NORTH SYDNEY GIRLS HIGH SCHOOL

2017 TRIAL HSC EXAMINATION

Mathematics Extension 1

General Instructions

- Reading Time – 5 minutes
- Working Time – 2 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet has been provided
- In Questions 11 – 14, show relevant mathematical reasoning and/or calculations

Total marks – 70

Section I Pages 3 - 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II Pages 7 – 13

60 Marks

- Attempt Questions 11 – 14
- Allow about 1 hour and 45 minutes for this section

NAME:______________________________ **TEACHER:**____________________

STUDENT NUMBER:__________________________

QUESTION	MARK
1–10	/10
11	/15
12	/15
13	/15
14	/15
TOTAL	/70

Section I

10 marks
Attempt Questions 1−10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 Consider the polynomial $P(x) = 3x^3 + 3x + a$.
If $x - 2$ is a factor of $P(x)$, what is the value of a?

(A) –30

(B) −18

(C) 18

(D) 30

2 Let α, β and γ be the roots of $P\left(x\right) = 2x^3 - 5x^2 + 4x - 9$.
Find the value of $\frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma}$.

(A) $\frac{5}{9}$

(B) $-\frac{5}{9}$

(C) $\frac{4}{9}$

(D) $-\frac{4}{9}$

3 Which expression is equal to $\int \sin^2 2x\,dx$?

(A) $\frac{1}{2}\left(x-\frac{1}{4}\sin 4x\right)+c$

(B) $\frac{1}{2}\left(x+\frac{1}{4}\sin 4x\right)+c$

(C) $\frac{1}{2}\left(x-\frac{1}{2}\sin 4x\right)+c$

(D) $\frac{1}{2}\left(x+\frac{1}{2}\sin 4x\right)+c$

4 Which of the following is equivalent to $\frac{\sin x}{1-\cos x}$?

(A) $\tan\left(\frac{x}{2}\right)$

(B) $-\tan\left(\frac{x}{2}\right)$

(C) $\cot\left(\frac{x}{2}\right)$

(D) $-\cot\left(\frac{x}{2}\right)$

5 What are the asymptotes of $y=\frac{3x}{(x+1)(x-2)}$?

(A) $y=0,\ x=-1,\ x=2$

(B) $y=0,\ x=1,\ x=-2$

(C) $y=3,\ x=-1,\ x=2$

(D) $y=3,\ x=1,\ x=-2$

6 Which of the following is the range of the function $y = 2\sin^{-1} x + \frac{\pi}{2}$?

(A) $-\pi \le y \le \pi$

(B) $-\pi \le y \le \frac{3\pi}{2}$

(C) $-\frac{\pi}{2} \le y \le \frac{\pi}{2}$

(D) $-\frac{\pi}{2} \le y \le \frac{3\pi}{2}$

7 If P divides the interval AB internally in the ratio $m:n$, in what ratio does A divide the interval BP?

(A) $(m+n):-n$

(B) $(m+n):-m$

(C) $-n:(m+n)$

(D) $-m:(m+n)$

8 What is a general solution of $\tan 2\theta \tan\theta = 1$?

.

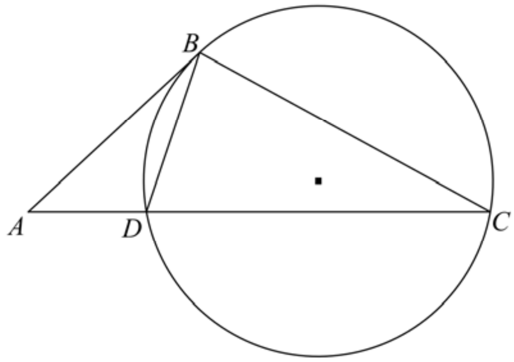
(A) $2n\pi \pm \frac{\pi}{3}$ where n is an integer.

(B) $(6n \pm 1)\frac{\pi}{6}$ where n is an integer.

(C) $(4n \pm 1)\frac{\pi}{6}$ where n is an integer.

(D) $2n\pi \pm \frac{\pi}{6}$ where n is an integer.

9 In the diagram below, AB is the tangent to the circle at B and ADC is a straight line. If $AB:AD = 2:1$, then what is the ratio of the area of ΔABD to the area of ΔCBD?



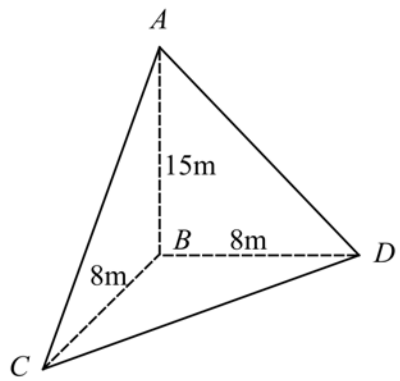
(A) $1:2$

(B) $1:3$

(C) $1:4$

(D) $2:3$

10 In the figure below, AB is a vertical pole standing on horizontal ground BCD, where $\angle CBD = 90°$. If the angle between the plane ACD and the horizontal ground is θ, then what is the value of θ closest to?



(A) 45°

(B) 53°

(C) 62°

(D) 69°

End of Section I

Section II

Total marks − 60
Attempt Questions 11−14
Allow about 1 hour 45 minutes for this section.

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 to 14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Differentiate $\tan^{-1}\sqrt{x}$ with respect to x. **2**

(b) Consider the function $f(x) = 1 + \dfrac{2}{x-3}$ for $x > 3$.

(i) What is the range of $f(x)$? **1**

(ii) Find the inverse function $f^{-1}(x)$ and state its domain. **1**

(c) Use the substitution $u = 3 + x$ to find $\displaystyle\int \frac{x+1}{\sqrt{3+x}}\,dx$. **3**

(d) Solve $\dfrac{4}{x+2} \geq \dfrac{1}{x}$. **3**

(e) Find $\displaystyle\lim_{x \to 0} \frac{1-\cos 2x}{x^2}$. Show all working. **2**

(f) (i) Neatly sketch the graph of $y = \sin^{-1} x$. **1**

(ii) By considering areas on the graph in (i), find the exact value of **2**

$$\int_0^{\frac{1}{2}} \sin^{-1} x\,dx.$$

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) During the early summer months, the rate of increase of the population P of cicadas is proportional to the excess of the population over 3000. The rate can be expressed by the differential equation $\frac{dP}{dt} = k(P - 3000)$ where t is the time in months and k is a constant. At the beginning of summer the population is 4000 and one month later it is 10 000.

(i) Show that $P = 3000 + Ae^{kt}$ is a solution of the differential equation, where A is a constant. **1**

(ii) Find the value of A. **1**

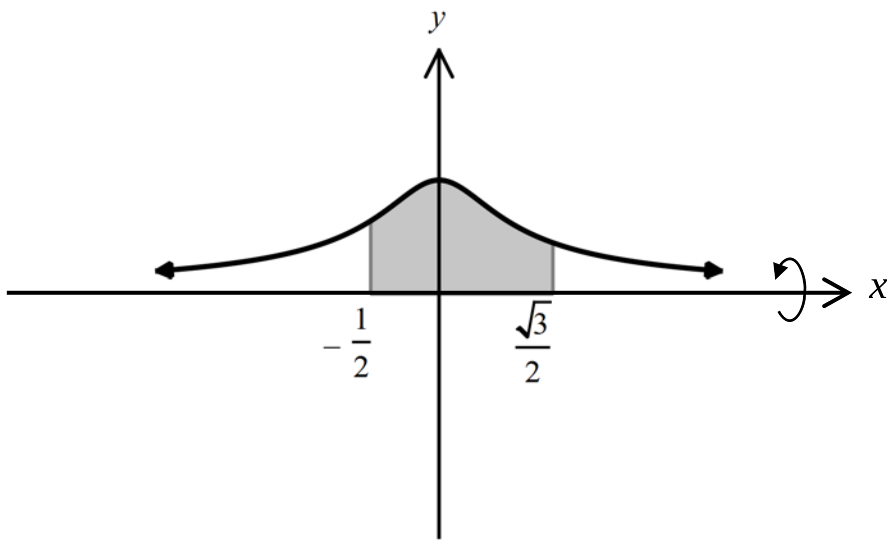
(iii) Show that the value of k is $\log_e 7$. **1**

(iv) After how many weeks will the population reach half a million? (Assume 52 weeks in a year). **2**

(b) The angle between the line $4x + 3y = 8$ and the line $ax + by + c = 0$ is $45°$. Find the possible values of the ratio $a : b$. **3**

Question 12 continues on page 9

(c) The graph of $y = \frac{1}{\sqrt{1+4x^2}}$ is shown below. **3**



The shaded region in the diagram is bounded by the curve $y = \frac{1}{\sqrt{1+4x^2}}$, the $x-$axis and the lines $x = -\frac{1}{2}$ and $x = \frac{\sqrt{3}}{2}$. Find the exact volume of the solid of revolution formed when the shaded region is rotated about the $x-$axis.

(d) (i) Express $3\sin x + \sqrt{3}\cos x$ in the form $A\sin(x+\alpha)$, where $0 < \alpha < \frac{\pi}{2}$. **2**

(ii) Hence, or otherwise, sketch the graph of $y = 3\sin x + \sqrt{3}\cos x$ where $0 \le x \le 2\pi$. **2**

End of Question 12

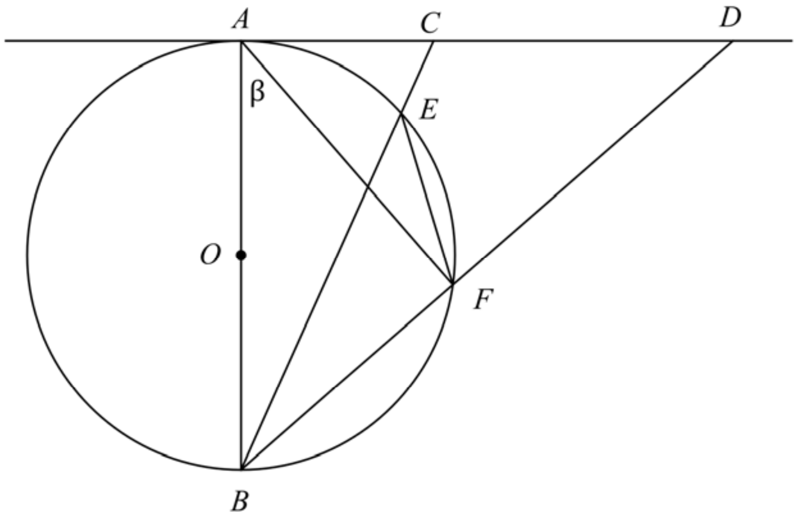
Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The acceleration of a particle as it moves in a straight line is given by $\frac{d^2x}{dt^2} = -12\cos 2t$ where x is the displacement in metres of the particle from the origin at time t seconds. The particle starts from rest at the point $x = 3$.

(i) Find the displacement, x, of the particle as a function of t. **2**

(ii) At what time is the particle at $x = 0$, and moving towards its initial position? **1**

(b) In the diagram below, the straight line ACD is a tangent at A to the circle with centre O. The interval AOB is a diameter of the circle The intervals BC and BD meet the circle at E and F respectively. Let $\angle BAF = \beta$.



(i) Explain why $\angle ABF = 90° - \beta$. **1**

(ii) Prove that the quadrilateral $CDFE$ is cyclic. **3**

Question 13 continues on page 11

(c) A ball on a spring is moving in simple harmonic motion with a vertical velocity v cms^{-1} given by $v^2 = -8 + 24y - 4y^2$ where y is the vertical displacement in cm.

(i) Find the acceleration of the ball in terms of y. **2**

(ii) Find the centre of motion of the ball. **1**

(iii) Find the period of the oscillation. **1**

(d) (i) Show that $n + (n+1) + (n+2) + ... + (2n+1) = \frac{(3n+1)(n+2)}{2}$ **1**

(ii) Hence prove by mathematical induction that for all integers $n \geq 1$, **3**

$$1 + (2+3) + (3+4+5) + ... + [n + (n+1) + (n+2) + ... + (2n-1)] = \frac{n^2}{2}(n+1).$$

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Show that the equation of the normal to the parabola $x^2 = 4y$ at the point $P(2p, p^2)$ is $x + py = 2p + p^3$. **2**

(ii) S is the focus of the parabola $x^2 = 4y$ and T is a point on the normal such that ST is perpendicular to the normal. Write down the equation of ST. **1**

(iii) Prove that the locus of T is a parabola and state its vertex and focal length. **3**

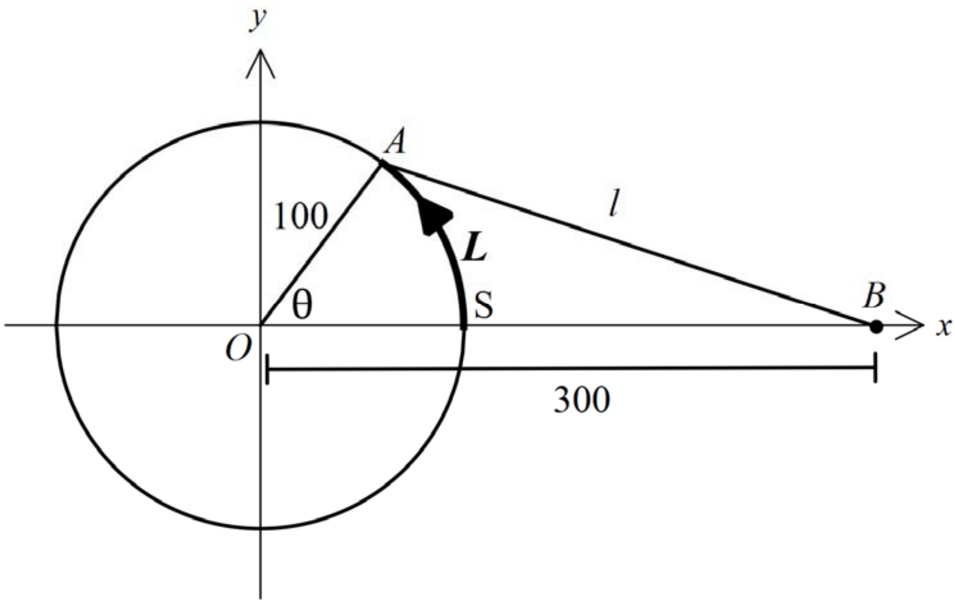
(b) (i) Show that $1 + e^{-x} = \dfrac{e^x + 1}{e^x}$. **1**

(ii) The velocity v of a particle moving along the x-axis is given by $\dfrac{dx}{dt} = 1 + e^{-x}$ where x is the displacement of the particle from the origin in metres. Initially the particle is at the origin. **3**

Find the time taken by the particle to reach a velocity of $1\frac{1}{2}$ ms^{-1}.

Question 14 continues on page 13

(c) A runner sprints in an anticlockwise direction around a circular track of radius 100 metres with centre O at a constant speed of 5 m/s. The runner's friend is standing at B, a distance of 300 metres from the centre of the track.



The runner starts at S and t seconds later is at point A. The distance AB between the two friends is l and the distance covered by the runner on the track is L.
Let the angle subtended by the arc SA be θ.

(i) From the diagram the coordinates of A are $(100\cos\theta, 100\sin\theta)$. **1**
Use the distance formula to show that $l = 100\sqrt{10 - 6\cos\theta}$.

(ii) At what rate is the distance between the friends changing at the moment when the runner is 250 metres from his friend and getting closer to him. **4**

End of paper

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2017 TRIAL HSC EXAMINATION

Mathematics Extension 1

SOLUTIONS

1	Consider the polynomial $P(x)=3x^3+3x+a$. If $x-2$ is a factor of $P(x)$, what is the value of a?

Answer: A

$P(2)=0 \qquad 3\times 2^3+3(2)+a=0$

$a=-30$

2	Let α, β and γ be the roots of $P(x)=2x^3-5x^2+4x-9$. Find the value of $\frac{1}{\alpha}+\frac{1}{\beta}+\frac{1}{\gamma}$.

Answer: C

$$\frac{\beta\gamma+\alpha\gamma+\alpha\beta}{\alpha\beta\gamma}=\frac{\frac{4}{2}}{\frac{9}{2}}=\frac{4}{9}$$

3	Which expression is equal to $\int \sin^2 2x\,dx$?

Answer: A

$$\int \sin^2 2x\,dx=\frac{1}{2}\int(1-\cos 4x)\,dx$$

$$=\frac{1}{2}\left(x-\frac{1}{4}\sin 4x\right)+c$$

4	Which of the following is equivalent to $\frac{\sin x}{1-\cos x}$?

Answer: C

Let $t=\tan\left(\frac{x}{2}\right)$

$$\frac{\frac{2t}{1+t^2}}{1-\frac{1-t^2}{1+t^2}}=\frac{2t}{1+t^2-1+t^2}$$

$$=\frac{1}{t}$$

$$=\cot\left(\frac{x}{2}\right)$$

5	What are the asymptotes of $y = \dfrac{3x}{(x+1)(x-2)}$?

Answer: A

Vertical asymptotes at $x = -1,\ x = 2$

$$\lim_{x\to\infty}\frac{\frac{3x}{x^2}}{\frac{x^2}{x^2}-\frac{x}{x^2}-\frac{2}{x^2}} = \lim_{x\to\infty}\frac{\frac{3}{x}}{1-\frac{1}{x}-\frac{2}{x^2}} \qquad \therefore \text{ horizontal asymptote at } y=0$$

$$= 0$$

6	Which of the following is the range of the function $y = 2\sin^{-1}x + \dfrac{\pi}{2}$?

Answer: D

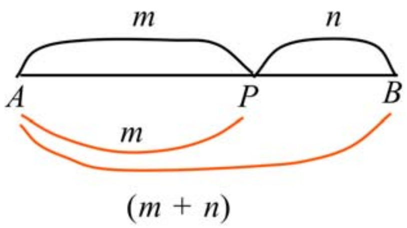
$-\frac{\pi}{2} \le \sin^{-1}x \le \frac{\pi}{2}$

$-\pi \le 2\sin^{-1}x \le \pi$

$-\frac{\pi}{2} \le 2\sin^{-1}x + \frac{\pi}{2} \le \frac{3\pi}{2}$

7	If P divides the interval AB internally in the ratio $m:n$, in what ratio does A divide the interval BP?

Answer: B



8	What is a general solution of $\tan 2\theta \tan\theta = 1$?

Answer: B

$\frac{2\tan\theta}{1-\tan^2\theta}\times\tan\theta = 1$

$2\tan^2\theta = 1-\tan^2\theta$

$3\tan^2\theta = 1$

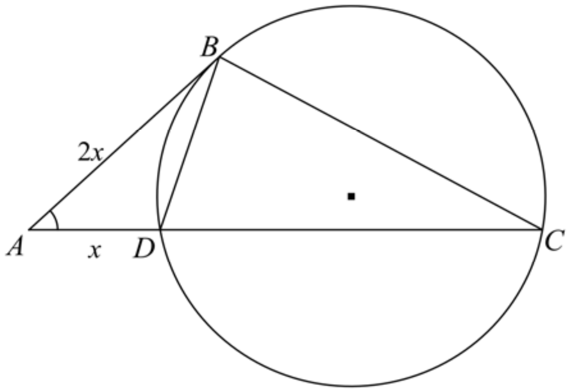
$\cdot\ \tan\theta = \pm\frac{1}{\sqrt{3}} \quad \Rightarrow \tan\theta = \tan\left(\pm\frac{\pi}{6}\right)$

$\theta = \pm\frac{\pi}{6} + n\pi$ where n is an integer

$= (6n\pm1)\frac{\pi}{6}$

9	In the diagram below, AB is the tangent to the circle at B and ADC is a straight line. If $AB:AD=2:1$, then what is the ratio of the area of ΔABD to the area of ΔCBD?

Answer: B



$AB^2 = AD \times AC$

$4x^2 = x \times AC \qquad \Rightarrow AC = 4x$

ΔABD is similar to ΔACB (two sides in ratio and included angle equal)

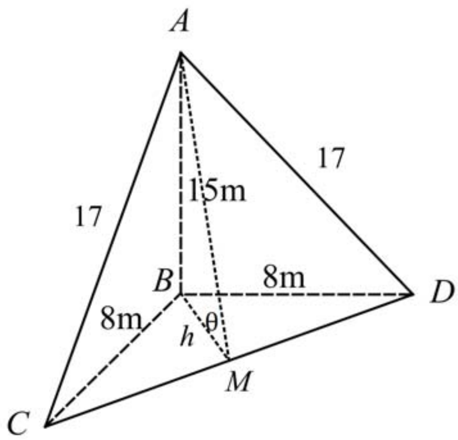
ratio of sides 1:2

ratio of areas 1:4

$\therefore$ area of ΔABD to area of ΔCBD is 1:3

10	In the figure below, AB is a vertical pole standing on horizontal ground BCD, where $\angle CBD = 90°$. If the angle between the plane ACD and the horizontal ground is θ, then what is the value of $\tan\theta$?

Answer: D



$CD^2 = \sqrt{8^2+8^2}$ Pythagoras Theorem

$= 8\sqrt{2}$

Area of $\Delta BCD = \frac{1}{2} \times 8 \times 8$

$= 32$

Area of $\Delta BCD = \frac{1}{2} \times 8\sqrt{2} \times h$

$\therefore \frac{1}{2} \times 8\sqrt{2} \times h = 32 \qquad \Rightarrow h = 4\sqrt{2}$

$\tan\theta = \frac{15}{4\sqrt{2}} \qquad \Rightarrow \theta \approx 69°$

Question 11 (15 marks)

(a)	Differentiate $\tan^{-1}\sqrt{x}$ with respect to x.	**2**

$$\frac{d}{dx}\left(\tan^{-1}\sqrt{x}\right)=\frac{1}{1+x}\times\frac{1}{2\sqrt{x}}$$

$$=\frac{1}{\sqrt{x}\left(2x+2\right)}$$

(b)	Consider the function $f(x)=1+\dfrac{2}{x-3}$ for $x>3$.	
	(i) What is the range of $f(x)$?	**1**
	(ii) Find the inverse function $f^{-1}(x)$ and state its domain.	**1**

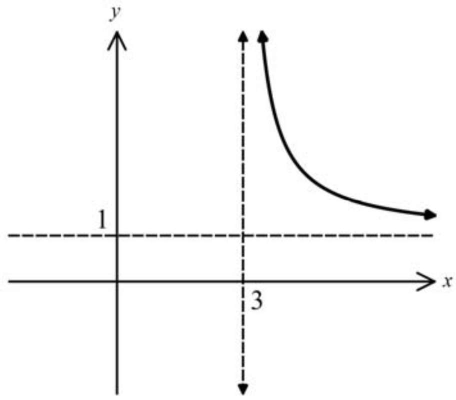
(i) $y>1$

(ii) $x=1+\dfrac{2}{y-3}$

$$x-1=\frac{2}{y-3}$$

$$y-3=\frac{2}{x-1}$$

$$y=\frac{2}{x-1}+3 \qquad \text{(domain: } x>1\text{)}$$



(c)	Use the substitution $u=3+x$ to find $\displaystyle\int\frac{x+1}{\sqrt{3+x}}\,dx$.	**3**

$$\int\frac{x+1}{\sqrt{3+x}}\,dx=\int\frac{u-2}{\sqrt{u}}\,du$$

$$=\int\left(u^{\frac{1}{2}}-2u^{-\frac{1}{2}}\right)du$$

$$=\frac{2}{3}u^{\frac{3}{2}}-2\times 2u^{\frac{1}{2}}+C$$

$$=\frac{2}{3}(3+x)^{\frac{3}{2}}-4(3+x)^{\frac{1}{2}}+C$$

$u=3+x$

$du=dx$

$x=u-3$

$x+1=u-2$

(d)	Solve $\dfrac{4}{x+2}\geq\dfrac{1}{x}$.	**3**

$$\frac{4}{x+2}\geq\frac{1}{x} \qquad x\neq 0,2$$

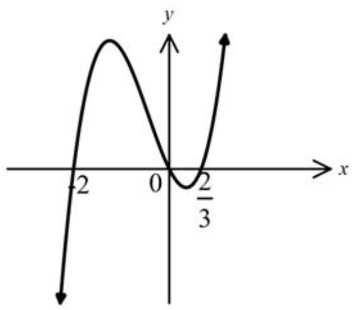
$$4(x+2)x^2\geq(x+2)^2x$$

$$4(x+2)x^2-x(x+2)^2\geq 0$$

$$x(x+2)\left[4x-(x+2)\right]\geq 0$$

$$x(x+2)(3x-2)\geq 0 \qquad x\neq 0,2$$

From the graph: $-2<x<0$, $x\geq\dfrac{2}{3}$

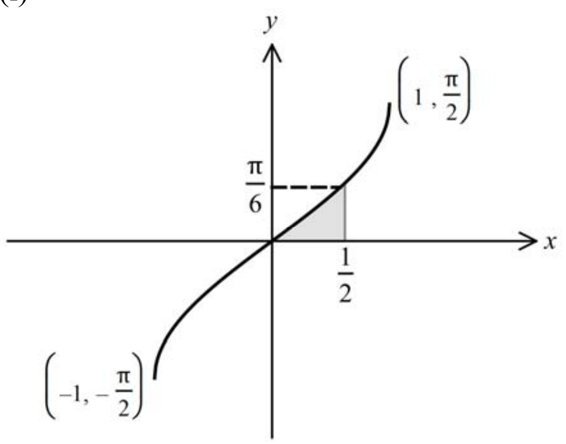


(e)	Find $\lim_{x\to 0}\frac{1-\cos 2x}{x^2}$. Show all working.	**2**

$$\lim_{x\to 0}\frac{1-\cos 2x}{x^2}=\lim_{x\to 0}\frac{1-\left(\cos^2 x-\sin^2 x\right)}{x^2}$$
$$=\lim_{x\to 0}\frac{\left(2\sin^2 x\right)}{x^2}$$
$$=2\times\lim_{x\to 0}\left(\frac{\sin x}{x}\right)^2$$
$$=2\times\lim_{x\to 0}\left(\frac{\sin x}{x}\right)\times\lim_{x\to 0}\left(\frac{\sin x}{x}\right)$$
$$=2\times 1\times 1$$
$$=2$$

(f)	(i)	Neatly sketch the graph of $y=\sin^{-1}x$.	**1**
	(ii)	By considering the graph in (i), find the exact value of $\int_0^{\frac{1}{2}}\sin^{-1}x\,dx$.	**2**

(i)



(ii) $$\int_0^{\frac{1}{2}}\sin^{-1}x\,dx=\frac{\pi}{6}\times\frac{1}{2}-\int_0^{\frac{\pi}{6}}\sin y\;dy$$
$$=\frac{\pi}{12}+\Big[\cos x\Big]_0^{\frac{\pi}{6}}$$
$$=\frac{\pi}{12}+\left(\frac{\sqrt{3}}{2}-1\right)$$
$$=\frac{\pi}{12}+\frac{\sqrt{3}}{2}-1$$

Question 12 (15 marks)

(a) During the early summer months, the rate of increase of the population P of cicadas is proportional to the excess of the population over 3000. The rate can be expressed by the differential equation $\frac{dP}{dt} = k(P - 3000)$ where t is the time in months and k is a constant. At the beginning of summer the population is 4000 and one month later, it is 10 000.

(i) Show that $P = 3000 + Ae^{kt}$ is a solution of the differential equation, where A is a constant. **1**

(ii) Find the value of A. **1**

(iii) Show that the value of k is $\log_e 7$. **1**

(iv) After how many weeks will the population reach half a million? (Assume 52 weeks in a year). **2**

(a) (i) $P = 3000 + Ae^{kt} \quad \Rightarrow \quad P - 3000 = Ae^{kt}$

$$\begin{aligned} \text{LHS} &= \frac{dP}{dt} \\ &= kAe^{kt} \\ &= k(P - 3000) \\ &= \text{RHS} \end{aligned}$$

$\therefore P = 3000 + Ae^{kt}$ is a solution.

(ii) When $t = 0,\ P = 4000$

$4000 = 3000 + A$

$A = 1000$

(iii) When P = 10 000 and $t = 1$, $10\,000 = 3000 + 1000e^{k}$

$$\begin{aligned} 7000 &= 1000e^{k} \\ e^{k} &= 7 \\ k &= \log_e 7 \end{aligned}$$

(iv)
$$\begin{aligned} P &= 500\,000 \\ 500\,000 &= 3000 + 1000e^{t\log_e 7} \\ 497000 &= 1000e^{t\log_e 7} \\ e^{t\log_e 7} &= 497 \\ t\log_e 7 &= \log_e 497 \\ t &= \frac{\log_e 497}{\log_e 7} \approx 3.19 \end{aligned}$$

$\frac{3.19}{12} \times 52 \approx 13.83$ weeks

It will take 14 weeks.

(b)	The angle between the line $4x+3y=8$ and the line $ax+by+c=0$ is $45°$. Find the possible values of the ratio $a:b$.	**3**

$m_1 = -\frac{4}{3}$ $\qquad m_2 = -\frac{a}{b}$

$$\tan 45° = \left| \frac{-\frac{4}{3} - \left(-\frac{a}{b}\right)}{1 + \frac{4}{3} \times \frac{a}{b}} \right|$$

$$1 = \left| \frac{\frac{-4b+3a}{3b}}{\frac{3b+4a}{3b}} \right|$$

$$1 = \frac{|-4b+3a|}{|3b+4a|}$$

$\therefore \ -4b+3a = 3b+4a$ or $-4b+3a = -(3b+4a)$

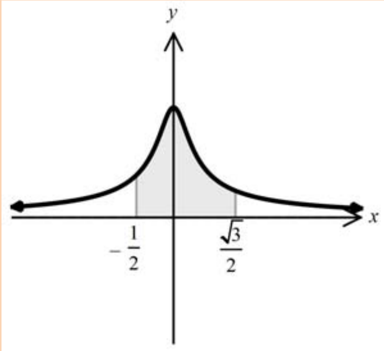
$a = -7b$ $\qquad 7a = b$

$\frac{a}{b} = -7$ $\qquad \frac{a}{b} = \frac{1}{7}$

Ratio of $a:b$ is -7:1 or 1:7

(c) The graph of $y = \dfrac{1}{\sqrt{1+4x^2}}$ is shown below. **3**

The shaded region in the diagram is bounded by the curve $y = \dfrac{1}{\sqrt{1+4x^2}}$, the $x-$axis and the lines $x = -\dfrac{1}{2}$ and $x = \dfrac{\sqrt{3}}{2}$. Find the volume of the solid of revolution formed when the shaded region is rotated about the $x-$axis.



$$V = \pi \int_a^b y^2 dx$$

$$= \pi \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{1+4x^2} dx$$

$$= \frac{\pi}{4} \int_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}} \frac{1}{\left(\frac{1}{2}\right)^2 + x^2} dx$$

$$= \frac{\pi}{4} \times 2 \left[\tan^{-1} 2x \right]_{-\frac{1}{2}}^{\frac{\sqrt{3}}{2}}$$

$$= \frac{\pi}{2} \left[\tan^{-1} \sqrt{3} - \tan^{-1}(-1) \right]$$

$$= \frac{\pi}{2} \left[\frac{\pi}{3} - \left(-\frac{\pi}{4} \right) \right]$$

$$= \frac{7\pi^2}{24} u^3$$

(d)	(i)	Express $3\sin x+\sqrt{3}\cos x$ in the form $A\sin(x+\alpha)$, where $0<\alpha<\dfrac{\pi}{2}$.	**2**
	(ii)	Hence, or otherwise, sketch the graph of $y=3\sin x+\sqrt{3}\cos x$ where $0\le x\le 2\pi$.	**2**

(i)

$$3\sin x+\sqrt{3}\cos x = A\sin x\cos\alpha + A\cos x\sin\alpha$$

equating coefficients: $A\cos\alpha=3$ $\quad$ $A\sin\alpha=\sqrt{3}$

$$A^2\cos^2\alpha + A^2\sin^2\alpha = 12$$

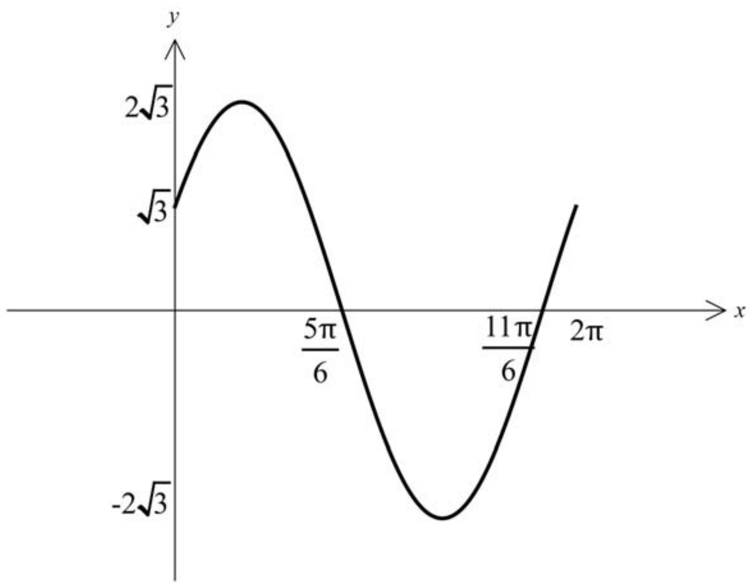
$$A^2 = 12 \quad\Rightarrow\quad A = 2\sqrt{3}$$

Also, $\dfrac{A\sin\alpha}{A\cos\alpha}=\dfrac{\sqrt{3}}{3}$

$$\tan\alpha=\frac{1}{\sqrt{3}} \quad\Rightarrow\quad \alpha=\frac{\pi}{6}$$

$$3\sin x+\sqrt{3}\cos x = 2\sqrt{3}\sin\left(x+\frac{\pi}{6}\right)$$

(ii)



Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The acceleration of a particle as it moves in a straight line is given by $\frac{d^2x}{dt^2} = -12\cos 2t$ and the particle started from rest at the point $x = 3$.

(i) Find the displacement, x, of the particle as a function of t. **2**

(ii) At what time is the particle first at $x = 0$, and moving towards its initial position? **1**

(i) $\ddot{x} = -12\cos 2t$

$\dot{x} = -\frac{12\sin 2t}{2} + c$

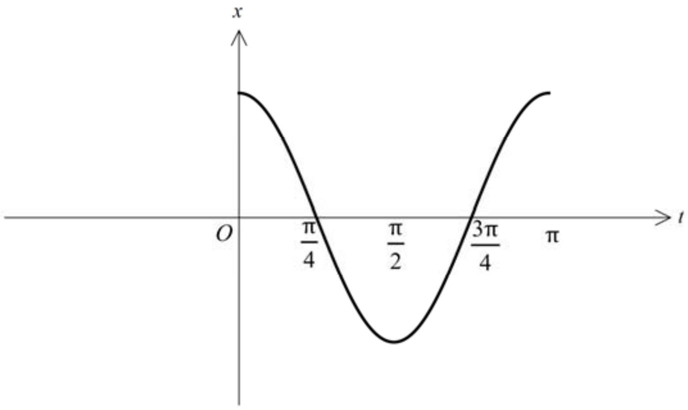
$= -6\sin 2t + c$ when $t = 0, v = 0$ $\Rightarrow c = 0$

$\dot{x} = -6\sin 2t$

$x = \frac{6\cos t2t}{2} + c_1$ when $t = 0, x = 3$ $\Rightarrow 3 = 3 + c_1$ $\therefore c_1 = 0$

$x = 3\cos 2t$

(ii)

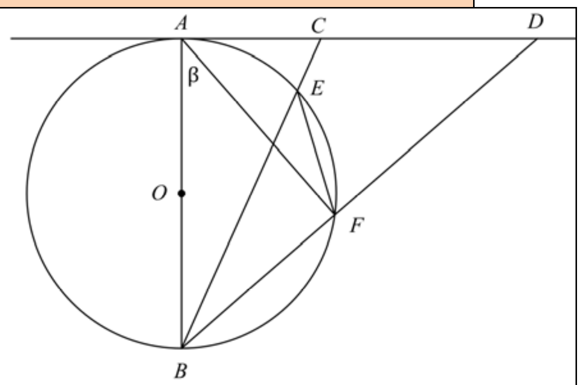


From the graph, $x = 0$ when particle is moving towards its initial position at $t = \frac{3\pi}{4}$ seconds.

(b) In the diagram below, the straight line ACD is a tangent at A to the circle with centre O. The interval AOB is a diameter of the circle The intervals BC and BD meet the circle at E and F respectively. Let $\angle BAF = \beta$.

(i) Explain why $\angle ABF = 90° - \beta$. **1**

(ii) Prove that the quadrilateral $CDFE$ is cyclic. **3**



(i) $\angle AFB = 90°$(angle in a semi-circle)

$\angle ABF + \beta + 90° = 180°$ (angle sum of triangle)

$\angle ABF = 90 - \beta$

(ii)

$\angle BAF = \angle BEF$ (angle in the same segment)

$= \beta$

$\angle BAD = 90°$ (tangent perpendicular to radius)

In ΔBAD, $\angle ADB + (90-\beta) + 90 = 180$

$\angle ADB = \beta$

$\angle CDF = \angle FEB = \beta$

$\therefore CDFE$ is cyclic (exterior angle of a cyclic quadrilateral)

c) A ball on a spring is moving in simple harmonic motion with a vertical velocity v cms^{-1} given by $v^2 = -8 + 24y - 4y^2$ where y is the vertical displacement in cm.

(i) Find the acceleration of the ball in terms of y. **2**

(ii) Find the centre of motion of the ball. **1**

(iii) Find the period of the oscillation. **1**

(i)

$$v^2 = -8 + 24y - 4y^2$$

$$\frac{1}{2}v^2 = -4 + 12y - 2y^2$$

$$\therefore \ddot{y} = \frac{d}{dy}(-4 + 12y - 2y^2)$$

$$= 12 - 4y$$

$$= -4(y-3)$$

(ii) centre of motion is at $x = 3$.

iii)

$$n^2 = 4$$

$$n = 2$$

$$\therefore T = \frac{2\pi}{n} = \frac{2\pi}{2} = \pi \text{ sec}$$

(d) (i) Show that $n + (n+1) + (n+2) + \ldots\ldots + (2n+1) = \frac{(3n+1)(n+2)}{2}$ **1**

(ii) Hence prove by mathematical induction that for all integers $n \geq 1$, **3**

$$1 + (2+3) + (3+4+5) + \ldots. + [n + (n+1) + (n+2) + \ldots + (2n-1)] = \frac{n^2}{2}(n+1).$$

(i) arithmetic series: $a = n$, $l = (2n+1)$, number of terms $= (2n+1) - n + 1$
$= n + 2$

$$\therefore n + (n+1) + (n+2) + \ldots(2n+1) = \frac{(n+2)}{2}\left(n + (2n+1)\right)$$

$$= \frac{(3n+1)(n+2)}{2}$$

(ii) Prove true for n=1

LHS=1 RHS=$\frac{1}{2}(1+1) = 1$

LHS=RHS $\therefore$ true for $n = 1$.

Assume true for $n = k$.

$$1 + (2+3) + (3+4+5) + \ldots. + [k + (k+1) + (k+2) + \ldots + (2k-1)] = \frac{k^2}{2}(k+1).$$

Prove true for $n = k+1$

Required to prove:

$$1 + (2+3) + \ldots. + [k + (k+1) + \ldots + (2k-1)] + [(k+1) + ((k+1)+1) + \ldots + (2(k+1)-1)] = \frac{(k+1)^2}{2}((k+1)+1).$$

$$1 + (2+3) + \ldots. + [k + (k+1) + \ldots + (2k-1)] + [(k+1) + (k+2) + \ldots + (2k+1))] = \frac{(k+1)^2}{2}(k+2)$$

LHS $= 1 + (2+3) + \ldots. + [k + (k+1) + \ldots + (2k-1)] + [(k+1) + (k+2) + \ldots + (2k+1)]$

$= \frac{k^2}{2}(k+1) + \left[(k+1) + (k+2) + ... + (2k+1)\right]$ from assumption

$= \frac{k^2}{2}(k+1) + \frac{(3k+1)(k+2)}{2} - k$ from part (i)

$= \frac{k^2(k+1) + 3k^2 + 7k + 2 - 2k}{2}$

$= \frac{k^2(k+1) + (3k+2)(k+1)}{2}$

$= \frac{(k+1)\left[k^2 + 3k + 2\right]}{2}$

$= \frac{(k+1)(k+1)(k+2)}{2}$

$= \frac{(k+1)^2(k+2)}{2}$

$=$ RHS

Hence, if the result is true for $n = k$, then it is true for $n = k+1$

$\therefore$ the result is true for all $n \geq 1$ by mathematical induction

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a)	(i)	Show that the equation of the normal to the parabola $x^2 = 4y$ at the point $P(2p, p^2)$ is $x + py = 2p + p^3$.	2
	(ii)	S is the focus of the parabola $x^2 = 4y$ and T is a point on the normal such that ST is perpendicular to the normal. Write down the equation of ST.	1
	(iii)	Prove that the locus of T is a parabola with vertex (0,1) and with focal length $\frac{1}{4}$ that of the parabola $x^2 = 4y$.	3

(i)

$$x^2 = 4y \qquad \text{(focal length } a = 1)$$

$$y = \frac{x^2}{4}$$

$$\frac{dy}{dx} = \frac{x}{2}$$

At P, $x = 2p$, $\frac{dy}{dx} = \frac{2p}{2} = p$ (gradient of tangent at P)

$\therefore$ Gradient of normal $= -\frac{1}{p}$

Equation of normal:

$$y - y_1 = m(x - x_1)$$

$$y - p^2 = -\frac{1}{p}(x - 2p)$$

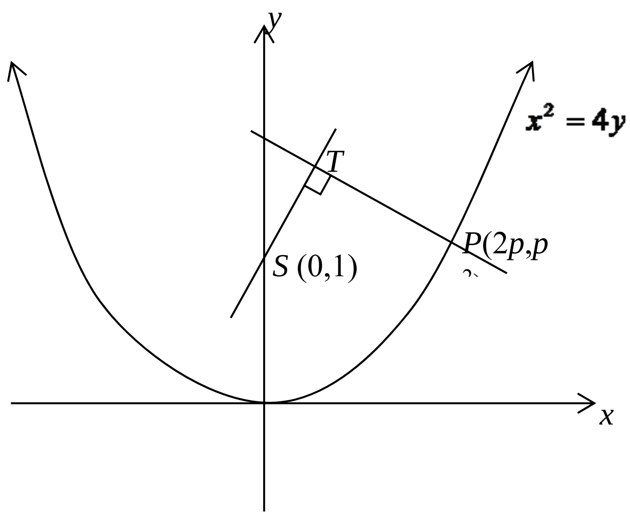
$$py - p^3 = -x + 2p$$

$$x + py = 2p + p^3$$

(ii)

Gradient of $ST = p$

Equation of ST: $y = px + 1$



(iii)

Since T is the intersection of ST and PT

$$\left.\begin{aligned} y &= px + 1 \\ x + py &= 2p + p^3 \end{aligned}\right\} \begin{aligned} &\text{..........(1)} \\ &\text{..........(2)} \end{aligned}$$

$$\underline{py = p^2x + p}\text{...........}1 \times p\text{......(3)}$$

$$x = p + p^3 - p^2x\text{....(2)} - \text{(3)}$$

$$x(1 + p^2) = p(1 + p^2)$$

$$x = p$$

$$y = p^2 + 1$$

$\therefore$ Cartesian equation of T:

$y = x^2 + 1$, *i.e.* $x^2 = y - 1$

Vertex (0,1), focal length $a = \frac{1}{4}$

(b) (i) Show that $1+e^{-x}=\dfrac{e^x+1}{e^x}$. **1**

(ii) The velocity v of a particle moving along the x-axis is given by $\dfrac{dx}{dt}=1+e^{-x}$ where x is the displacement of the particle from the origin in metres. Initially the particle is at the origin. **3**

Find the time taken by the particle to reach a velocity of $1\frac{1}{2}$ ms^{-1}.

(i) $\text{RHS}=\dfrac{e^x+1}{e^x}$

$=\dfrac{e^x}{e^x}+\dfrac{1}{e^x}$

$=1+e^{-x}$

$=\text{LHS}$

(ii) $\dfrac{dx}{dt}=1+e^{-x}$

$\dfrac{dt}{dx}=\dfrac{1}{1+e^{-x}}$

$=\dfrac{e^x}{1+e^x}$

$t=\displaystyle\int\frac{e^x}{1+e^x}\,dx \quad =\log(1+e^x)+c$

At $t=0, x=0$ $\quad 0=\log_e(1+1)+c \quad \Rightarrow c=-\log_e 2$

$t=\log_e(1+e^x)-\log_e 2$

$=\log_e\left(\dfrac{1+e^x}{2}\right)$

when $v=\dfrac{3}{2}$, $\quad \dfrac{3}{2}=1+e^{-x}$

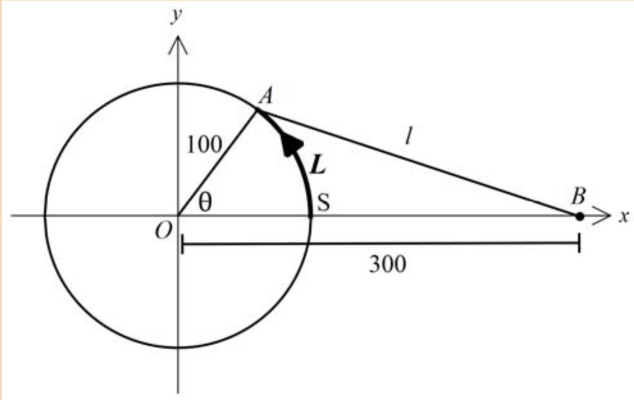
$e^{-x}=\dfrac{1}{2} \quad \Rightarrow e^x=2 \Rightarrow x=\log_e 2$

when $x=\log_e 2$, $t=\log_e(1+e^{\ln 2})-\log_e 2$

$=\log_e\dfrac{3}{2}$

It will take $\log_e\left(\dfrac{3}{2}\right)$ seconds.

(c) A runner sprints in an anticlockwise direction around a circular track of radius 100 metres with centre O at a constant speed of 5 m/s. The runner's friend is standing at B, a distance of 300 metres from the centre of the track.



The runner starts at S and t seconds later is at point A. The distance AB between the two friends is l and the distance covered by the runner on the track is L.
Let the angle subtended by the arc SA be θ.

(i) From the diagram the coordinates of A are $(100\cos\theta, 100\sin\theta)$. **1**
Use the distance formula to show that $l = 100\sqrt{10-6\cos\theta}$.

(ii) At what rate is the distance between the friends changing at the moment when the runner is 250 metres from his friend and getting closer to him. **4**

(i)

$A\ (100\cos\theta, 100\sin\theta) \qquad B\ (300, 0)$

$$l^2 = (100\cos\theta - 300)^2 + (100\sin\theta - 0)^2$$
$$= 100^2\cos^2\theta - 2\times100\times300\cos\theta + 300^2 + 100\sin^2\theta$$
$$= 100^2(\cos^2\theta + \sin^2\theta) + 300^2 - 2\times100\times300\cos\theta$$
$$l = \sqrt{100000 - 60000\cos\theta}$$
$$= 100\sqrt{10-6\cos\theta}$$

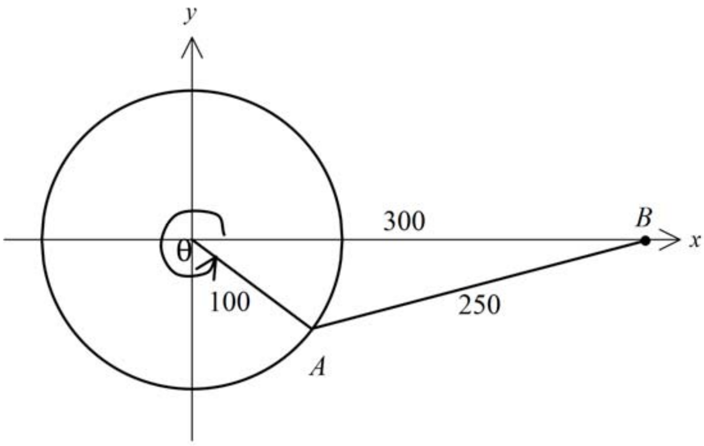
(ii) $\dfrac{dL}{dt} = 5 \qquad \dfrac{dl}{d\theta} = \dfrac{600\sin\theta}{2\sqrt{10-6\cos\theta}}$

$L = 100\theta$ (arc length) $\Rightarrow \dfrac{dL}{d\theta} = 100$

$$\frac{dl}{dt} = \frac{dl}{d\theta}\times\frac{d\theta}{dL}\times\frac{dL}{dt}$$
$$= \frac{600\sin\theta}{2\sqrt{10-6\cos\theta}}\times\frac{1}{100}\times5$$
$$= \frac{15\sin\theta}{\sqrt{10-6\cos\theta}}$$

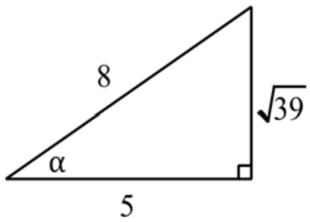
When $l = 250$, $\quad 250 = 100\sqrt{10-6\cos\theta}$

$$\left(\frac{5}{2}\right)^2 = 10 - 6\cos\theta \qquad \Rightarrow \cos\theta = \frac{5}{8}$$



θ is in the 4th quadrant.

$\cos\theta = \frac{5}{8},\ \sin\theta = -\frac{\sqrt{39}}{8}$



$$\frac{dl}{dt} = \frac{15\left(-\frac{\sqrt{39}}{8}\right)}{\sqrt{10-6\times\frac{5}{8}}}$$

$$= -\frac{\left(\frac{15\sqrt{39}}{8}\right)}{\sqrt{\frac{25}{4}}} = -\frac{15\sqrt{39}}{8}\times\frac{2}{5}$$

$$= -\frac{3\sqrt{39}}{4}$$

The distance between the two friends is decreasing a rate of $\frac{3\sqrt{39}}{4}$ms $\approx$ 4.7 ms