Student Name: ___________________

HUNTERS HILL HIGH SCHOOL
EXTENSION 2
MATHEMATICS
HSC TRIAL 2017

Hunters Hill
High School

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A Reference Sheet is provided
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks - 100

Section I

10 marks

- Attempt Questions 1-10
- Allow about 15 minutes for this section

Section II

90 marks

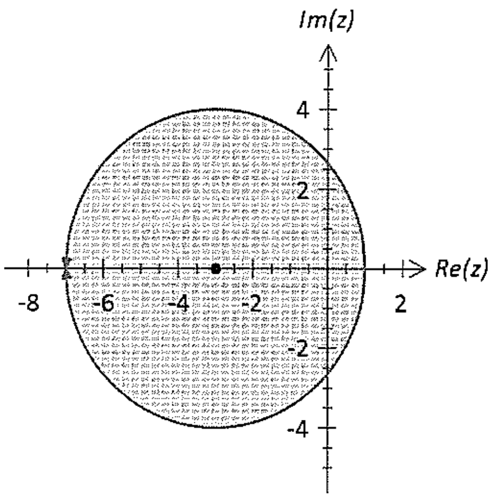
- Attempt Questions 11-16
- Allow about 2 hour 45 minutes for this section

Section I

10 marks
Attempt Questions 1-10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1-10.

1



Which of the following inequalities describes the region above?

(A) $|z-3| \le 2$

(B) $|z+3| \le 2$

(C) $|z-3| \le 4$

(D) $|z+3| \le 4$

2 What is the value of i^{2017} ?

(A) i

(B) 1

(C) $-i$

(D) -1

3 Multiplying a non-zero complex number by $\dfrac{1+i}{1-i}$ results in a rotation about the origin on an Argand diagram.
What is the rotation?

(A) Clockwise by $\dfrac{\pi}{4}$

(B) Clockwise by $\dfrac{\pi}{2}$

(C) Anticlockwise by $\dfrac{\pi}{4}$

(D) Anticlockwise by $\dfrac{\pi}{2}$

4 What is the eccentricity of $\dfrac{x^2}{9}+\dfrac{y^2}{16}=1$?

(A) $\dfrac{7}{16}$

(B) $\dfrac{\sqrt{7}}{4}$

(C) $\dfrac{9}{16}$

(D) $\dfrac{7}{9}$

5 The gradient of the curve $x^2y-xy^2+6=0$ at the point $P(2,3)$ is equal to:

(A) -5 (B) $\dfrac{3}{8}$

(C) $\dfrac{9}{8}$ (D) 1

6 The directrices of the hyperbola $\frac{y^2}{9} - \frac{x^2}{16} = 1$ are

(A) $x = \pm\frac{9}{5}$ (B) $y = \pm\frac{9}{5}$

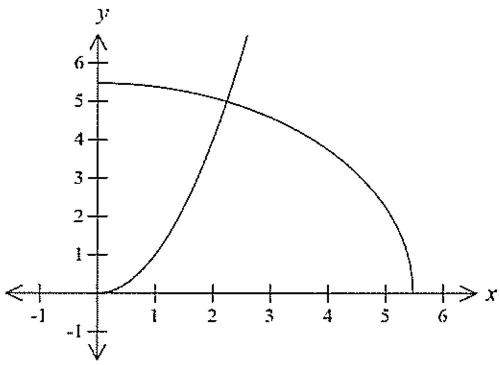
(C) $y = \pm 5$ (D) $x = \pm 5$

7 Which of the following is the correct expression for $\int \frac{1}{\sqrt{8 + 2x - x^2}}\,dx$?

(A) $\sin^{-1}\frac{x-1}{9} + C$ (B) $\sin^{-1}\frac{x-1}{3} + C$

(C) $\sin^{-1}\frac{x+1}{3} + C$ (D) $\sin^{-1}\frac{x+1}{9} + C$

8 What is the volume of the solid formed when the region bounded by the curves $y = x^2$, $y = \sqrt{30 - x^2}$ and the y-axis is rotated about the y-axis? Use the method of slicing.



What is the correct expression for volume of this solid using the method of cylindrical shells?

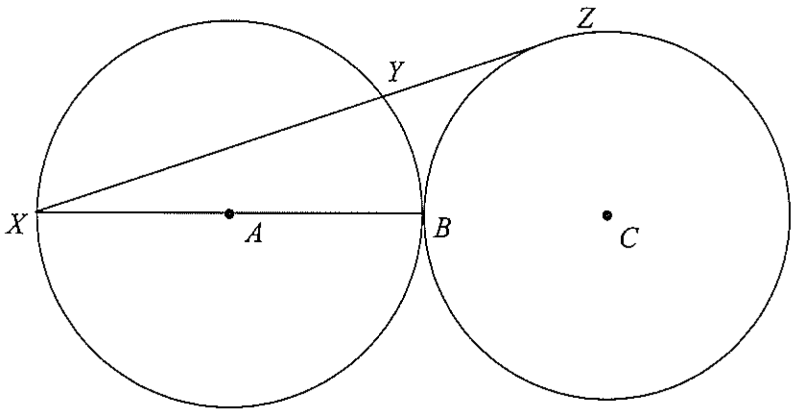
(A) $V = \int_0^{\sqrt{5}} 2\pi\left(x^2 - \sqrt{30 - x^2}\right)dx$

(B) $V = \int_0^{\sqrt{5}} 2\pi x\left(x^2 - \sqrt{30 - x^2}\right)dx$

(C) $V = \int_0^{\sqrt{5}} 2\pi\left(\sqrt{30 - x^2} - x^2\right)dx$

(D) $V = \int_0^{\sqrt{5}} 2\pi x\left(\sqrt{30 - x^2} - x^2\right)dx$

9 Two equal circles touch externally at B. XB is a diameter of one circle. XZ is the tangent from X to the other circle and cuts the first circle at Y.



Which is the correct expression that relates XZ to XY?

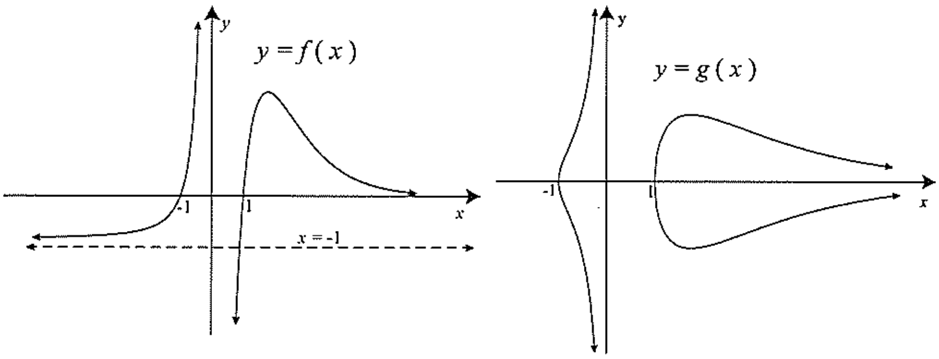
(A) $3XZ = 4XY$

(B) $XZ = 2XY$

(C) $2XZ = 3XY$

(D) $2XZ = 5XY$

10 What statement is true for these graphs?



(A) $|f(x)| = g(x)$

(B) $g(x) = \ln[f(x)]$

(C) $f(x) = \pm\ln|g(x)|$

(D) $g(x) = \pm\sqrt{f(x)}$

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find

(i) $\displaystyle\int \frac{\cos\theta}{\sin^5\theta}\,d\theta$ **2**

(ii) $\displaystyle\int \frac{dx}{x^2+2x+2}$ **2**

(b) Evaluate $\displaystyle\int_0^1 \frac{2x+1}{x^2+1}\,dx$ **3**

(c) Evaluate the following definite integrals:

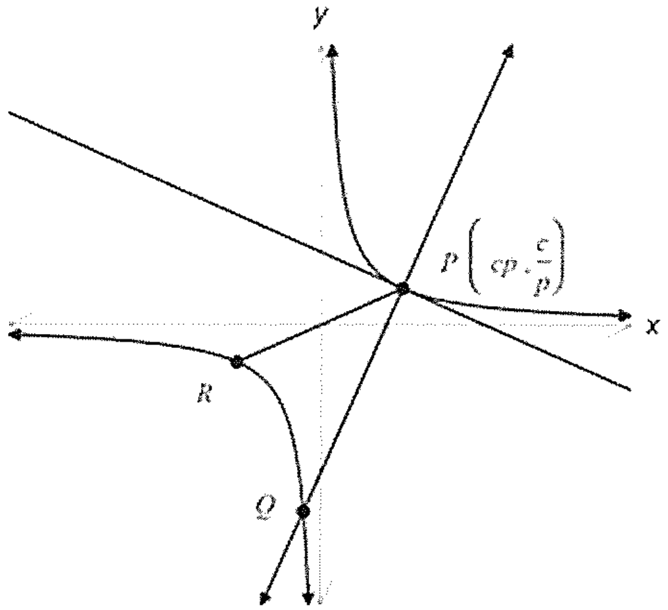
(i) $\displaystyle\int_0^1 \cos^{-1} x\,dx$ **2**

(ii) $\displaystyle\int_1^2 x(\ln x)^2\,dx$ **3**

(d) Use the substitution $x = \dfrac{1}{u}$ to evaluate $\displaystyle\int_{\frac{1}{e}}^{e} \frac{\log_e x}{(1+x)^2}\,dx$ **3**

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) PQ is a variable chord of the rectangular hyperbola $xy = c^2$.



(i) Show that the equation of the normal at the point $P\,(cp\,, \frac{c}{p})$ on the rectangular hyperbola $xy = c^2$ is given by $p^3x - py = c(p^4 - 1)$ **2**

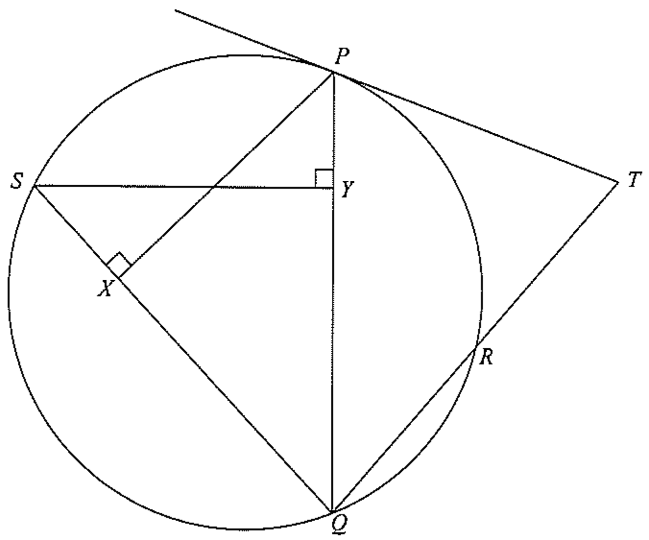
(ii) Prove that the normal cuts the hyperbola again at the point $Q\left(\frac{-c}{p^3}, -cp^3\right)$ **3**

(iii) If R is the opposite end of the diameter of the hyperbola through P, show that PR is perpendicular to RQ. **2**

Question 12 continued on next page.

Question 12 (continued)

(b) In the diagram below, TP is the tangent of the circle at P, and TQ is a secant cutting the circle at R. SQ is a chord of the circle such that PX and SY are perpendicular to SQ and PQ, respectively.



(i) Prove that $\angle TRP = \angle TPQ$ **3**

(ii) Explain why $SPYX$ is a cyclic quadrilateral and state the diameter of the circle. **1**

(iii) Prove $\angle PYX = \angle PRQ$ **2**

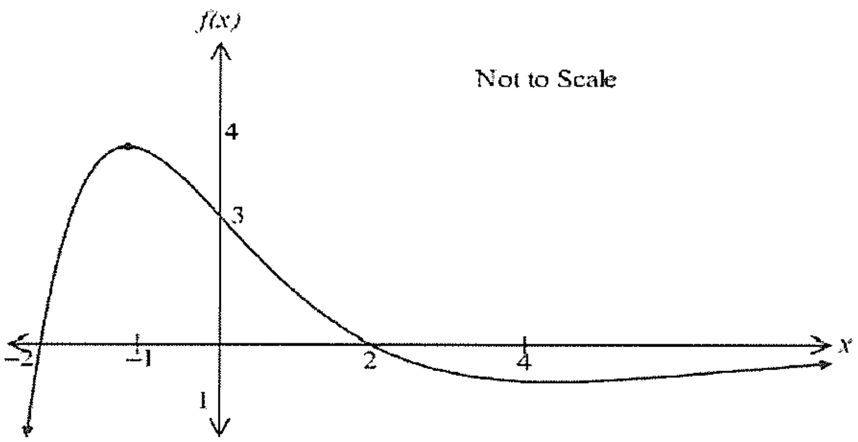
(c) If P and Q represent the complex numbers z and w, where **2**

$$w = \frac{1}{z-2} + \frac{3}{2},$$

find the Cartesian equation of the locus of Q as P moves on the circle $|z-2| = 3$.

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a)



The diagram shows the graph of the function, which has a horizontal asymptote at $y = 0$. On separate diagrams sketch the following graphs, showing clearly any intercepts on the coordinate axes and the equations of any asymptotes:

(i) $y = f\left(x^2\right)$ **2**

(ii) $y = \dfrac{1}{f(x)}$ **2**

(iii) $y = \ln\left(f(x)\right)$ **2**

(b) Sketch $|y| = x^2 - 4x$, showing all important features. **4**

(c) If $z_1 = 1 - i$, $z_2 = 2z_1$ and $z_3 = -2iz_1$ clearly on an Argand diagram the points represented by

(iii) z_1, z_2 *and* z_3 **3**

(iv) $z_3 - z_2$ **2**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $x^4 + 2x^3 - 7x^2 - 20x - 12 = 0$ has a double root. Find this root and hence solve this equation. **3**

(b) The equation $x^3 + px + 5 = 0$ has roots α, β and γ.

(i) Find in terms of p, $\alpha^2 + \beta^2 + \gamma^2$ **2**

(ii) Show that $\alpha^2\beta^2 + \beta^2\gamma^2 + \gamma^2\alpha^2 = p^2$ **2**

(iii) Find the cubic polynomial with integer coefficients, whose roots are **2**

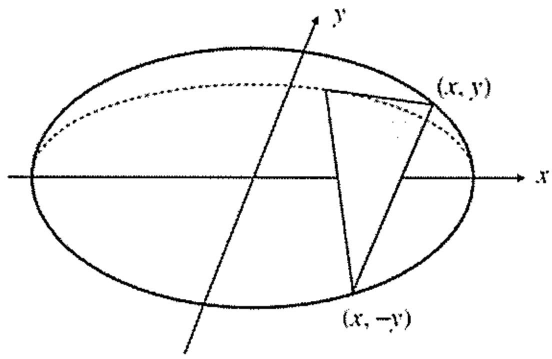
$$\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}.$$

(c) Solve the equation $x^4 + 2x^3 + x^2 - 1 = 0$, given that one root is **3**

$$-\frac{1}{2} + i\frac{\sqrt{3}}{2}$$

(d) The base of a solid is in the shape of an ellipse with equation $\frac{x^2}{9} + \frac{y^2}{4} = 1$.

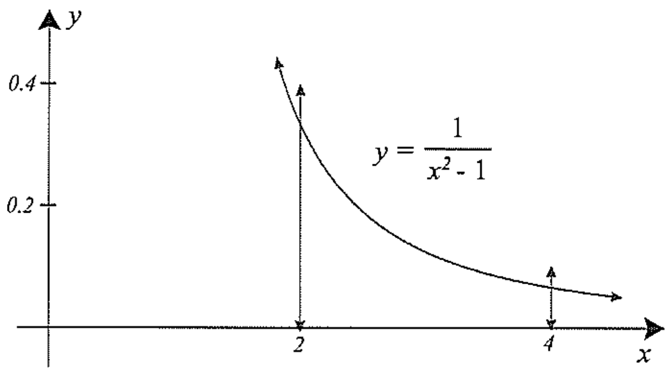
Sections parallel to the y-axis are equilateral triangles, with one side sitting in the base of the solid, as shown in the diagram.



Find the volume of this solid. **3**

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The region bounded by the curve $y = \dfrac{1}{x^2 - 1}$ and the x – axis between $x = 2$ and $x = 4$ is rotated through one revolution about the line $x = 2$.



(i) Use the method of cylindrical shells to show that the volume, V, of the solid formed is given by $V = 2\pi \int_2^4 \dfrac{x-2}{x^2-1}\, dx$. **2**

(ii) Hence find the exact value of V in simplest form. **3**

(b) If $a > 0$, $b > 0$, $c > 0$ and $a + b + c = 1$,(use in part (ii) only)

(i) show that $(a+b)(b+c)(c+d) \geq 8abc$ **3**

(ii) hence, or otherwise, prove that **2**

$(1-a)(1-b)(1-c) \geq 8abc$

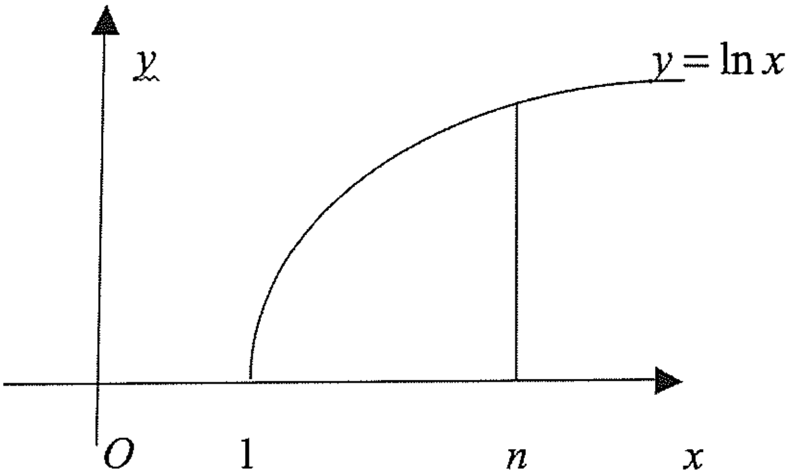
(c) (i) Prove that $\int_0^a F(x)\, dx = \int_0^a F(a-x)\, dx$. **2**

(ii) Hence, or otherwise, evaluate $\int_0^{\frac{\pi}{4}} \log_e \left(1 + \tan\theta\right)\, d\theta$ **3**

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Twelve different books are made into four parcels of three each. How many different sets of parcels could be made? **3**

(b)



(i) Use the trapezoidal rule with n function values to approximate $\int_1^n \ln x \, dx$. **2**

(ii) Show that $\frac{d}{dx}(x\ln x - x) = \ln x$ and hence find the exact value of $\int_1^n \ln x \, dx$. **2**

(i) Deduce that $\ln n! < \left(n + \frac{1}{2}\right)\ln n - n + 1$. **2**

(b) $I_n = \int_1^e (1 - \ln x)^n \, dx$, $n = 1, 2, 3, \ldots$

(i) Show $I_n = -1 + nI_{n-1}$, $n = 1, 2, 3, \ldots$ **2**

(ii) Hence evaluate $\int_1^e (1 - \ln x)^4 \, dx$. **2**

(iii) Show that $\frac{I_n}{n!} = e - \sum_{r=0}^{n} \frac{1}{r!}$, $n = 1, 2, 3, \ldots$ **2**

END OF EXAMINATION

MXX TRIAL 2018

SECTION I

1. D

2. A $\qquad 2017 = 4\times 504 + 1$

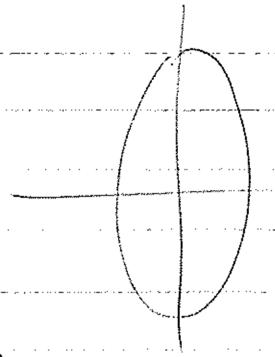
3. D $\qquad \dfrac{1+i}{1-i}\times\dfrac{1+i}{1+i} = \dfrac{2i}{1+1}$

$= i$

4. B $\qquad e^2 = 1-\dfrac{b^2}{a^2}$

$= 1-\dfrac{9}{16}$

$= \dfrac{7}{16}$



5. B $\qquad 2xy + x^2\dfrac{dy}{dx} - y^2 - 2xy\dfrac{dy}{dx} = 0$

$\dfrac{dy}{dx}(x^2-2xy) = y^2-2xy$

$\dfrac{dy}{dx} = \dfrac{3^2-2(2)(3)}{2^2-2(2)(3)}$

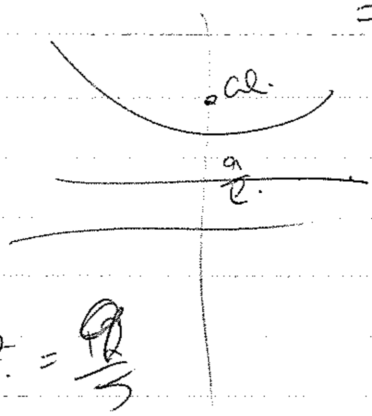
$= \dfrac{9-12}{4-12} = \dfrac{-3}{-8}$

6. B $\qquad e^2 = 1+\dfrac{16}{9}$

$= \dfrac{25}{9}$

$e = \dfrac{5}{3}$

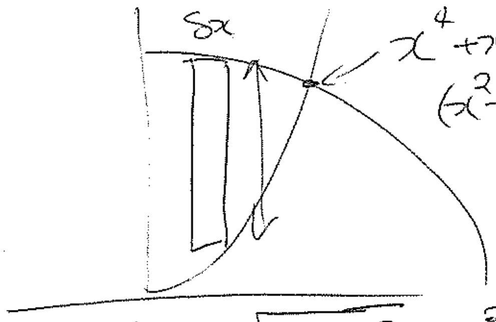
$\dfrac{3\times 4}{5} = \dfrac{12}{5}$



7. B.

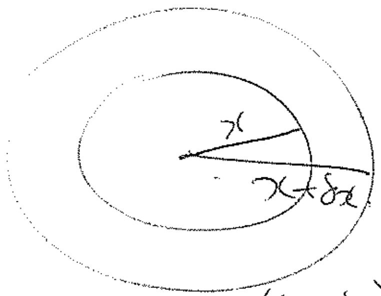
$$\frac{1}{\sqrt{8+2x-x^2}} = \frac{1}{\sqrt{9-1+2x-x^2}}$$
$$= \frac{1}{\sqrt{9-(x-1)^2}}.$$

8. D



$x^4+x^2-30=0$
$(x^2+6)(x^2-5)=0$
$x=\pm\sqrt{5}$

$h=\sqrt{30-x^2}-x^2$



$\pi((x+\delta x)^2-(x)^2)$
$=\pi(2x\delta x)$

9. C

$$\frac{x4}{x7}=\frac{2}{3}$$

10. D.

SECTION II

Question 11

a) i $I = \int \frac{\cos\theta}{\sin^5\theta} \, d\theta$

let $u = \sin\theta$.

$du = \cos\theta \, d\theta$.

$I = \int u^{-5} \, du$

$= \frac{u^{-4}}{-4} + c$

$= -\frac{1}{4\sin^4\theta} + c$

ii. $I = \int \frac{dx}{x^2+2x+2}$

$= \int \frac{dx}{(x+1)^2+1}$

$= \tan^{-1}(x+1) + c$

b) $\int_0^1 \frac{2x+1}{x^2+1} \, dx = \int_0^1 \left(\frac{2x}{x^2+1} + \frac{1}{x^2+1} \right) dx$

$= \left[\ln(x^2+1) + \tan^{-1} x \right]_0^1$

$= \ln(1^2+1) + \tan^{-1} 1 - \ln(0^2+1) - \tan^{-1} 0$

$= \ln 2 + \frac{\pi}{4}$

c) i) $I=\int_0^1 \cos^{-1}x\,dx$

let $u=\cos^{-1}x$ $\quad du=\frac{-1}{\sqrt{1-x^2}}\,dx$

$dv=dx$ $\quad v=x.$

$=\left[x\cos^{-1}x\right]_0^1+\int_0^1\frac{x}{\sqrt{1-x^2}}\,dx$

let $u=1-x^2$

$du=-2x\,dx.$

$I=\left(1\cos^{-1}1-0\cos^{-1}0\right)-\frac{1}{2}\int_1^0\frac{du}{\sqrt{u}}$

$=\frac{1}{2}\left[\frac{2}{1}u^{\frac{1}{2}}\right]_0^1$

$=1.$

ii) $\int_1^2 x(\ln x)^2\,dx$ let ~~$x=e^u$~~ $u=\ln x.$

$du=\frac{dx}{x}.$

~~$e^u du=dx$~~

~~$I=\int_0^{\ln 2} e^u u^2 e^u\,du.$~~

let $u=(\ln x)^2$ $\quad du=\frac{2\ln x}{x}\,dx$

$dv=x\,dx$ $\quad v=\frac{x^2}{2}$

$I=\left[\frac{x^2}{2}(\ln x)^2\right]_1^2-\int_1^2\frac{\cancel{2}\ln x}{\cancel{x}}\frac{x^{\cancel{2}}}{\cancel{2}}\,dx$

$=\left[\frac{2^2}{2}(\ln 2)^2-\frac{1^2}{2}(\ln 1)^2\right]-\int_1^2 x\ln x\,dx.$ $\quad u=\ln x$ $\quad du=\frac{dx}{x}$

$dv=x\,dx$ $\quad v=\frac{x^2}{2}$

$=2(\ln 2)^2-\left[\frac{x^2}{2}\ln x\right]_1^2+\int_1^2\frac{x^{\cancel{2}}}{2}\frac{dx}{\cancel{x}}$

$=2(\ln 2)^2-\frac{2^2}{2}\ln 2+\frac{1^2}{2}\ln 1+\left[\frac{x^2}{4}\right]_1^2=2(\ln 2)^2-2\ln 2+\frac{3}{4}$

d) $\int_{\frac{1}{e}}^{e} \frac{\log_e x}{(1+x)^2}\,dx$

let $x = \frac{1}{u}$ $\qquad$ $\log_e x = \log_e \frac{1}{u}$
$= -\log_e u.$

$x = e, \; u = \frac{1}{e}.$
$x = \frac{1}{e}, \; u = e.$

$$I = \int_{e}^{\frac{1}{e}} \frac{-\log_e u}{\left(1+\frac{1}{u}\right)^2} \left(\frac{-1}{u^2}\right) du$$

$$= +\int_{e}^{\frac{1}{e}} \frac{\log_e u}{\cancel{u^2 \frac{1}{u^2}}(u+1)^2}\,du$$

$$= -\int_{\frac{1}{e}}^{e} \frac{\log_e u}{(1+u)^2}\,du. \quad = -I$$

$\therefore \; 2I = 0$
$I = 0$

Question 12.

a) i)

$$xy = c^2$$

$$y = \frac{c^2}{x}$$

$$\frac{dy}{dx} = -\frac{c^2}{x^2}$$

at P

$$m_T = \frac{-c^2}{(cp)^2} \quad \Rightarrow \quad m_N = p^2.$$

$$= -\frac{1}{p^2}$$

by point-gradient

$$y - \frac{c}{p} = p^2(x - cp)$$

$$py - c = p^3x - cp^4.$$

$$\therefore p^3x - py = cp^4 - c$$

so $p^3x - py = c(p^4 - 1)$ is the equation of the normal.

ii) $Q\left(\frac{-c}{p^3}, -cp^3\right)$

Test in normal

$$LHS = p^3\left(\frac{-c}{p^3}\right) - p(-cp^3)$$

$$= -c + cp^4$$

$$= c(p^4 - 1)$$

$$= RHS$$

Test in hyperbola

$$LHS = \left(-\frac{c}{p^3}\right)(-cp^3)$$

$$= c^2$$

$$= RHS$$

as Q satisfies both, it is intersection of normal and hyperbola.

iii) as P is $\left(cp, \frac{c}{p}\right)$,

R is $\left(-cp, -\frac{c}{p}\right)$

$$M_{PR} \cdot M_{QR} = \frac{\frac{c}{p} - \left(-\frac{c}{p}\right)}{cp - (-cp)} \cdot \frac{-cp^3 - \left(-\frac{c}{p}\right)}{-\frac{c}{p^3} - (-cp)}$$

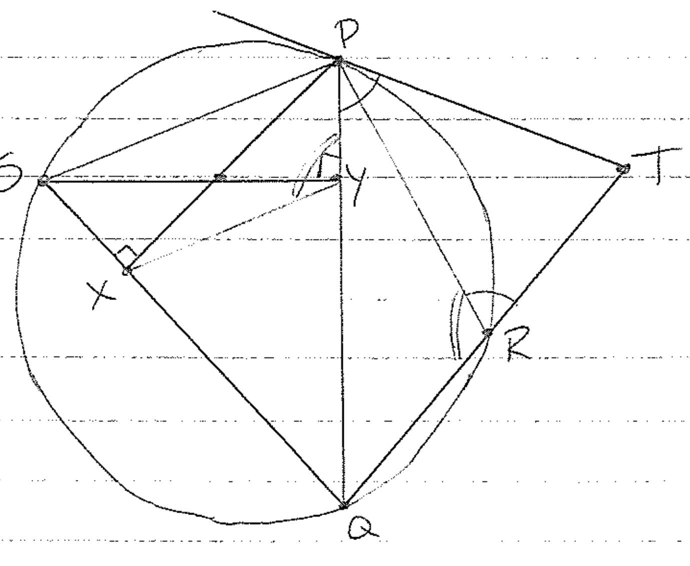
$$= \frac{\cancel{2}c}{p}\left(\frac{1}{\cancel{2}cp}\right) \cdot \frac{-cp^4 + c}{-c + cp^4} \cdot \frac{\frac{1}{p}}{\frac{1}{p^3}}$$

$$= \frac{1}{p^2} \cdot -p^2$$

$$= -1$$

$\therefore$ $PR \perp QR$.

b)



i) In $\triangle TPR$ and $\triangle TQP$

$\angle T = \angle T$ (common)

$\angle TPR = \angle TQP$ (angle in alternate segment)

$\therefore \triangle TPR \parallel\!\mid \triangle TQP$ (equiangular)

and $\angle TRP = \angle TPQ$ (corresponding angles of similar triangles)

ii) as $\angle SXP = \angle SYP = 90°$,
S, P, Y, X are concyclic,
Hence, SPYX is a cyclic quadrilateral
with diameter SP.

iii) $\angle PYX = 180° - \angle PSX$ (opposite angles of cyclic quadrilateral SPYX are supplementary)

Similarly in SPRQ.

$\angle PRQ = 180° - \angle PSX$

Hence $\angle PRQ = \angle PYX$.

c)

$$\omega = \frac{1}{z-2} + \frac{3}{2} \qquad \frac{1}{z-2} = \frac{2\omega - 3}{2}.$$

$$|z-2| = \left|\frac{2}{2\omega-3}\right|$$

$$= \frac{1}{z-2} \cdot \frac{\overline{(z-2)}}{\overline{(z-2)}} + \frac{3}{2}$$

$$= \frac{x-2-iy}{|z-2|} + \frac{3}{2}$$

$$3 = \left|\frac{2}{2x-3+2iy}\right|$$

$$\frac{3}{2} = \left|\frac{2x-3}{(2x-3)^2-(2y)^2} - \frac{2iy}{(2x-3)^2-(2y)^2}\right|$$

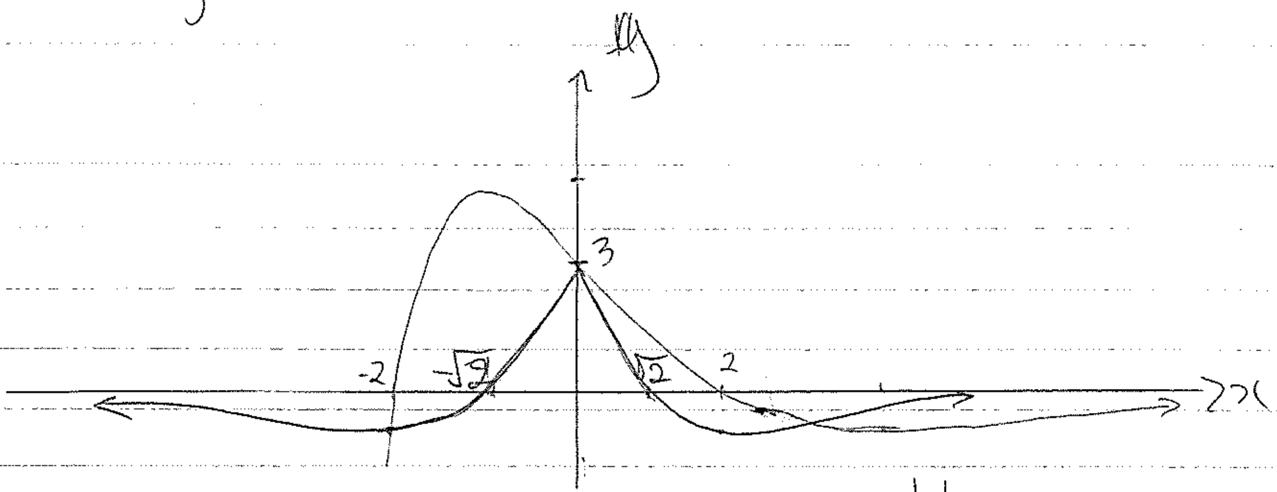
$$\frac{9}{4} = \frac{(2x-3)^2 + (2y^2)^2}{((2x-3)^2+(2y)^2)^2}$$

$$(2x-3)^2 + (2y)^2 = \frac{4}{9}$$

$$\left(x-\frac{3}{2}\right)^2 + y^2 = \frac{1}{9}$$

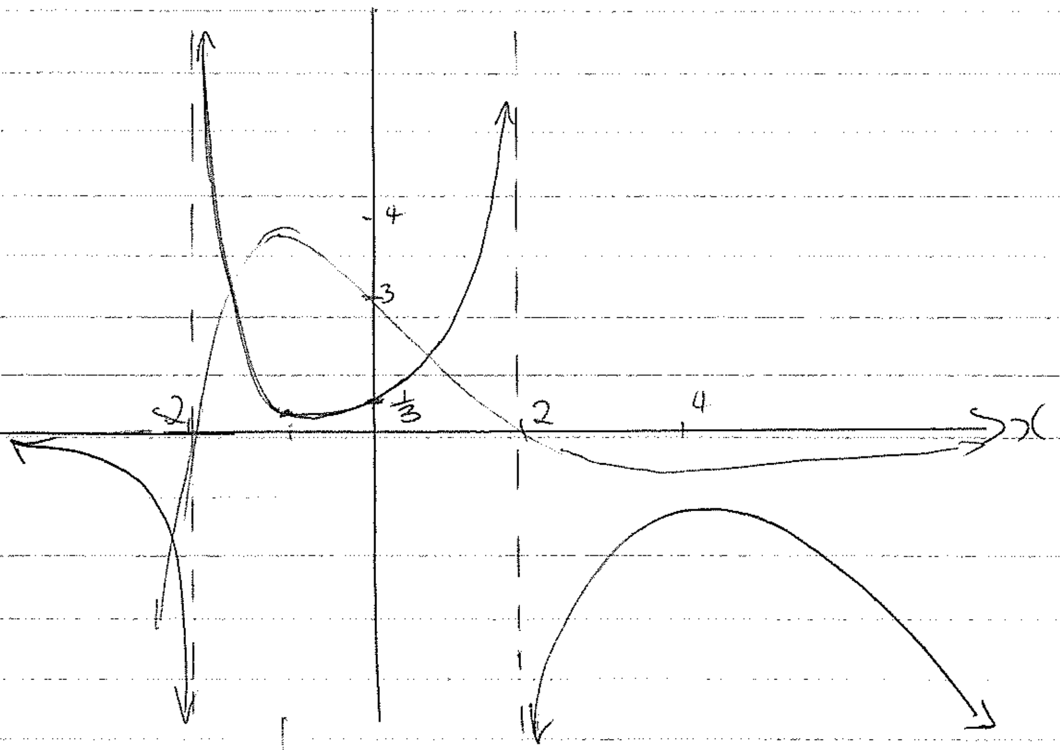
Question 13

a) $y = f(x^2)$

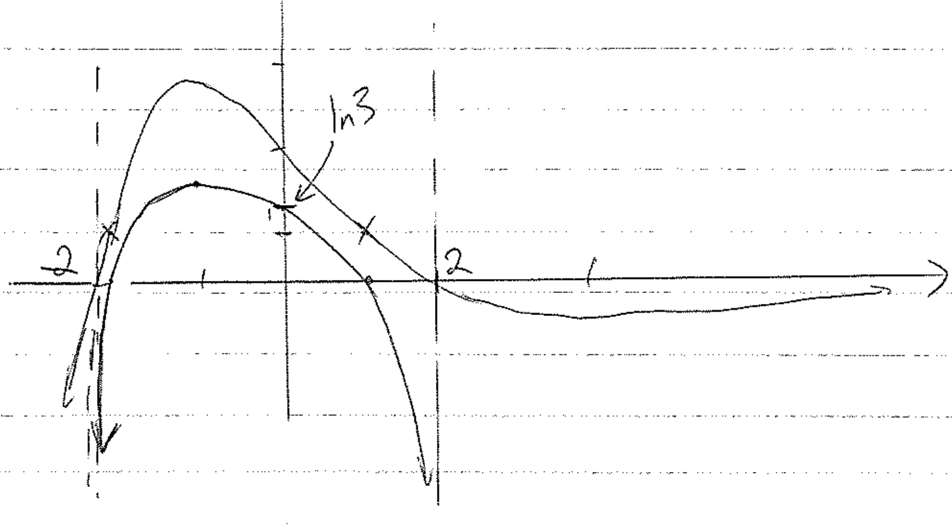


asymptote: $y = 0$

b)

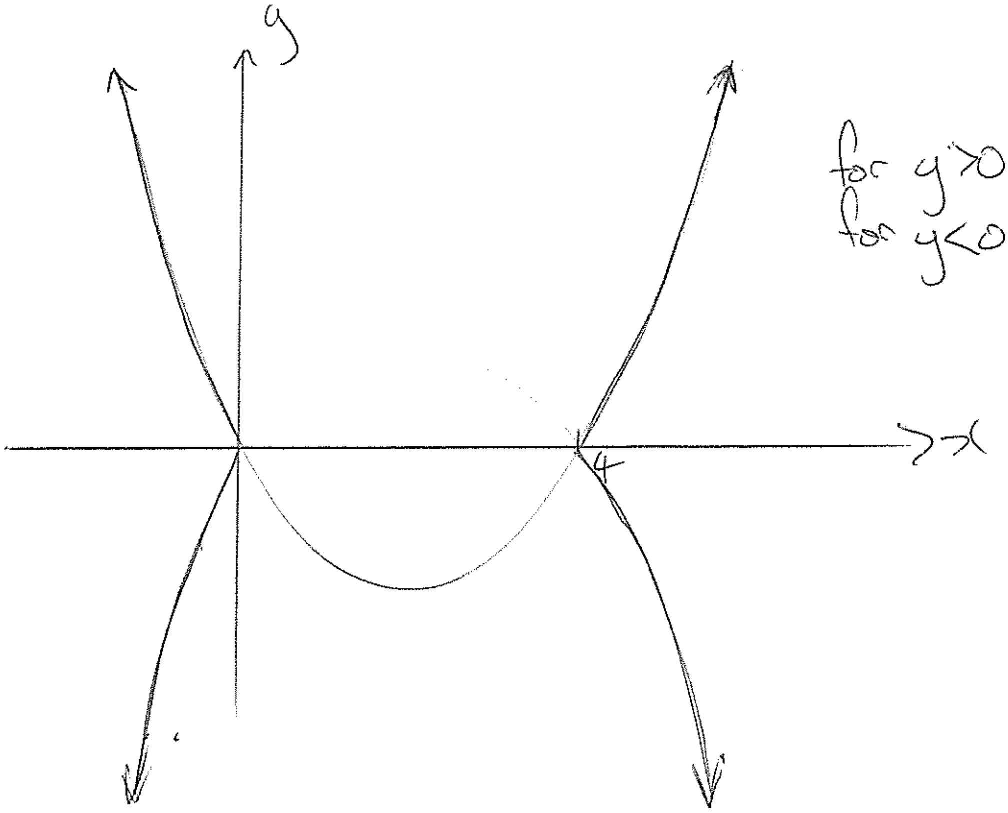


c



b). $y = x^2 - 4x$
$= x(x-4)$

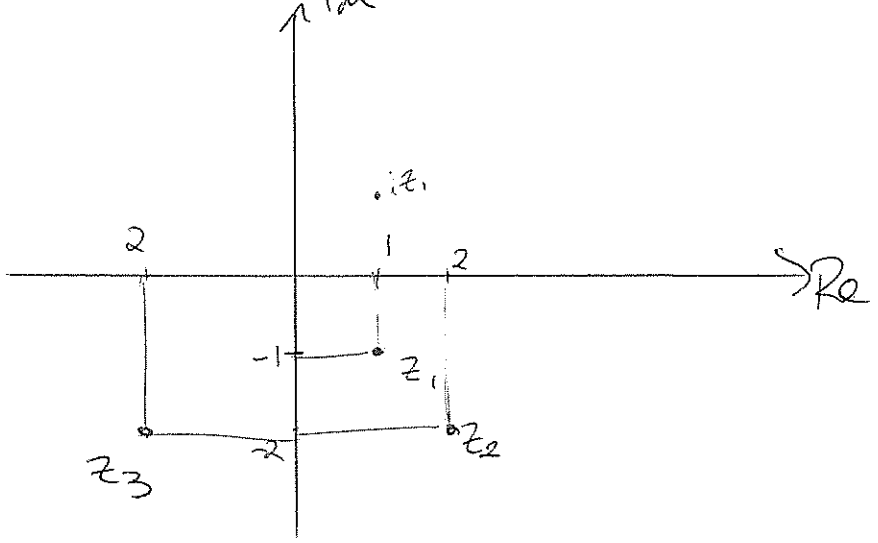
x-ints $x = 0, 4$.
y-int 0.



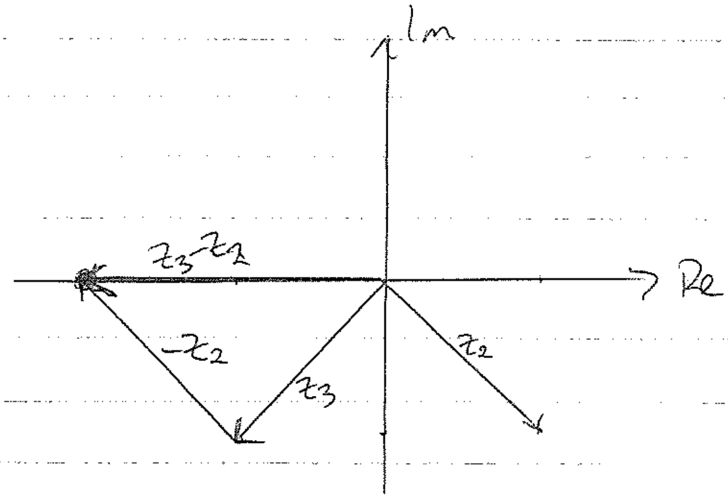
for $y > 0$, $|y| = x^2 - 4x$
for $y < 0$, $|y| = -(x^2 - 4x)$

c) $z_1 = 1 - i$, $z_2 = 2z_1$, $z_3 = -2iz_1$.

iii)



iv)



Question 14

a) (i) $P(x)=x^4+2x^3-7x^2-20x-12$

$P'(x)=4x^3+6x^2-14x-20$

test for roots in $P'(x)$

$P'(2)=4(2)^3+6(2)^2-14(2)-20$

$=8$

$P'(-2)=4(-2)^3+6(-2)^2-14(-2)-20$

$=0$

test $x=-2$ in $P(x)$

$P(-2)=(-2)^4+2(-2)^3-7(-2)^2-20(-2)-12$

$=0$

$\therefore$ $x=-2$ is the double root as it satisfies $P(x)=0$ and $P'(x)=0$

$$
\begin{array}{r}
x^2-2x-3 \\
x^2+4x+4 \,\big)\, x^4+2x^3-7x^2-20x-12 \\
x^4+4x^3+4x^2 \\
\hline
-2x^3-11x^2-20x \\
-2x^3-8x^2-8x \\
\hline
-3x^2-12x-12 \\
-3x^2-12x-12 \\
\hline
0
\end{array}
$$

$\therefore x^4+2x^3-7x^2-20x-12 = (x+2)^2(x^2-2x-3)$
$= (x+2)^2(x-3)(x+1)$

$\therefore (x+2)^2(x-3)(x+1)=0$
$x=-2,-1,3.$ is the solution.

b) $x^3+px+5=0$ roots are α,β,γ.

i) $\alpha^2+\beta^2+\gamma^2 = (\alpha+\beta+\gamma)^2-2(\alpha\beta+\beta\gamma+\gamma\alpha)$
$= \left(-\frac{0}{1}\right)^2-2\left(\frac{p}{1}\right)$
$= -2p$

ii) $\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2 = (\alpha\beta)^2+(\beta\gamma)^2+(\gamma\alpha)^2$
$= (\alpha\beta+\beta\gamma+\gamma\alpha)^2-2(\alpha\beta.\beta\gamma+\beta\gamma.\gamma\alpha+\gamma\alpha\alpha\beta)$
$= (\alpha\beta+\beta\gamma+\gamma\alpha)^2-2\alpha\beta\gamma(\alpha+\beta+\gamma)$
$= (p)^2-2\left(\frac{-5}{1}\right)(0)$
$= p^2.$

iii) roots $\frac{\alpha}{\beta\gamma}, \frac{\beta}{\gamma\alpha}, \frac{\gamma}{\alpha\beta}.$

sum of roots: $\frac{\alpha}{\beta\gamma}+\frac{\beta}{\gamma\alpha}+\frac{\gamma}{\alpha\beta} = \frac{\alpha^2+\beta^2+\gamma^2}{\alpha\beta\gamma}$
$= \frac{-2p}{-5}$
$= \frac{2p}{5}$

sum x2 of roots: $\frac{\alpha}{\beta\gamma}\cdot\frac{\beta}{\gamma\alpha}+\frac{\beta}{\gamma\alpha}\cdot\frac{\gamma}{\alpha\beta}+\frac{\gamma}{\alpha\beta}\cdot\frac{\alpha}{\beta\gamma}$

$$= \frac{1}{\gamma^2}+\frac{1}{\alpha^2}+\frac{1}{\beta^2}$$

$$= \frac{\alpha^2\beta^2+\beta^2\gamma^2+\gamma^2\alpha^2}{\alpha^2\beta^2\gamma^2}$$

$$= \frac{p^2}{(-5)^2}$$

$$= \frac{p^2}{25}$$

product of roots: $\frac{\alpha}{\beta\gamma}\cdot\frac{\beta}{\gamma\alpha}\cdot\frac{\gamma}{\alpha\beta} = \frac{1}{\alpha\beta\gamma}$

$$= \frac{1}{(-5)}$$

$$= -\frac{1}{5}.$$

polynomial is

$$x^3+\left(\frac{-2p}{5}\right)x^2+\frac{p^2}{25}x+\left(\frac{1}{5}\right)=0.$$

$$\therefore 25x^3-10px^2+p^2x+5=0$$

is the polynomial with roots $\frac{\alpha}{\beta\gamma},\frac{\beta}{\gamma\alpha},\frac{\gamma}{\alpha\beta}$.

c) $x^4+2x^3+x^2-1=0$

as $-\frac{1}{2}+i\frac{\sqrt{3}}{2}$ is a root, so is $-\frac{1}{2}-i\frac{\sqrt{3}}{2}$.

$$\left(x-\left(-\frac{1}{2}+i\frac{\sqrt{3}}{2}\right)\right)\left(x-\left(-\frac{1}{2}-i\frac{\sqrt{3}}{2}\right)\right)$$

$$=\left(\left(x+\frac{1}{2}\right)-\frac{i\sqrt{3}}{2}\right)\left(\left(x+\frac{1}{2}\right)+\frac{i\sqrt{3}}{2}\right)$$

$$=\left(x+\frac{1}{2}\right)^2+\frac{3}{4}$$

$$=x^2+x+\frac{1}{4}+\frac{3}{4}$$

$=x^2+x+1$ is a factor of polynomial

$$\begin{array}{r|l} & x^2+x-1 \\ \hline x^2+x+1 & x^4+2x^3+x^2-1 \\ & x^4+x^3+x^2 \\ \hline & x^3 \\ & x^3+x^2+x \\ \hline & -x^2-x-1 \\ & -x^2-x-1 \\ \hline & 0 \end{array}$$

$\therefore (x^2+x+1)(x^2+x-1)=0$

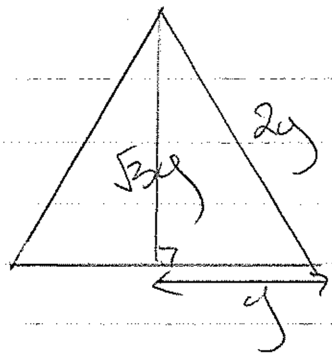
$$x^2+x-1=0$$

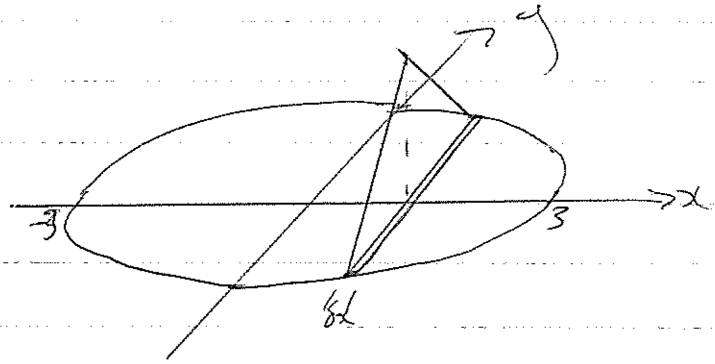
$$x=\frac{-1\pm\sqrt{1^2-4(1)(-1)}}{2(1)}$$

$$=\frac{-1}{2}\pm\frac{\sqrt{5}}{2}.$$

$\therefore x=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2},\ -\frac{1}{2}\pm\frac{\sqrt{5}}{2}$

d) $\frac{x^2}{9}+\frac{y^2}{4}=1$





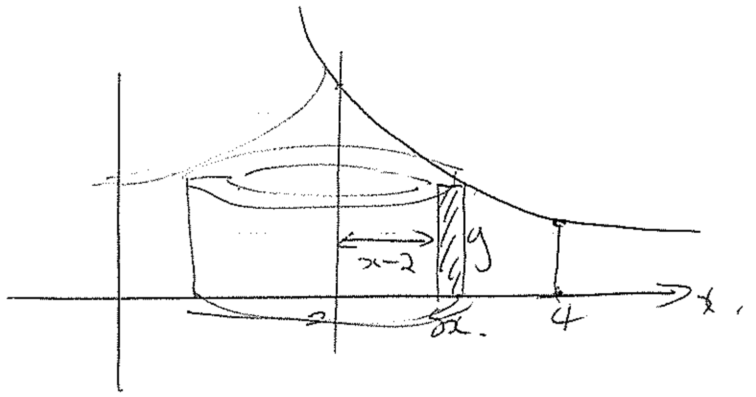
$$A=\frac{1}{2}bh$$
$$=\frac{1}{2}\times 2y\times\sqrt{3}y$$
$$=\sqrt{3}\,y^2$$

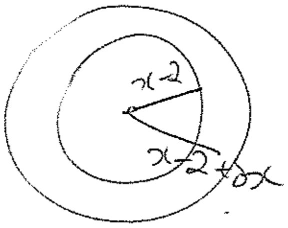
$$\delta V=\sqrt{3}\,y^2\,\delta x$$
$$=\sqrt{3}\,\frac{4}{9}(9-x^2)\,\delta x$$

$$V=\lim_{\delta x\to 0}\sum_{x=-3}^{3}\frac{4\sqrt{3}}{9}(9-x^2)\,\delta x$$
$$=\frac{4\sqrt{3}}{9}\int_{-3}^{3}(9-x^2)\,dx$$
$$=\frac{4\sqrt{3}}{9}\cdot 2\int_{0}^{3}(9-x^2)\,dx \qquad f(x)=9-x^2 \Rightarrow \text{even}$$
$$=\frac{8\sqrt{3}}{9}\left[9x-\frac{x^3}{3}\right]_0^3$$
$$=\frac{8\sqrt{3}}{9}\left(9(3)-\frac{3^3}{3}-9(0)+\frac{0^3}{3}\right)$$
$$=\frac{8\sqrt{3}}{9}(27-9)$$
$$=16\sqrt{3}\text{ units}^3$$

Question 15

a) i)





$$\delta A = \pi\left[(x-2+\delta x)^2 - (x-2)^2\right]$$

$$= \pi\left[\cancel{(x-2)^2} + 2(x-2)\delta x + \cancel{\delta x^2} - \cancel{(x-2)^2}\right]$$

$$= 2\pi(x-2)\,\delta x$$

$$\delta V = 2\pi\,(x-2)\delta x \cdot y$$

$$= 2\pi\,(x-2)\delta x \cdot \frac{1}{x^2-1}$$

$$= 2\pi\,\frac{x-2}{x^2-1}\cdot\delta x$$

$$V = \lim_{\delta x \to 0} \sum_{x=2}^{4} 2\pi\,\frac{x-2}{x^2-1}\,\delta x$$

$$= 2\pi\int_2^4 \frac{x-2}{x^2-1}\,dx$$

ii) $$V = \pi\int_2^4 \frac{2x}{x^2-1}\,dx - 4\pi\int_2^4 \frac{1}{(x-1)(x+1)}\,dx,$$

$$\frac{1}{(x-1)(x+1)} \equiv \frac{A}{x-1} + \frac{B}{x+1}$$

$$1 \equiv A(x+1) + B(x-1) \Rightarrow x=1,\ A=\tfrac{1}{2}$$

$$x=-1,\ B=-\tfrac{1}{2}$$

$$\therefore V = \pi \int_2^4 \frac{2x}{x^2-1}\,dx - \frac{4\pi}{2}\int_2^4 \left(\frac{1}{x-1} - \frac{1}{x+1}\right) dx$$

$$= \pi \Big[\ln(x^2-1)\Big]_2^4 - 2\pi \Big[\ln(x-1) - \ln(x+1)\Big]_2^4$$

$$= \pi(\ln 15 - \ln 3) - 2\pi\big(\ln 3 - \ln(5) - (\ln 1 - \ln 3)\big)$$

$$= \pi \ln 5 - 2\pi(2\ln 3 - \ln 5)$$

$$= \pi\left(\ln 5 - \ln \frac{9^2}{5^2}\right)$$

$$= \pi\left(\ln \frac{5^3}{3^4}\right)$$

$$= \pi \ln \frac{125}{81}$$

b) $a>0,\ b>0,\ c>0$ $\quad a+b+c=1$

i)

$$(x-y)^2 \geq 0$$
$$x^2 - 2xy + y^2 \geq 0$$
$$\therefore x^2 + y^2 \geq 2xy$$

let $x=\sqrt{a}$ and $y=\sqrt{b}$.

so $a+b \geq 2\sqrt{ab}$ $\quad$ -i

similarly, $b+c \geq 2\sqrt{bc}$ $\quad$ -ii

$c+a \geq 2\sqrt{ca}$. $\quad$ -iii

i × ii × iii

$$(a+b)(b+c)(c+a) \geq 2^3\sqrt{ab\,bc\,ca}$$
$$\geq 8\sqrt{a^2b^2c^2} \geq 8abc$$

iii)

$a+b+c=1$

so $a+b=1-c$

$b+c=1-a$

$c+a=1-b$

so $(1-a)(1-b)(1-c)=(b+c)(c+a)(a+b)$

$\geq 8abc.$

e). i). $LHS=\int_0^a F(x)\,dx$

$=\left[f(x)\right]_0^a$ where $\frac{d}{dx}f(x)=F(x)$

$=f(a)-f(0)$

$RHS=\int_0^a F(a-x)\,dx$

$=\left[(-1)f(a-x)\right]_0^a$

$=-f(a-a)-(-f(a-0))$

$=-f(0)+f(a)$

$=LHS$

ii) $I=\int_0^{\frac{\pi}{4}} \log_e(1+\tan\theta)\,d\theta$

$=\int_0^{\frac{\pi}{4}} \log_e\left(1+\tan\left(\frac{\pi}{4}-\theta\right)\right)d\theta$ using (i)

$=\int_0^{\frac{\pi}{4}} \log_e\left(1+\frac{\tan\frac{\pi}{4}-\tan\theta}{1+\tan\frac{\pi}{4}\cdot\tan\theta}\right)d\theta$.

$$= \int_0^{\frac{\pi}{4}} \log_e\left(1 + \frac{1-\tan\theta}{1+\tan\theta}\right) d\theta$$

$$= \int_0^{\frac{\pi}{4}} \log_e\left(\frac{2}{1+\tan\theta}\right) d\theta.$$

$$= \int_0^{\frac{\pi}{4}} \log_e 2 \, d\theta - \int_0^{\frac{\pi}{4}} (1+\tan\theta) \, d\theta.$$

$$= \int_0^{\frac{\pi}{4}} \log_e 2 \, d\theta - I$$

$$2I = \left[\log_e 2 \times \theta\right]_0^{\frac{\pi}{4}}.$$

$$I = \frac{\pi}{8} \ln 2.$$

Question 16

a) 12! ways of arranging all books
3! for each package,
4! packages

$$\text{ways} = \frac{12!}{(3!)^4 \cdot 4!}$$

$$= 15400$$

b) each trapezia width is $\frac{n-1}{n-1} = 1$ — n function values

$$\int_1^n \ln x \, dx = \frac{1}{2}\left[\ln 1 + \ln(n) + 2\left(\ln 2 + \ln 3 + \ln 4 + \ldots + \ln(n-1)\right)\right]$$

$$= \frac{1}{2}\left[0 + \ln(n) + 2\ln\left(2.3.4.\ldots(n-1)\right)\right]$$

$$= \frac{1}{2}\left[\cancel{\ln(n)}\ 2\ln(n-1)!\right]$$

$$= \frac{1}{2}\left[2\ln(n!) - \ln(n)\right]$$

$$= \ln(n!) - \frac{1}{2}\ln(n)$$

ii) $\frac{d}{dx}(x\ln x - x) = \cancel{x} \cdot \frac{1}{\cancel{x}} + 1.\ln x - 1$

$$= \ln x.$$

$$\int_1^n \ln x\,dx = \left[x\ln x - x\right]_1^n$$

$$= n\ln(n) - n - (1.\ln(1) - 1)$$

$$= n.\ln(n) - n + 1$$

iii) as $y = \ln x$ is concave down, the sum of areas of trapezia will be less that the exact area ~~of~~ under the curve.

Hence

$$\ln(n!) - \frac{1}{2}\ln(n) < n.\ln(n) - n + 1$$

$$\ln(n!) < \frac{1}{2}\ln(n) + n\ln(n) - n + 1$$

$$< (n + \tfrac{1}{2})\ln(n) - n + 1$$

c) i) $I_n = \int_1^e (1-\ln x)^n \, dx.$

let $u = (1-\ln x)^n$, $du = n(1-\ln x)^{n-1} \cdot \left(-\frac{1}{x}\right) dx$

$dv = dx$ $\qquad v = x$

$$I_n = \Big[x(1-\ln x)\Big]_1^e - \int_1^e x \cdot n(1-\ln x)^{n-1} \cdot \left(-\frac{1}{x}\right) dx$$

$$= \Big(e(1-\ln e) - 1(1-\ln 1)\Big) + n\int_1^e (1-\ln x)^{n-1} dx$$

$$= e(1-1) - 1(1-0) + n\, I_{n-1}$$

$$= -1 + n I_{n-1}$$

ii) $I_3 = -1 + 3I_2.$

$I_2 = -1 + 2I_1$

$I_1 = -1 + I_0$

$$I_0 = \int_1^e (1-\ln x)^0 dx$$
$$= \Big[x\Big]_1^e$$
$$= e - 1$$

$I_1 = -1 + e - 1$
$= e - 2$

$I_2 = -1 + 2(e-2)$
$= 2e - 5$

$I_3 = -1 + 3(2e-5)$
$= 6e - 16$

iii) Show $\frac{I_n}{n!} = e - \sum_{r=0}^{n} \frac{1}{r!}$, $n = 1, 2, 3, \ldots$

Prove true for $n=1$

$LHS = \frac{I_1}{1!}$ $\quad$ $RHS = e - \sum_{r=0}^{1} \frac{1}{r!}$

$= \frac{e-2}{1}$ $\quad$ $= e - \left(\frac{1}{0!} + \frac{1}{1!}\right)$

$= e-2$ $\quad$ $= e - \left(\frac{1}{1} + \frac{1}{1}\right)$

$= e - 2$

$= LHS$

$\therefore$ true for $n=1$

Assume true for $n=k$.

$\therefore \frac{I_k}{k!} = e - \sum_{r=0}^{k} \frac{1}{r!}$

Prove true for $n=k+1$

i.e. $\frac{I_{k+1}}{(k+1)!} = e - \sum_{r=0}^{k+1} \frac{1}{r!}$

$LHS = \frac{I_{k+1}}{(k+1)!}$

$= \frac{-1 + (k+1) I_n}{(k+1)!}$

$= \frac{-1}{(k+1)!} + \frac{I_n}{k!}$

$= \frac{-1}{(k+1)!} + e - \sum_{r=0}^{k} \frac{1}{r!}$

$= e - \left(\sum_{r=0}^{k} \frac{1}{r!} + \frac{1}{(k+1)!}\right)$ $\quad$ $\therefore$ by induction, statement is true.

$= e - \sum_{r=0}^{k+1} \frac{1}{r!} = RHS$