac{U^2}{g} \cdot \frac{1}{U} \tan^{-1}\frac{v}{U} + C$$

$t = 0,\ v = U \qquad 0 = -\frac{U}{g}\tan^{-1} 1 + C$

$$C = U \cdot \pi$$

$$\therefore t = \frac{U}{g}\left\{\frac{\pi}{4} - \tan^{-1}\frac{v}{U}\right\}$$

$$\frac{gt}{U} = \frac{\pi}{4} - \tan^{-1}\frac{v}{U}$$

1 $$\tan^{-1}\frac{v}{U} = \frac{\pi}{4} - \frac{gt}{U}$$

$$\frac{v}{U} = \tan\left(\frac{\pi}{4} - \frac{gt}{U}\right)$$

iii) $$v = U\tan\left(\frac{\pi}{4} - \frac{gt}{U}\right)$$

$$\frac{dx}{dt} = U\tan\left(\frac{\pi}{4} - \frac{gt}{U}\right)$$

$$= U\,\frac{\sin\left(\frac{\pi}{4} - \frac{gt}{U}\right)}{\cos\left(\frac{\pi}{4} - \frac{gt}{U}\right)}$$

1 $$x = \frac{U^2}{g}\ln\cos\left(\frac{\pi}{4} - \frac{gt}{U}\right) + C_2$$

or $$\frac{x}{U} = \frac{U}{g}\ln\cos\left(\frac{\pi}{4} - \frac{gt}{U}\right) + c$$

$$\left(\text{as } \frac{d}{dt}\cos\left(\frac{\pi}{4} - \frac{gt}{U}\right) = -\cdot\frac{-U}{g}\sin\left(\frac{\pi}{4} - \frac{gt}{U}\right)\right)$$

$t=0$, $x=0$ $$0 = \frac{U^2}{g}\ln\cos\frac{\pi}{4} + C_2$$

$$C_2 = -\frac{U^2}{g}\ln\frac{1}{\sqrt{2}} = \frac{U^2}{g}\ln\sqrt{2}$$

1 $$\frac{x}{U} = \frac{U}{g}\ln\cos\left(\frac{\pi}{4} - \frac{gt}{U}\right) + \frac{U}{g}\ln\sqrt{2}$$

$$\frac{x}{U} = \frac{U}{g} \ln \cos\sqrt{2}\left(\frac{\pi}{4} - \frac{gt}{U}\right)$$

iv) $t = T$ max height $v = 0$

$$H = \frac{U^2}{g} \ln \sqrt{2} \cos\left(\frac{\pi}{4} - \frac{g}{U}T\right)$$

$$\frac{gT}{U} = \frac{\pi}{4}$$

$$H = \frac{U^2}{g} \ln \sqrt{2} \cos 0$$

$$H = \frac{U^2}{g} \ln \sqrt{2} \quad \text{or} \quad \frac{H}{U} = \frac{U}{2g} \ln 2$$

$$t = \tfrac{1}{2}T \Rightarrow \frac{g}{U}\left(\tfrac{1}{2}T\right) = \frac{\pi}{8}$$

$$t = \tfrac{1}{2}T \Rightarrow \frac{x}{U} = \frac{U}{g} \ln \sqrt{2} \cos\frac{\pi}{8}$$

$$= \frac{U}{2g} \ln 2\cos^2\frac{\pi}{8}$$

$$= \frac{U}{2g} \ln\left(1 + \cos\frac{\pi}{4}\right)$$

$$= \frac{U}{2g} \ln\left(1 + \frac{1}{\sqrt{2}}\right)$$

$$t = T \Rightarrow \frac{H}{U} = \frac{U}{g} \ln \sqrt{2} = \frac{U}{2g} \ln 2$$

$$t = \tfrac{1}{2}T \Rightarrow \frac{x}{H} = \frac{\ln\left(1 + \frac{1}{\sqrt{2}}\right)}{\ln 2}$$

$$\approx 0.77 \quad \text{ie } 77\%$$

of max ht during 1st half of ascent time

**Q16(cont)**
**b. Outcomes assessed: E5**

**Marking Guidelines**

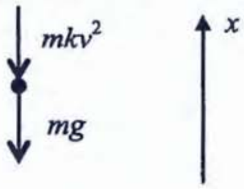
| | Criteria | Marks |
|---|---|---|
| i | • considers the forces on a falling particle to deduce the value of the square of $U$ | 1 |
| | • considers the forces on a rising particle to deduce its equation of motion in the form required | 1 |
| ii | • finds $t$ as a function of $v$ by integration | 1 |
| | • rearranges to find an expression for $v$ as a function of $t$ | 1 |
| iii | • integrates with respect to $t$ finding the primitive function | 1 |
| | • evaluates constant and simplifies to obtain required expression for $x$ as a function of $t$ | 1 |
| iv | • finds an expressions for $T$, and hence $H$, in terms of $g$ and $U$ | 1 |
| | • finds an expression for $x$ in terms of $g$ and $U$ when half the ascent time has elapsed | 1 |
| | • calculates the percentage of the maximum height attained at this time | 1 |

**Answer**

i. *Forces on a falling particle*


$$\ddot{x} \to 0 \text{ as } mkv^2 \to mg$$
$$\therefore kU^2 = g$$
$$U^2 = \frac{g}{k}$$

*Forces on a rising particle*




$$\ddot{x} = -g - kv^2$$
$$= -k\left(\frac{g}{k} + v^2\right)$$
$$= -\frac{g}{U^2}\left(U^2 + v^2\right)$$

ii)
$$\frac{dv}{dt} = -\frac{g}{U^2}\left(U^2 + v^2\right)$$
$$\frac{dt}{dv} = -\frac{U}{g} \cdot \frac{U}{U^2 + v^2}$$
$$\frac{-g}{U}t = \tan^{-1}\left(\frac{v}{U}\right) + c$$
$$\left.\begin{array}{l} t = 0 \\ v = U \end{array}\right\} \Rightarrow \begin{array}{l} 0 = \tan^{-1}1 + c \\ 0 = \frac{\pi}{4} + c \end{array}$$
$$\therefore \frac{g}{U}t = \frac{\pi}{4} - \tan^{-1}\left(\frac{v}{U}\right)$$
$$\tan^{-1}\left(\frac{v}{U}\right) = \frac{\pi}{4} - \frac{g}{U}t$$
$$\therefore \frac{v}{U} = \tan\left(\frac{\pi}{4} - \frac{g}{U}t\right)$$

iii)
Hence $\displaystyle \frac{1}{U}\frac{dx}{dt} = \tan\left(\frac{\pi}{4} - \frac{g}{U}t\right)$
$$\frac{1}{U}\frac{dx}{dt} = \frac{\sin\left(\frac{\pi}{4} - \frac{g}{U}t\right)}{\cos\left(\frac{\pi}{4} - \frac{g}{U}t\right)}$$
$$\frac{x}{U} = \frac{U}{g}\log_e\left\{\cos\left(\frac{\pi}{4} - \frac{g}{U}t\right)\right\} + c_1$$
$$\left.\begin{array}{l} t = 0 \\ x = 0 \end{array}\right\} \Rightarrow \begin{array}{l} 0 = \frac{U}{g}\log_e\left(\cos\frac{\pi}{4}\right) + c_1 \\ 0 = \frac{U}{g}\log_e\frac{1}{\sqrt{2}} + c_1 \end{array}$$
$$\frac{x}{U} = \frac{U}{g}\log_e\left\{\sqrt{2}\cos\left(\frac{\pi}{4} - \frac{g}{U}t\right)\right\}$$

iv) $t = T \Rightarrow v = 0 \quad \therefore \frac{g}{U}T = \frac{\pi}{4}$ and $\frac{g}{U}\left(\frac{1}{2}T\right) = \frac{\pi}{8}$
$$\therefore t = \tfrac{1}{2}T \Rightarrow \frac{x}{U} = \frac{U}{g}\log_e\left\{\sqrt{2}\cos\frac{\pi}{8}\right\}$$
$$= \frac{U}{2g}\log_e\left(2\cos^2\frac{\pi}{8}\right)$$
$$= \frac{U}{2g}\log_e\left(1 + \cos\frac{\pi}{4}\right)$$
$$= \frac{U}{2g}\log_e\left(1 + \frac{1}{\sqrt{2}}\right)$$

$$t = T \Rightarrow \frac{x}{U} = \frac{U}{g}\log_e\sqrt{2} \qquad \therefore \frac{H}{U} = \frac{U}{2g}\log_e 2$$
$$\therefore t = \tfrac{1}{2}T \Rightarrow \frac{x}{H} = \frac{\log_e\left(1 + \frac{1}{\sqrt{2}}\right)}{\log_e 2} \approx 0 \cdot 77$$

Hence particle gains 77% of its maximum height during the first half of its ascent time.

16 b)

Tangent at P is $x + p^2 y = 2cp$ — (1)

Tangent at Q is $x + q^2 y = 2cq$ — (2)

(1) − (2) $(p^2 - q^2)y = 2c(p-q)$

$p \neq q$ $\therefore (p+q)\,y = 2c$

$$y = \frac{2c}{p+q}$$

Sub into (1)

$$x + p^2 \frac{2c}{p+q} = 2cp$$

$$(p+q)x + 2cp^2 = 2cp^2 + 2cpq$$

1 $$x = \frac{2cpq}{p+q}$$

Tangents meet at $R(x_0, y_0)$

$$y_0 = \frac{2c}{p+q} \quad \text{or} \quad p+q = \frac{2c}{y_0}$$

1 and $x_0 = \dfrac{2cpq}{2c/y_0} = pq\,y_0$

$$\therefore pq = \frac{x_0}{y_0}$$

ii) Distance PQ is $d$

1 $$d^2 = (cp - cq)^2 + \left(\frac{c}{p} - \frac{c}{q}\right)^2$$

$$= c^2(p-q)^2 + c^2\left(\frac{q-p}{pq}\right)^2$$

1 $$d^2 = c^2(p-q)^2 \left\{1 + \frac{1}{p^2q^2}\right\}$$

iii) Note $(p-q)^2 = (p+q)^2 - 4pq$

$$= \left(\frac{2c}{y_0}\right)^2 - \frac{4x_0}{y_0}$$

Substitute into $d^2$

$$d^2 = c^2\left\{\frac{4c^2}{y_0^2} - \frac{4x_0}{y_0}\right\}\left\{1 + \frac{y_0^2}{x_0^2}\right\}$$

Multiply by $x_0^2 y_0^2$

$$x_0^2 y_0^2 d^2 = 4c^2(c^2 - x_0 y_0)(x_0^2 + y_0^2)$$

$(x_0, y_0)$ lies on curve with equation

$$x^2 y^2 d^2 = 4c^2(c^2 - xy)(x^2 + y^2)$$

$\therefore$ this is the locus of $R$ as $c$ & $d$ are constants.