Student Name: ______________________

HUNTERS HILL HIGH SCHOOL
EXTENSION 1
MATHEMATICS
HSC TRIAL 2017

Hunters Hill

High School

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using black pen
- Board-approved calculators may be used
- A Reference Sheet is provided
- In Questions 11–14, show relevant mathematical reasoning and/or calculations

Total marks - 70

Section I

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II

60 marks

- Attempt Questions 11-14
- Allow about 1 hour 45 minutes for this section

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

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1 Which of the following is the domain of $y = \cos^{-1}\left(\frac{x}{2}\right)$?

(A) $-2 \le x \le 2$

(B) $0 < x < 2\pi$

(C) $-\frac{1}{2} \le x \le \frac{1}{2}$

(D) $0 \le x \le \frac{\pi}{2}$

2 The point dividing the interval from $A(-3,1)$ to $B(1,-1)$externally in the ratio 3:1 is:

(A) $\left(0,-\frac{1}{2}\right)$

(B) $\left(-1,\frac{1}{2}\right)$

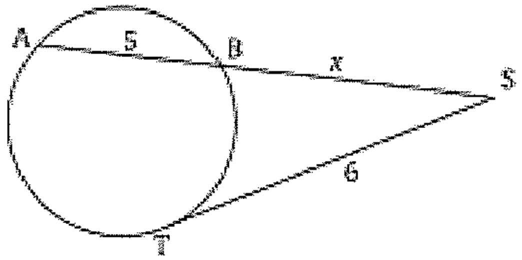
(C) $(-5,2)$

(D) $(3,-2)$

3 Find the remainder when $P(x) = 2x^3 + x^2 - 13x + 6$ is divided by$(x - 1)$.

(A) 18 (B) 6 (C) 4 (D) -4

4 The line DT is a tangent to the circle at T and AS is a secant meeting the circle at A and B. Given that ST = 6, AB = 5 and SB = x, which of the following is the value of x?



(A) $x=4$ (B) $x=5$ (C) $x=6$ (D) $x=9$

5 Using the auxillary angle method, or otherwise, which expression is equal to $\cos x - \sin x$?

(A) $\sqrt{2}\cos\left(x+45^0\right)$ (B) $\sqrt{2}\cos\left(x-45^0\right)$

(C) $2\cos\left(x+45^0\right)$ (D) $2\cos\left(x-45^0\right)$

6 The equation of the chord of contact of the parabola $y = x^2$ from the point $(1,-1)$ is

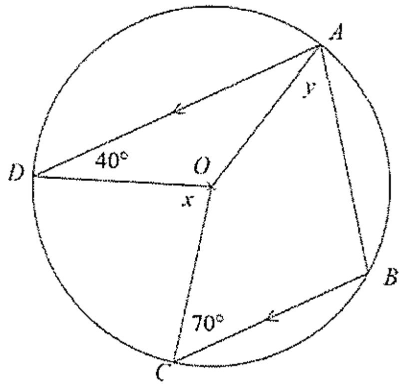
(A) $x+8y+8=0$ (B) $x-8y+8=0$

(C) $2x+y+1=0$ (D) $2x-y+1=0$

7 A particle is moving along the x-axis. Its velocity v at position x is given by $v=\sqrt{8x-x^2}$. What is the acceleration when $x = 3$?

(A) 1 (B) 2 (C) 3 (D) 4

8 A, B, C and D are points on a circle with centre O. $\angle ADO = 40^\circ$ and $\angle BCO = 70^\circ$.



What are the values of x and y?

(A) $x = 80^\circ$ and $y = 15^\circ$

(B) $x = 80^\circ$ and $y = 30^\circ$

(C) $x = 110^\circ$ and $y = 15^\circ$

(D) $x = 110^\circ$ and $y = 35^\circ$

9 Let $x = 1$ be a first approximation to the root of the equation $\cos x = \log_e x$. What is a better approximation to the root using Newton's method?

(A) 1.28

(B) 1.29

(C) 1.30

(D) 1.31

10 How many solutions does the equation $\sin 2x = 4\cos x$ have for $0 \le x \le 2\pi$.

(A) 1 (B) 2 (C) 3 (D) 4

Section II

60 marks
Attempt Questions 11-14
Allow about 1 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Differentiate $y = \tan^{-1}\left(\frac{x}{3}\right)$ **1**

(b) Given $f(x) = \sin^{-1} 3\theta$

(i) Determine the domain and range. **2**

(ii) Find $f'(x)$ **1**

(iii) Sketch $y = f(x)$. **2**

(c) Evaluate $\int_1^4 x\sqrt{x-1}\,dx$ using the substitution $u = x - 1$. **3**

(d) Evaluate the following definite integrals:

(i) $\int_0^{\frac{\pi}{2}} \cos^2\left(\frac{x}{2}\right) dx$ **3**

(ii) $\int_1^2 x\left(x^2-1\right)^2 dx,\quad let\ u = x^2 - 1$ **3**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) If $f(x) = e^{x+1}$ find the inverse function $f^{-1}(x)$ and hence show **2**

$$f\left[f^{-1}(x)\right] = f^{-1}\left[f(x)\right] = x$$

(b) If α, β and γ are the roots of the equation $x^3 + 2x^2 + 3x + 5 = 0$, find the value of :

(i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ **2**

(ii) $\alpha^2 + \beta^2 + \gamma^2$ **2**

(c) When a polynomial $P(x)$ is divided by $x - 2$ the remainder is 4 and when it is divided by $x - 3$ the remainder is 7. Find the remainder when $P(x)$ is divided by $(x - 2)(x - 3)$ **3**

(d) Solve $\frac{2x}{x+1} \leq x$ **3**

(e) Use mathematical induction to prove that for all positive integers, n **3**

$$\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2$$

Question 13 (15 marks) Use a SEPARATE writing booklet.

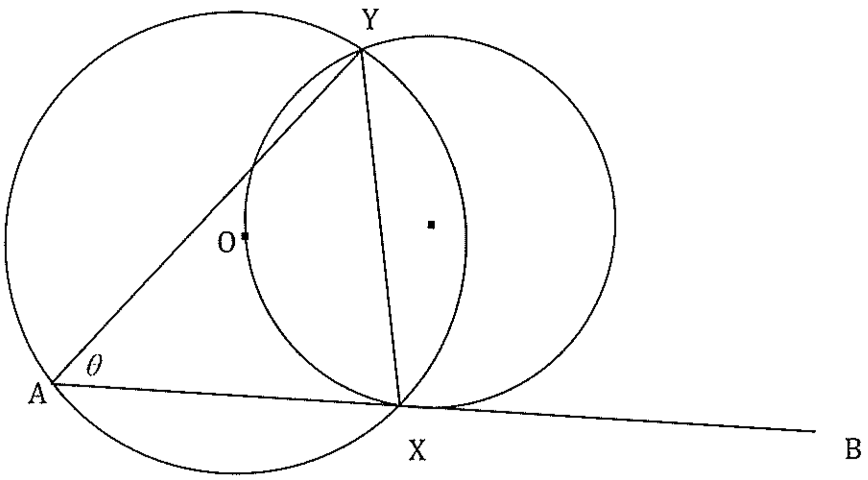
(a) Find the coefficient of x^6 in the expansion $\left(x^2+4\right)^{12}$ **2**

(b) A particle is moving along the x-axis so that its acceleration after t seconds is by $\ddot{x} = 4x(x^2-2)$. The particle starts at the origin with an initial velocity of $\sqrt{6}$ cm/s.

(i) If v is the velocity of the particle, find v^2 as a function of x. **2**

(ii) Explain why the motion is confined to $-1 \le x \le 1$. **2**

(c) O is the centre of the larger circle. The two circles intersect at the points X and Y. AXB is a tangent to the smaller circle at point X. O is on the circumference of the smaller circle.



Copy or trace the diagram onto your answer paper.

(i) Find $\angle XOY$ in terms of θ.
Give a reason for your answer. **1**

(ii) Explain why $\angle BXY = 2\theta$ **1**

(iii) Prove AX=YX. **2**

Question 13 continued on next page.

Question 13 (continued)

(d) (i) By considering the terms in x^r on both sides of the identity **2**

$(1+x)^{m+n} = (1+x)^m (1+x)^n$, show that ${}^{m+n}C_r = \sum_{k=0}^{r} {}^{m}C_k \, {}^{n}C_{r-k}$

for $0 \le r \le m$ and $0 \le r \le n$.

(ii) Hence show that **3**

$${}^{m+1}C_0 \, {}^{n}C_2 + {}^{m+1}C_1 \, {}^{n}C_1 + {}^{m+1}C_2 \, {}^{n}C_0 = {}^{m}C_0 \, {}^{n+1}C_2 + {}^{m}C_1 \, {}^{n+1}C_1 + {}^{m}C_2 \, {}^{n+1}C_0$$

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) Find the exact value of $\sin\left[\cos^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\left(\frac{5}{12}\right)\right]$. **3**
Show all working.

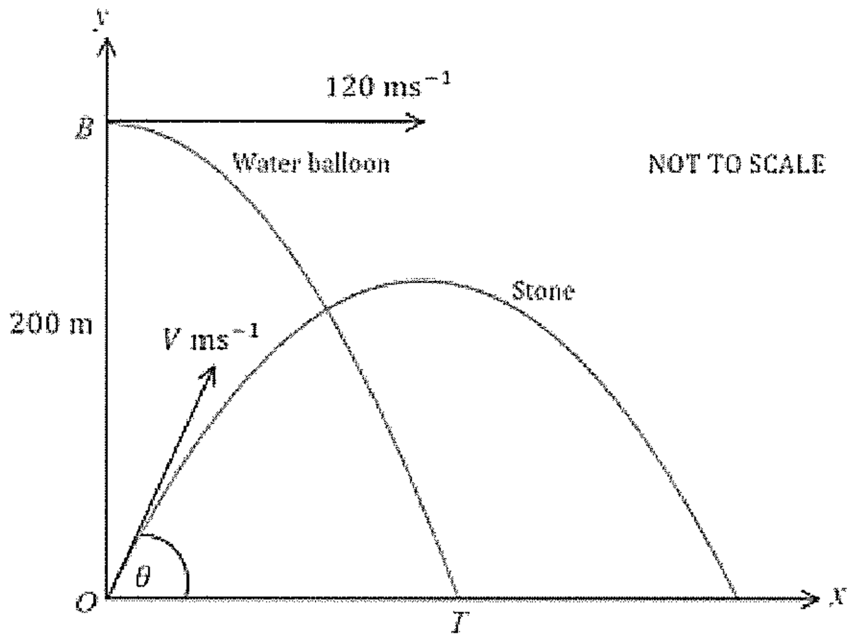
(b) At time t years the number N of individuals in a population is given by
$N = 5000 - 4250e^{-kt}$ for some $k > 0$.

(i) Find the initial population. **1**

(ii) Sketch the graph of N as a function of t showing clearly the initial and limiting populations. **2**

(iii) Find the value of k if $\frac{dN}{dt} = 250$ when N is three times the initial population. **2**

(c) A water balloon is fired horizontally by a cannon from the point B with a velocity of 120 ms^{-1} to reach a target at T.
At the same time, a stone is launched from the point O with a velocity of V ms^{-1} and an angle of projection of θ in order to burst the water balloon in the air
The point O is 200 metres directly below the point B and $\theta = \tan^{-1}\left(\frac{3}{4}\right)$.
Take the acceleration due to gravity as 10 ms^{-2}.



(i) For the water balloon, show that the equations of motion of the water balloon are given by $x = 120t$ and $y = -5t^2 + 200$. **2**

Question 14 continued on next page.

Question 14 (continued) 2

(ii) For the stone, assume that the equations of motion are given by $x = Vt\cos\theta$ and $y = -5t^2 + Vt\sin\theta$. (Do NOT prove this.) 3

Show that in order for the stone to successfully burst the water balloon in the air, it must be launched at a velocity of $150\ \text{ms}^{-1}$.

(iii) How high above the ground does the collision occur? Give your answer correct to the nearest metre.

END OF EXAMINATION

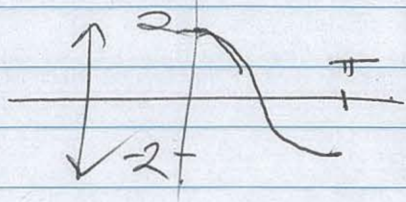
MAX TRIAL SOLUTIONS - 2017

SECTION I

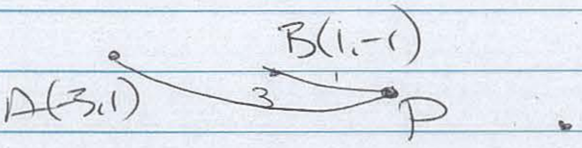
Q.1 **A**

$y = \cos^{-1}\left(\frac{x}{2}\right)$

$x = 2\cos y$



Q2. **D**



Q3 **D**

$P(1) = 2 + 1 - 13 + 6$

$= -4$

Q4. **A**

$ST^2 = SB.SA.$

$6^2 = x(x+5)$

$x^2 + 5x - 36 = 0$

$(x+9)(x-4) = 0$

$x = 4$

Q5. **A**

Q6. **D**

$xx_0 = 2a(y+y_0)$ $(1,-1)$

$x^2 = 4ay$ $4a = 1$

$a = \frac{1}{4}$

$x(1) = 2\left(\frac{1}{4}\right)(y-1)$

$2x = y-1$

Q7. $\underline{\underline{A}}$

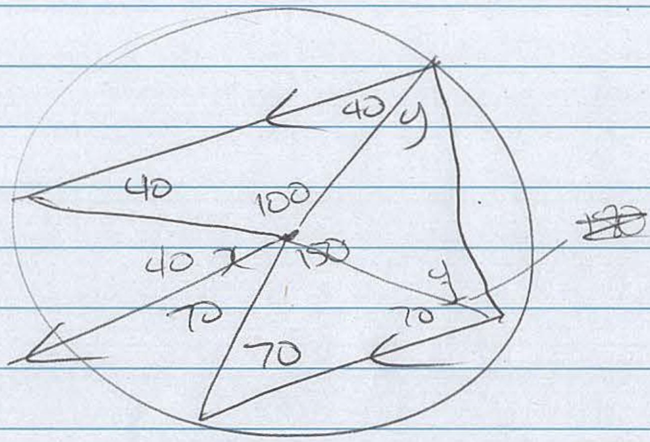
$$v = \sqrt{8x - x^2}$$

$$\frac{1}{2}v^2 = \frac{1}{2}(8x - x^2)$$

$$\frac{d}{dx}: \ a = \frac{1}{2}(8 - 2x)$$

$$= 4 - x.$$

Q8. $\underline{\underline{D}}$



$$x = 110 \qquad 2y + 140 + 150 = 360$$

Q9. $\underline{\underline{B}}$

$$\text{let } f(x) = \cos x - \ln x \cancel{= 0}$$

$$f'(x) = -\sin x - \frac{1}{x}$$

$$x = 1 - \frac{\cos 1 - \ln 1}{-\sin 1 - 1}$$

Q10. $\underline{\underline{B}}$

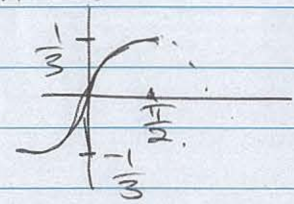
SECTION II

Question 11.

a) $y = \tan^{-1}\left(\frac{x}{3}\right)$

$y' = \dfrac{3}{9+x^2}$

b) $f(x) = \sin^{-1} 3\theta$. $\theta = \frac{1}{3}\sin f(x)$



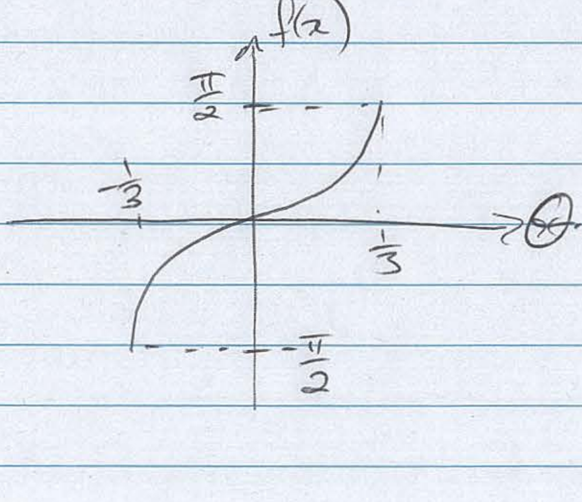
i domain: $\{\cancel{\theta}: -\frac{1}{3} \le \cancel{\theta} \le \frac{1}{3}\}$

range: $\{y: -\frac{\pi}{2} \le \cancel{f(\theta)} \le \frac{\pi}{2}\}$

ii $f'(x) = \dfrac{1}{\sqrt{1-(3x)^2}} \cdot 3$

$= \dfrac{3}{\sqrt{1-9x^2}}$

iii.



c) $I = \int_1^4 x\sqrt{x-1}\,dx$

let $u = x-1$

$du = dx$.

for $x=1$, $u = 1-1$

$= 0$

$x=4$ $u = 4-1$

$= 3$

$$I = \int_0^3 (u+1)\sqrt{u}\,du$$

$$= \int_0^3 \left(u^{\frac{3}{2}} + u^{\frac{1}{2}}\right) du$$

$$= \left[\frac{2u^{\frac{5}{2}}}{5} + \frac{2u^{\frac{3}{2}}}{3}\right]_0^3$$

$$= \frac{2}{5}\cdot 3^{\frac{5}{2}} + \frac{2}{3}\cdot 3^{\frac{3}{2}} - \left(\frac{2}{5}(0)^{\frac{5}{2}} + \frac{2}{3}(0)^{\frac{3}{2}}\right)$$

$$= \frac{2}{5}\cdot 9\sqrt{3} + \frac{2}{\cancel{3}}\cdot \cancel{3}\sqrt{3}$$

$$= \frac{2\sqrt{3}}{5}\left(9 + 5\right)$$

$$= \frac{28\sqrt{3}}{5} = \cancel{13.16358614}$$

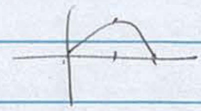
9.699484522

d). i) $\displaystyle\int_0^{\frac{\pi}{2}} \cos^2\left(\frac{x}{2}\right)dx$ $\qquad \cos 2\theta = 2\cos^2\theta - 1$

$= \dfrac{1}{2}\displaystyle\int_0^{\frac{\pi}{2}} \left(1 + \cos x\right) dx$

$= \dfrac{1}{2}\left[x + \sin x\right]_0^{\frac{\pi}{2}}$

$= \dfrac{1}{2}\left(\dfrac{\pi}{2} + \sin\dfrac{\pi}{2} - 0 - \sin 0\right)$



$= \dfrac{1}{4}\left(\pi + 2\right)$

ii) $I = \displaystyle\int_1^2 x\left(x^2-1\right)^2 dx$

let $u = x^2 - 1$ $\qquad x=1 \Rightarrow u=0$

$du = 2x\,dx$ $\qquad x=2 \Rightarrow u=3$

$I = \dfrac{1}{2}\displaystyle\int_0^3 u^2\,du$

$= \dfrac{1}{2}\left[\dfrac{u^3}{3}\right]_0^3$

$= \dfrac{1}{2}\left(\dfrac{3^3}{3} - \dfrac{0^3}{3}\right)$

$= \dfrac{9}{2}$

Question 12

a) $f(x) = e^{x+1}$

f^{-1}: $x = e^{f^{-1}(x)+1}$

$\ln x = f^{-1}(x) + 1$

$f^{-1}(x) = \ln x - 1$

$f[f^{-1}(x)] = e^{(\ln x - 1)+1}$
$= e^{\ln x}$
$= x$

$f^{-1}[f(x)] = \ln(e^{x+1}) - 1$
$= x + 1 - 1$
$= x$

$\therefore f[f^{-1}(x)] = f^{-1}[f(x)] = x$

b) $x^3 + 2x^2 + 3x + 5 = 0$

i) $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} = \frac{\alpha\beta + \beta\gamma + \gamma\alpha}{\alpha\beta\gamma}$

$= \frac{3}{-5}$

$= -\frac{3}{5}$

ii) $\alpha^2 + \beta^2 + \gamma^2 = (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \gamma\alpha)$

$= (-2)^2 - 2(3)$

$= -2$

2½

c)

$$P(2)=4 \quad , P(3)=7$$

$$P(x)=(x-2).Q(x)+4$$

when $x=3$

$$P(3)=(3-2)Q(3)+4.$$

$$7 = Q(3)+4$$

$$Q(3)=3$$

$$\therefore Q(x)=(x-3)M(x)+3$$

$$\text{So, } P(x)=(x-2)\big((x-3)M(x)+3\big)+4$$

$$=(x-2)(x-3)M(x) + 3(x-2)+4$$

$$+3x-2$$

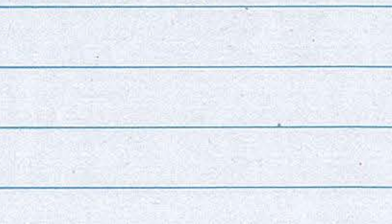
$\therefore$ remainder is $3x-2$.

d) $\dfrac{2x}{x+1} \leq x \quad , x \neq -1$

$$2x(x+1) \leq x(x+1)^2$$

$$x(x+1)\left[x+1-2\right] \geq 0$$

$$x(x-1)(x+1) \geq 0$$



$\therefore -1 < x \leq 0 \quad , x \geq 1$ is the solution

e) $\sum_{r=1}^{n} r^3 = \frac{n^2}{4}(n+1)^2$, $n \geq 1$

Prove true for $n=1$

$LHS = \sum_{r=1}^{1} r^3$ $\quad RHS = \frac{1^2}{4}(1+1)^2$

$= 1^3$

$= 1 \quad\quad = \frac{4}{4}$

$= 1 = LHS$

$\therefore$ true for $n=1$.

~~Prove~~ Assume true for $n=k$.

$$1^3 + 2^3 + 3^3 + \ldots + k^3 = \frac{k^2}{4}(k+1)^2$$

Prove true for $n=k+1$

i.e. $1^3 + 2^3 + 3^3 + \ldots + k^3 + (k+1)^3 = \frac{(k+1)^2}{4}((k+1)+1)^2$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$LHS = \frac{k^2}{4}(k+1)^2 + (k+1)^3$$

$$= \frac{(k+1)^2}{4}\left[k^2 + 4(k+1)\right]$$

$$= \frac{(k+1)^2}{4}(k^2+4k+4)$$

$$= \frac{(k+1)^2(k+2)^2}{4}$$

$$= RHS$$

$\therefore$ by induction, statement is true for all $n \geq 1$

Question 13

a) $(x^2+4)^{12}$

general term. ${}^{12}C_k (x^2)^{12-k} 4^k$

for x^6, $(x^2)^{12-k} = x^6$

$24-2k=6$

$2k=18$

$k=9$

coefficient.

$= {}^{12}C_9 \, 4^9$

$= 57\ 671\ 680$

b) $\ddot{x} = 4x(x^2-2)$ at $t=0$, $x=0$, $\dot{x}=\sqrt{6}$

i) $\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 4x(x^2-2)$

$\frac{1}{2}v^2 = \int 4x(x^2-2)\,dx$

$v^2 = 8\int x(x^2-2)\,dx$

let $u=x^2-2$

$du = 2x\,dx$

$v^2 = 4\int u\,du$

$= 4\frac{u^2}{2} + c$

$= 2u^2 + c$

$= 2(x^2-2)^2 + c$

at $x=0$, $\dot{x}=\sqrt{6}$

$$(\sqrt{6})^2 = 2(0^2-2)^2 + c.$$

$$6 = 8 + c$$

$$c = -2$$

$$\therefore v^2 = 2(x^2-2)^2 - 2$$

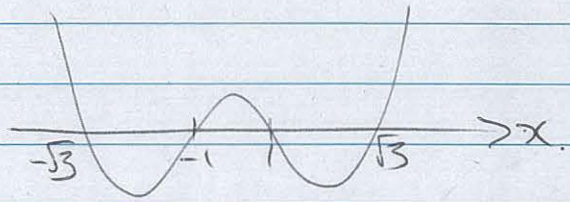
$$= 2(x^2-3)(x^2-1).$$

ii) as initial velocity is positive,

$$v^2 \geq 0$$

so, $(x^2-3)(x^2-1) \geq 0$

$$\therefore (x-\sqrt{3})(x+\sqrt{3})(x-1)(x+1) \geq 0$$



hence, $-1 \leq x \leq 1$

c) i) $\angle XOY = 2\theta$ (angle at centre twice angle at circumference)

ii) angle in alternate segment equal to angle between tangent and chord

iii) $\angle YAX + \angle AYX = \angle BXY$ (exterior angle of triangle)

$$\theta + \angle AYX = 2\theta$$

$$\angle AYX = \theta$$

$$= \angle YAX$$

$\therefore \triangle XYA$ is isosceles (two equal angles)

hence, $AX = YX$ (equal sides of isosceles triangle)

d) i) $(1+x)^{m+n} = (1+x)^m (1+x)^n$

coefficient of x^r in $(1+x)^{m+n}$ is ${}^{m+n}C_r$.

x^r terms from $(1+x)^m(1+x)^n$

$x^r.1 + x^{r-1}.x + x^{r-2}.x^2 + \ldots + x^2.x^{r-2} + x.x^{r-1} + 1x^r$

coefficients ${}^mC_r.{}^nC_0 + {}^mC_{r-1}.{}^nC_1 + {}^mC_{r-2}.{}^nC_2 + \ldots + {}^mC_2\,{}^nC_{r-2} + {}^mC_1.{}^nC_{r-1} + {}^mC_0.{}^nC_r$

$$= \sum_{k=0}^{r} {}^mC_k\,{}^nC_{r-k}$$

Hence $(1+x)^{m+n} = \sum_{k=0}^{r} {}^mC_k\,{}^nC_{r-k}$

ii) from (i), if $m \geq 2$, $n \geq 2$ $(r=2)$

$${}^{(m+1)+n}C_2 = {}^{m+1}C_0\,{}^nC_2 + {}^{m+1}C_1\,{}^nC_1 + {}^{m+1}C_2\,{}^nC_0$$

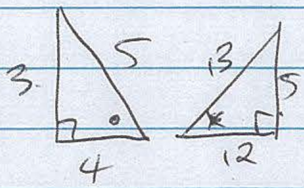
also $${}^{m+(n+1)}C_2 = {}^mC_0\,{}^{n+1}C_2 + {}^mC_1\,{}^{n+1}C_1 + {}^mC_2\,{}^{n+1}C_0$$

hence, ${}^{m+1}C_0\,{}^nC_2 + {}^{m+1}C_1\,{}^nC_1 + {}^{m+1}C_2\,{}^nC_0 = {}^mC_0\,{}^{n+1}C_2 + {}^mC_1\,{}^{n+1}C_1 + {}^mC_2\,{}^{n+1}C_0$

Question 14

a) $\sin\left[\cos^{-1}\left(\frac{4}{5}\right) - \tan^{-1}\left(\frac{5}{12}\right)\right]$

$= \sin\left(\cos^{-1}\left(\frac{4}{5}\right)\right) \cdot \cos\left(\tan^{-1}\left(\frac{5}{12}\right)\right) - \sin\left(\tan^{-1}\left(\frac{5}{12}\right)\right) \cos\left(\cos^{-1}\left(\frac{4}{5}\right)\right)$



$= \frac{3}{5} \cdot \frac{12}{13} - \frac{5}{13} \cdot \frac{4}{5}$

$= \frac{36 - 20}{65}$

$= \frac{16}{65}$

b. $N = 5000 - 4250e^{-kt}$

i) at $t = 0$

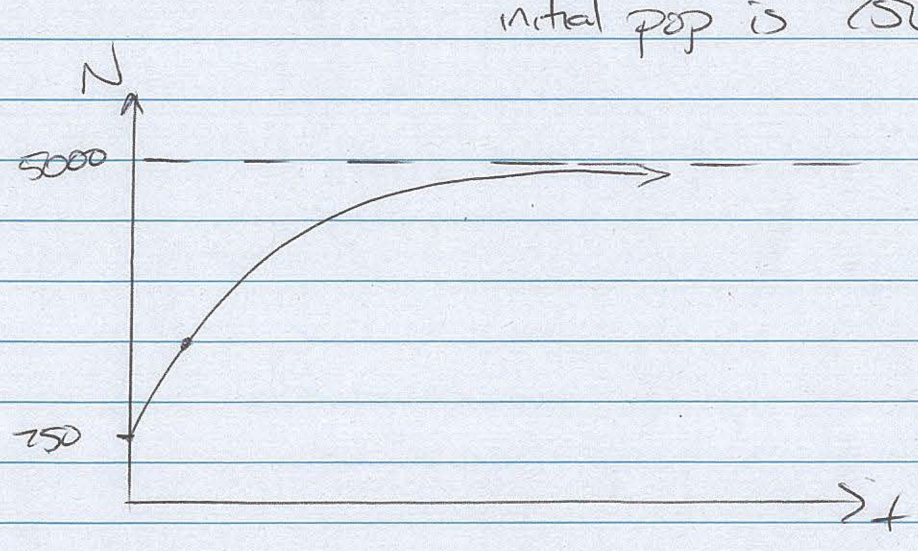
$N = 5000 - 4250e^{-k(0)}$

$= 5000 - 4250$

$= 750$

initial pop is 750

ii)



iii).
$$\frac{dN}{dt} = 4250\, ke^{-kt}$$
$$= k(5000 - N)$$
$$250 = k(5000 - 3\times 750)$$
$$k = \frac{1}{11}$$

c) i
$$\ddot{x} = 0 \quad \text{- horizontally.}$$
$$\dot{x} = 120$$

$$x = \int 120\,dt$$
$$= 120t + c$$
at $t=0, x=0$

so $c = 0$

$\therefore x = 120t.$

$$\ddot{y} = -10 \quad \text{-vertically}$$
$$\dot{y} = \int -10\,dt$$
$$= -10t + c$$
at $t=0, \dot{y}=0$

so $c = 0$

$\therefore \dot{y} = -10t$

$$y = \int -10t\,dt$$
$$= -5t^2 + c$$
at $t=0, y=200$

so, $c = 200$

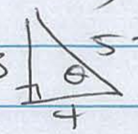
and $y = -5t^2 + 200$

ii)

$x = Vt\cos\theta$

$y = -5t^2 + Vt\sin\theta.$

$\theta = \tan^{-1}\left(\frac{3}{4}\right)$



for $V = 150$

$x = 150t\cos\theta$

$= 150t\left(\frac{4}{5}\right)$

$= 120t.$

~~range of balloon~~ time of balloon, $y = 0 \Rightarrow -5t^2 + 200 = 0$

$t^2 = 40$

$t = 2\sqrt{10}$

time of flight of stone, $y = 0 \Rightarrow -5t^2 + 150t\left(\frac{3}{5}\right) = 0$

$5t(-t + 18\ \ ) = 0$

$t = 0, 18$

as balloon and stone have same horizontal motion equate, and stone travels for longer, stone will intersect water balloon

iii) collision at

equate vertical

$$-\cancel{5t^2} + 200 = -\cancel{5t^2} + 90t$$

$$t = \frac{20}{9}.$$

$$y = -5\left(\frac{20}{9}\right)^2 + 200$$

$$= \frac{14200}{81} \doteq 175 \quad \text{m above ground (0 dp)}$$