Ext 1

NORTH SYDNEY BOYS HIGH SCHOOL

2017 HSC ASSESSMENT TASK 3 (TRIAL HSC)

Mathematics
Extension 1

General Instructions

- Reading time – 5 minutes
- Working time – 2 hours
- Write using blue or black pen
- Board approved calculators may be used
- All necessary working should be shown in every question
- Each new question is to be started on a **new page.**
- Attempt all questions

Class Teacher:
(Please tick or highlight)

- O Mr Berry
- O Mr Hwang
- O Mr Ireland
- O Dr Jomaa
- O Ms Lee
- O Mr Lin
- O Ms Ziaziaris

Student Number:

(To be used by the exam markers only.)

Question No	1-10	11	12	13	14	Total	Total
Mark	$\frac{\quad}{10}$	$\frac{\quad}{15}$	$\frac{\quad}{15}$	$\frac{\quad}{15}$	$\frac{\quad}{15}$	$\frac{\quad}{70}$	$\frac{\quad}{100}$

Section I

10 Marks

Attempt Questions 1 – 10

Use the multiple choice answer sheet for Questions 1 – 10

1 $\int \frac{dx}{\sqrt{64-9x^2}}$ is equal to

(A) $\frac{1}{9}\sin^{-1}\left(\frac{9x}{64}\right)$ (B) $\frac{1}{3}\sin^{-1}\left(\frac{3x}{8}\right)$ (C) $\frac{1}{9}\sin^{-1}\left(\frac{3x}{8}\right)$ (D) $\frac{1}{3}\sin^{-1}\left(\frac{9x}{64}\right)$

2 The domain and range of the function $f(x) = 3\sin^{-1}(4x-1)$ are respectively

(A) $0 \le x \le \frac{1}{2}$ *and* $-\frac{3\pi}{2} \le y \le \frac{3\pi}{2}$ (B) $-\frac{1}{2} \le x \le 0$ *and* $-\frac{\pi}{2} \le y \le \frac{5\pi}{2}$

(C) $0 \le x \le \frac{1}{2}$ *and* $-\frac{\pi}{2} \le y \le \frac{5\pi}{2}$ (D) $-\frac{1}{2} \le x \le 0$ *and* $\frac{-3\pi}{2} \le y \le \frac{3\pi}{2}$

3 If $y = f(x)$ is a linear function with slope $\frac{1}{2}$, then the slope of $y = f^{-1}(x)$ is

(A) 2 (B) $\frac{1}{2}$ (C) -2 (D) $-\frac{1}{2}$

4 The derivative of $y = \cos^{-1} 3x + x\cos^{-1} 3x$ is

(A) $\cos^{-1} 3x - \frac{x+1}{\sqrt{9-x^2}}$ (B) $\cos^{-1} 3x - \frac{3(x+1)}{\sqrt{1-9x^2}}$

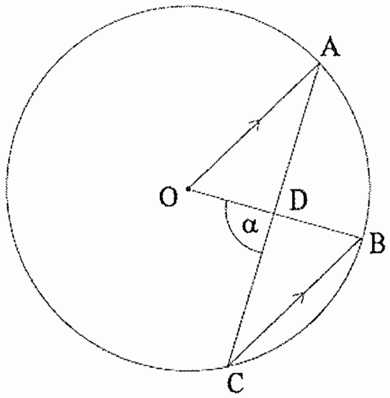
(C) $\cos^{-1} 3x + \frac{x-1}{\sqrt{9-x^2}}$ (D) $\cos^{-1} 3x + \frac{3(x-1)}{\sqrt{1-9x^2}}$

5 Given that α, β, γ are roots of $3x^3 - 2x^2 + x - 1 = 0$, then $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$ is equal to

(A) 2 (B) -1 (C) 1 (D) -2

6 The points A, B and C lie on a circle with centre O. OA is parallel to CB.

AC intersects OB at D, and $\angle ODC = \alpha$. What is the size of $\angle OAD$ in terms of α ?



(A) $\frac{\alpha}{2}$ (B) $\frac{\alpha}{3}$ (C) $\frac{2\alpha}{3}$ (D) 3α

7 Find the acute angle (to the nearest degree) between the lines $2x-3y+6=0$ and $x+2y-12=0$

(A) 7° (B) 60° (C) 14° (D) 30°

8 The point P divides the interval AB joining $A(-4,-3)$ and $B(1,5)$ externally in the ratio 3:2. The coordinates of P are

(A) $(-14,-19)$ (B) $(-11,-21)$ (C) $(11,21)$ (D) $(14,19)$

9 The equation of the normal to the parabola $x=6t,\ y=3t^2$ at the point where $t=-2$ is

(A) $x-3y+24=0$ (B) $x-2y+36=0$

(C) $2x+y+12=0$ (D) $2x-y-12=0$

10 The solution to the inequality $\frac{|3x+1|}{x} \le 1$ is

(A) $-\frac{1}{2} \le x < -\frac{1}{4}$ (B) $x<0$ (C) $x \le -\frac{1}{2}$ (D) No solution

Section II

60 Marks
Attempt Questions 11-14

Answer each question on a NEW page. Extra writing booklets are available.

In Questions 11 – 14, your responses should include all relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Start a NEW page

(a) Evaluate $\lim_{x \to 0} \dfrac{\sin 2x \cos 2x}{3x}$. 1

(b) Find the exact value of $\cos^{-1} \cos \left(\dfrac{5\pi}{4}\right)$. 1

(c) Find $\int \cos^2 9x \; dx$. 2

(d) Solve $\sin x + \cos x = 1$ for $0 \le x \le 2\pi$ 3

(e) When the polynomial $P(x)$ is divided by $1 - x^2$ it gives $4 - x$ as the remainder.

What is the remainder when $P(x)$ is divided by $1 + x$? 2

(f) If the three roots of $x^3 - 6x^2 + 3x + k = 0$ form an arithmetic progression,

find the value of k. 3

(g) Prove using mathematical induction that $n^3 + (n+1)^3 + (n+2)^3$ is divisible 3

by 9 for $n = 1, 2, 3, \ldots$.

Question 12 (15 marks) Start a NEW page

(a) Use the substitution $x = u^2 - 1$ for $u > 0$ to evaluate $\int_3^8 \frac{x-1}{\sqrt{x+1}}\, dx$. 4

(b) A bar of gold is initially at a temperature of -8 degrees Celsius.

It is taken into a nearby room where the air temperature is 22 degrees Celsius.

The rate at which the bar warms follows Newton's Law, that is, $\frac{dT}{dt} = -k(T-22)$,

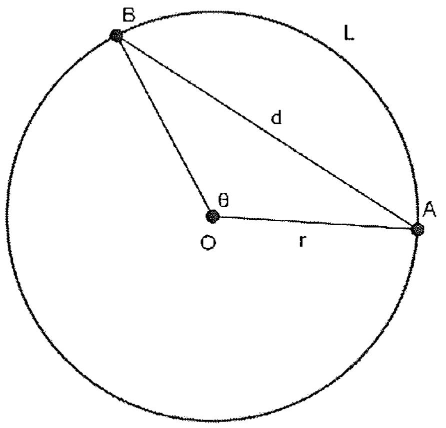
where k is a constant, time t is measured in minutes and temperature T is in degrees Celsius.

(i) Show that $T = 22 - Ae^{-kt}$ is a solution of the equation above, and evaluate A. 2

(ii) Given that the bar's temperature reaches 4 degrees Celsius in 90 minutes, find the exact value of k. 2

(iii) Find the temperature of the bar after another 90 minutes. 1

(c) An arc AB of a circle subtends an angle of θ radians at the centre of a circle of radius r. The arc's length is L, and the associated chord's length is d. (See diagram).



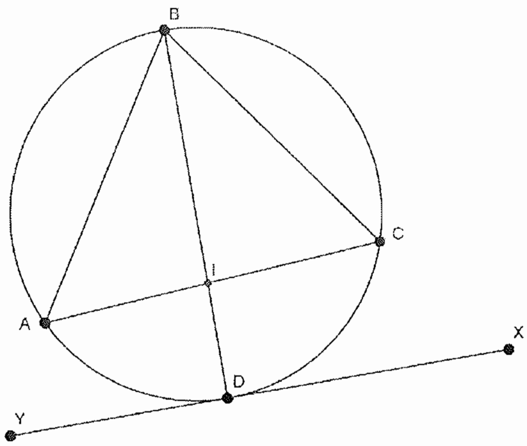
(i) If $L : d = 4 : 3$, show that $3\theta = 8\sin\frac{\theta}{2}$. 3

(ii) Using $\theta = 2 \cdot 5$ as a first approximation, use Newton's method once to find a second approximation (to 3 decimal places) for θ. 3

Question 13 (15 marks) Start a NEW page

(a) In the circle, $\angle ABD + \angle BCA = 90°$, and XY is tangent to the circle at D.

The chords AC and BD intersect at I. (Copy or trace the diagram in your writing booklet.)



(i) Prove $\angle BCD = 90°$ and hence that BD is a diameter of the circle. 2

(ii) Prove that if ΔABC is isosceles with $AB = BC$, then $AC \,||\, XY$. 2

(b) The velocity of a particle moving in a straight line is given by $\dfrac{dx}{dt} = 1 + x^2$, where

x is the displacement in metres from the origin, and t is the time in seconds. Initially the particle is at $x = 1$.

(i) Find an expression for the acceleration of the particle in terms of x. 2

(ii) Find an expression for the displacement of the particle in terms of t. 2

(c) Assume that a spherical snowball melts so that its volume V decreases at a rate proportional to its surface area S (and also that it stays spherical as it melts).

(i) If r is its radius in centimetres at time t hours, show that the rate of change of r is constant. 1

(ii) Given that it takes 3 hours for the snowball to decrease to half its original volume, show that $r = kt + R$, where R is the initial radius and

$$k = \frac{R}{3}\left(\frac{1}{\sqrt[3]{2}} - 1\right).$$ 3

(iii) How much longer will it take for the snowball to melt completely? 1

(d) Find the value of the following limit: $\displaystyle\lim_{x \to -5} \frac{\sqrt{20 - x} - 5}{5 + x}$ 2

Question 14 (15 marks) Start a NEW page

(a) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ lie on the parabola $x^2 = 4ay$.

(i) Find the coordinates of the point of intersection T of the tangents to the parabola at P and Q. 2

(ii) Given that the tangents at P and Q intersect at $45°$ show that

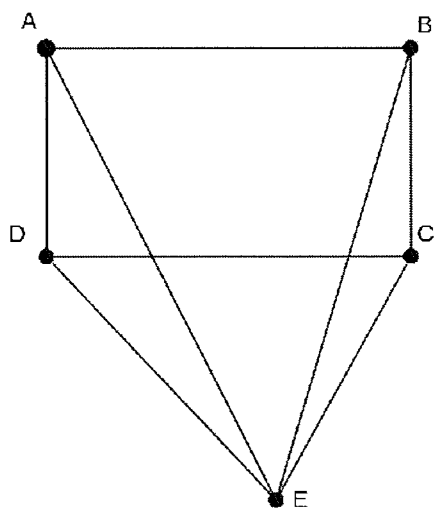
$p - q = 1 + pq$ where $p > q$. 2

(iii) Find the equation of the locus of T when the tangents at P and Q intersect as given in (ii). 2

(b) A plane flying horizontally at a speed of 180 km/h is observed from point E on the ground (see diagram). Initially it is observed to be at A, on a bearing 337°T and at an elevation of 34°. After 2 minutes it is observed to be at B, bearing 018°T at an elevation of 27°.

(i) Show that $\angle DEC = 41°$ 1

(ii) Find the height h (in metres) of the plane. 3



(c) Given the function $f(x) = e^x + e^{2x}$,

(i) Write down the domain and range of $f(x)$. 1

(ii) Hence write down the domain and range of its inverse function $f^{-1}(x)$. 1

(iii) Find the equation of $f^{-1}(x)$. 3

END OF EXAMINATION

2017 Y12 Ext.1 Test 3 (Trial) — Suggested Solutions

Multiple Choice

1. B
2. A
3. A
4. B
5. C
6. B
7. B
8. C
9. B
10. B

/10

Q $\boxed{11}$

(a) $\lim_{x \to 0} \frac{\sin 2x \cos 2x}{3x} = \frac{2}{3}$ ✓

(b) $\cos^{-1}\cos\left(\frac{5\pi}{4}\right) = \frac{3\pi}{4}$ ✓

(c) $\int \cos^2 9x \, dx = \int \frac{1}{2}(1+\cos 18x)\, dx$ ✓

$= \frac{x}{2} + \frac{\sin 18x}{36} + C$ ✓

(d) $\sin x + \cos x = \sqrt{2}\sin\left(x+\frac{\pi}{4}\right)$ ✓

$\therefore \sqrt{2}\sin\left(x+\frac{\pi}{4}\right) = 1, \quad 0 \le x \le 2\pi$

$\therefore \sin\left(x+\frac{\pi}{4}\right) = \frac{1}{\sqrt{2}}, \quad \frac{\pi}{4} \le x+\frac{\pi}{4} \le 2\pi+\frac{\pi}{4}$

$\therefore x+\frac{\pi}{4} = \frac{\pi}{4}, \frac{3\pi}{4}, \frac{\pi}{4}+2\pi$

$\therefore x = 0, \frac{\pi}{2}, 2\pi.$ ✓✓

[alt: can use t-method:

$\frac{2t}{1+t^2} + \frac{1-t^2}{1+t^2} = 1 \rightarrow t = 0$ or 1

$\rightarrow \tan\frac{x}{2} = 0$ or 1 $\quad \therefore \frac{x}{2} = 0, \pi, \frac{\pi}{4}$

$\therefore x = 0, 2\pi, \frac{\pi}{2}$ (*) must test $x = \pi$] ← (*)

$\boxed{Q11}$ ctd.

(e) $P(x) = (1-x)(1+x) \cdot Q(x) + 4 - x$

remainder when divided by $1+x$, i.e. by $x+1$, is $P(-1)$

$\therefore$ rem. $= P(-1) = 4-(-1) = 5$ ✓✓

(f) $P(x) = x^3 - 6x^2 + 3x + k = 0$

Let roots be $a-d$, a, $a+d$

$\therefore$ sum roots $= 3a = 6$ $\quad\therefore a = 2$ ✓

$\therefore$ Since 2 is a root, $P(2) = 0$ ✓

$\therefore 2^3 - 6 \cdot 2^2 + 3 \cdot 2 + k = 0$ ✓

$-10 + k = 0$ $\quad\therefore k = 10$

(g) When $n=1$, $n^3 + (n+1)^3 + (n+2)^3 = 1^3 + 2^3 + 3^3$

$= 36$

$= 9 \times 4$

$\therefore$ true for $n=1$. ✓

Assume true for $n=k$,

i.e. assume $k^3 + (k+1)^3 + (k+2)^3 = 9M$, M an integer.

Then $(k+1)^3 + (k+2)^3 + (k+3)^3 = (9m - k^3) + (k+3)^3$, by assumption ✓

$= 9m - \cancel{k^3} + \cancel{k^3} + 3 \cdot k^2 \cdot 3 + 3 \cdot k \cdot 3^2 + 3^3$

$= 9m + 9k^2 + 27k + 27$

$= 9[m + k^2 + 3k + 3]$

$= 9N$, N an integer, as M & k are integers.

$\therefore$ true for $n=k+1$ if true for $n=k$ ✓ (no fudging!)

Since true for $n=1$, $\therefore$ true for $n=1, 2, 3, \ldots$ by induction.

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2017 Y12 Ext.1 Test 3 (Trial) — Suggested Solutions

Q12

(a) $I = \int_3^8 \frac{x-1}{\sqrt{x+1}}\,dx$

Let $x = u^2 - 1$, $u > 0$. $\quad \begin{cases} x=3 \to u=2 \\ x=8 \to u=3 \end{cases}$ ✓

$\therefore dx = 2u\,du$ $\quad \begin{cases} x-1 = u^2-2 \\ \sqrt{x+1} = u \end{cases}$

$\therefore I = \int_2^3 \frac{u^2-2}{u} \cdot 2u\,du$

$= 2\int_2^3 (u^2-2)\,du$ ✓

$= 2\left[\frac{u^3}{3} - 2u\right]_2^3 = 2\left[\left(\frac{27}{3}-6\right) - \left(\frac{8}{3}-4\right)\right]$

$= \frac{26}{3}$ ✓✓

(b) (i) $T = 22 - Ae^{-kt}$ $\quad \therefore \frac{dT}{dt} = kAe^{-kt}$

$= k(22-T)$ ✓

$= -k(T-22)$ ✓

when $t=0$, $T=-8$ $\quad \therefore -8 = 22 - A$ $\quad \therefore A = 30$ ✓

(ii) when $t=90$, $T=4$ $\quad \therefore 4 = 22 - 30e^{-90k}$

$\therefore e^{-90k} = \frac{18}{30} = \frac{3}{5}$ ✓

$\therefore -90k = \ln\frac{3}{5}$ $\quad \therefore k = -\frac{1}{90}\ln\left(\frac{3}{5}\right)$

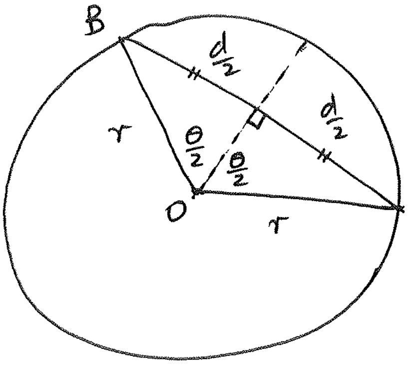
(iii) when $t = 180$,

$T = 22 - 30e^{-180k}$

$= 22 - 30e^{2\ln\left(\frac{3}{5}\right)} = 22 - 30 \times \frac{9}{25} = 11.2^\circ C$ ✓

Q12 - ctd

(c) (i)



Join O to the midpoint of AB. It meets it at 90° (circle geom. theorem)

$$\therefore \sin\frac{\theta}{2} = \frac{\frac{d}{2}}{r} = \frac{d}{2r}$$

$$\therefore d = 2r\sin\left(\frac{\theta}{2}\right)$$

Also, $L = r\theta$

Since $\frac{L}{d} = \frac{4}{3}$ $\quad \therefore \frac{\cancel{r}\theta}{2\cancel{r}\sin\left(\frac{\theta}{2}\right)} = \frac{4}{3}$

$$\therefore 3\theta = 8\sin\left(\frac{\theta}{2}\right)$$

(ii) Let $f(\theta) = 3\theta - 8\sin\frac{\theta}{2}$

then $f'(\theta) = 3 - 4\cos\frac{\theta}{2}$

Thus $\theta_2 = \theta_1 - \frac{f(\theta_1)}{f'(\theta_1)}$

$$= 2.5 - \frac{3(2.5) - 8\sin(1.25)}{3 - 4\cos 1.25}$$

$$\doteqdot 2.5528420\ldots$$

$\therefore \theta_2 = 2.553$ (3 d.p.)

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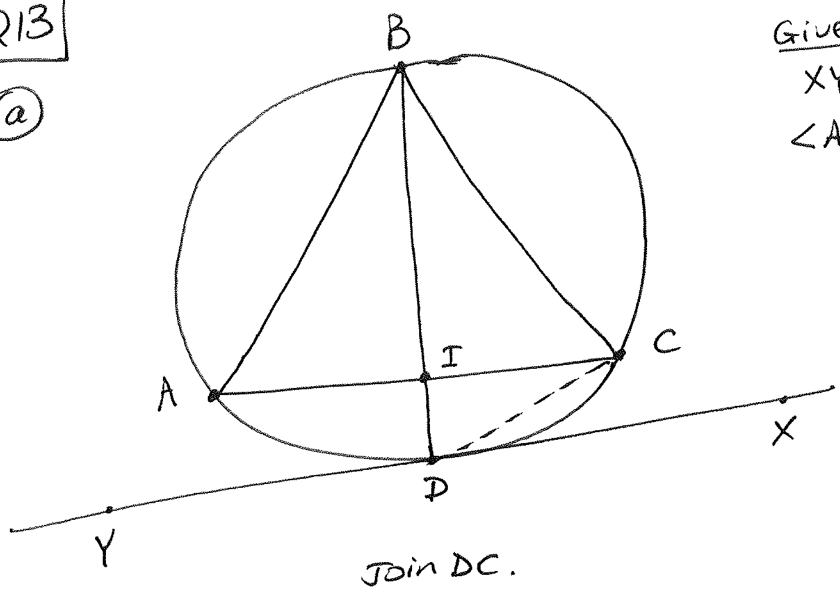
Q13

(a)



Given:
XY is tangent,
$\angle ABD + \angle BCA = 90°$

Join DC.

(i) $\angle BCD = \angle BCA + \angle ACD$. ✓

But $\angle ACD = \angle ABD$ (angles in the same segment)

$\therefore \angle BCD = \angle BCA + \angle ABD$

$= 90°$ (given)

Hence BD is a diameter (converse of "angle in a semicircle is 90°" theorem) ✓

(ii) Since $AB = BC$,

$\therefore \angle BAC = \angle BCA$ (equal angles opposite to equal sides in ΔABC)

So $\angle AID = \angle BAC + \angle ABD$ (exterior angle of triangle equals sum of opposite interior angles)

$= \angle BCA + \angle ABD$

$= 90°$ (from (i)).

But $\angle IDX = 90°$ (tangent is perpendicular to radius to point of contact)

$\therefore AC \parallel XY$ (alternate angles equal) ✓✓

[note: alternate routes possible to solution here].

Q13 ctd

(b) (i) $v = 1 + x^2 \quad \therefore \quad \frac{1}{2}v^2 = \frac{1}{2}(1+x^2)^2$

$\therefore \quad a = \frac{d}{dx}\left(\frac{1}{2}v^2\right) = \frac{1}{2} \cdot 2 \cdot 2x(1+x^2)$

$\therefore a = 2x(1+x^2)$ ✓✓

(ii) $\frac{dx}{dt} = 1 + x^2 \quad \therefore \quad \frac{dt}{dx} = \frac{1}{1+x^2}$

$\therefore \quad t = \tan^{-1}x + C$ ✓

at $t = 0$, $x = 1$ $\quad \therefore \quad 0 = \tan^{-1}1 + C$

$= \frac{\pi}{4} + C \qquad \therefore \quad C = -\pi/4$

$\therefore \quad t = \tan^{-1}x - \frac{\pi}{4}$

$\therefore \quad \tan^{-1}x = t + \frac{\pi}{4}$

$\therefore \quad x = \tan\left(t + \frac{\pi}{4}\right)$ ✓

(c)

$\frac{dV}{dt} \propto S \quad \therefore \quad \frac{dV}{dt} = kS$, for some constant k.

But $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$\therefore \quad kS = 4\pi r^2 \cdot \frac{dr}{dt} \qquad$ (as $V = \frac{4}{3}\pi r^3$)

But $S = 4\pi r^2$

$\therefore \quad k = \frac{dr}{dt}$

ie $\frac{dr}{dt}$ is constant. ✓

Q13 ctd

(c) (ii) from (i), $\frac{dr}{dt} = k$

$\therefore r = kt + c$

at $t=0$, $r=R$ $\therefore R = 0 + c$ $\therefore c = R$

$\therefore r = kt + R$ ✓

$\therefore$ at $t=3$, $r = 3k + R$.

So we have: $\frac{4}{3}\pi(3k+R)^3 = \frac{1}{2}\cdot\frac{4}{3}\pi R^3$ (from the given data)

$3k + R = \frac{1}{\sqrt[3]{2}}\cdot R$

$3k = R\left(\frac{1}{\sqrt[3]{2}} - 1\right)$ $\therefore k = \frac{R}{3}\left(\frac{1}{\sqrt[3]{2}} - 1\right)$ ✓✓

(iii) $r = \frac{R}{3}\left(\frac{1}{\sqrt[3]{2}} - 1\right)t + R$

When fully melted, $r = 0$

$\therefore \frac{1}{3}\left[\frac{1}{\sqrt[3]{2}} - 1\right]t + 1 = 0$

$\therefore t\left(\frac{1}{\sqrt[3]{2}} - 1\right) = -3$ $\therefore t = \frac{-3}{\left(\frac{1}{\sqrt[3]{2}} - 1\right)} \doteqdot 14.54196\ldots$

So extra time taken is $\doteqdot$ 11.54 hours ✓
(ie. 11 h 33 mins).

(d) $\lim_{x\to -5} \frac{\sqrt{20-x} - 5}{5+x} = \lim_{x\to -5} \frac{\sqrt{20-x} - 5}{5+x} \times \frac{\sqrt{20-x} + 5}{\sqrt{20-x} + 5}$ ✓

$= \lim_{x\to -5} \frac{20 - x - 25}{(5+x)\sqrt{20-x} + 5}$

$= \lim_{x\to -5} \frac{-1}{\sqrt{20-x} + 5} \doteqdot \frac{-1}{10}$ ✓

(needs working for full marks)

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Q14 (a)

(i) Using the formula sheet, at T we have

$$px - ap^2 = qx - aq^2$$
$$(p-q)x = ap^2 - aq^2$$
$$\therefore x = a(p+q) \quad \text{as } p \neq q$$
$$\therefore y = pa(p+q) - ap^2 = apq$$
$$\therefore T = [a(p+q), apq]$$

(some working required) ✓✓

(ii) The tangents have gradients p, q.

$$\therefore \tan 45^\circ = \left| \frac{p-q}{1+pq} \right|$$

✓ Needs:
- $\tan 45^\circ$
- abs. value.
- $p>q$

$$\therefore 1 = \frac{p-q}{1+pq} \quad (p>q)$$

ie. $p - q = 1 + pq$ ✓

(iii) from (i), $x = a(p+q)$ $\therefore p+q = \frac{x}{a}$

$y = apq$ $\therefore pq = \frac{y}{a}$

Now $(p-q)^2 = (p+q)^2 - 4pq$

$\therefore (1+pq)^2 = (p+q)^2 - 4pq$ from (ii)

✓ uses part (ii) in way that could lead to solution

$$\therefore \left(1 + \frac{y}{a}\right)^2 = \left(\frac{x}{a}\right)^2 - \frac{4y}{a}$$

$$1 + \frac{2y}{a} + \frac{y^2}{a^2} = \frac{x^2}{a^2} - \frac{4y}{a}$$

$$\frac{x^2}{a^2} = \frac{y^2}{a^2} + 1 + \frac{6y}{a}$$

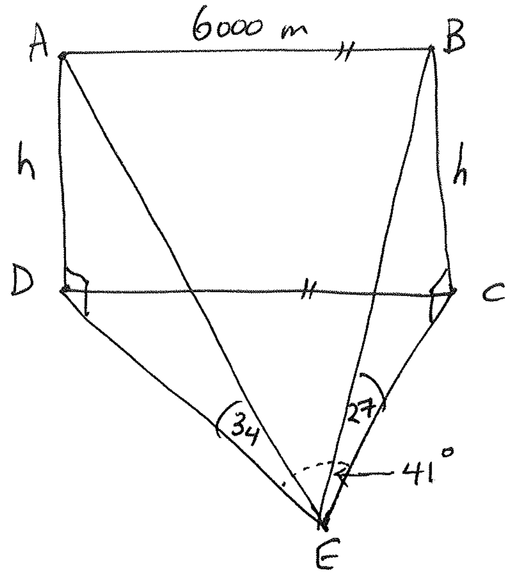
ie $x^2 = y^2 + a^2 + 6ay$

✓ or equivalent expression

2017 4U2 Ext.1 Test 3 (Trial) Suggested Solutions

Q14 - ctd.

(b)



(i) The bearing of A (& hence of D) is 337°T, & the bearing of B (& hence of C) is 018°T.

$\therefore \angle DEC = (360° - 337°) + 018°$
$= 23° + 18° = 41°$

(must not assume flies W–E) ✓

(ii) $\frac{h}{DE} = \tan 34°$ $\therefore DE = \frac{h}{\tan 34°}$ $\therefore DE = h \cot 34°$

Likewise, $CE = h \cot 27°$ ✓

(in 2mins @ 180 km/h, it goes 6km = 6000 m)

By cosine rule in ΔDEC,

$$6000^2 = h^2\cot^2 34 + h^2\cot^2 27 - 2h\cot 34 \cdot h\cot 27 \cdot \cos 41$$
$$= h^2\left[\cot^2 34 + \cot^2 27 - 2\cot 34 \cot 27 \cos 41\right]$$

(incorrect cosine rule terminates)
(cannot use sine rule) ✓

$$\therefore h = \frac{6000}{\sqrt{\cot^2 34 + \cot^2 27 - 2\cot 34 \cot 27 \cos 41}}$$

$$= \frac{6000}{\sqrt{\tan^2 56 + \tan^2 63 - 2\tan 56 \tan 63 \cos 41}}$$

$\therefore h \doteqdot 4659.87$ m
$h = 4660$ m (nearest metre) ✓

$\boxed{Q14}$ - ctd.

(c) $f(x) = e^x + e^{2x}$, i.e. $y = e^x + e^{2x}$

(i) For f, D: all real x
R: $y > 0$ ✓

(ii) For f^{-1}, D: $x > 0$
R: all real y ✓

(iii) for inverse, $x = e^y + e^{2y}$ ✓

i.e. $e^{2y} + e^y - x = 0$

Let $u = e^y$

$\therefore u^2 + u - x = 0$

$u = \dfrac{-1 \pm \sqrt{u^2 + 4x}}{2}$

$\therefore e^y = \dfrac{-1 + \sqrt{1+4x}}{2}$ or $\dfrac{-1 - \sqrt{1+4x}}{2}$ ✓

But $e^y > 0$ $\therefore$ $e^y = \dfrac{-1 + \sqrt{1+4x}}{2}$ must reject one solution

$\therefore y = \ln\left[\dfrac{\sqrt{1+4x} - 1}{2}\right]$ ✓

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