Penrith Selective High School

2017

Trial Higher School Certificate Examination

Mathematics Extension 1

General Instructions
- Reading time – 5 minutes
- Working time – 2 hours
- Write using black or blue pen Black pen is preferred
- Board-approved calculators may be used
- A Reference Sheet is provided
- In Questions 11-14, show relevant mathematical reasoning and/or calculations in the writing booklets provided
- All diagrams are not to scale

Total Marks – 70

Section I Pages 3–6

10 marks
- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 7–12

60 marks
- Attempt Questions 11–14
- Allow about 1 hour 45 minutes for this section

Assessor: X. Chirgwin

Student Number: ____

Students are advised that this is a trial examination only and cannot in any way guarantee the content or format of the 2017 Higher School Certificate Examination.

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

Q1. A is the point $(2,-3)$ and B is the point $(-5,11)$. Which of the following are the coordinates of a point that divides AB internally in the ratio of $4:3$?

(A) $(3,-1)$

(B) $(-1,3)$

(C) $(5,-2)$

(D) $(-2,5)$

Q2. What is the natural domain of the function $f(x) = 2\sqrt{x+1} - \sqrt{x-2}$?

(A) $x \leq -1$

(B) $x \geq 2$

(C) $x \leq -1$ or $x \geq 2$

(D) $-1 \leq x \leq 2$

Q3. What is the acute angle between the tangents drawn to the curve $y = e^{3x}$ at the points where $x = 0$ and $x = 1$ to the nearest degree?

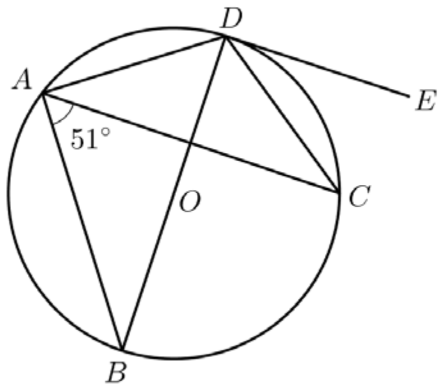
(A) 17°

(B) 19°

(C) 42°

(D) 44°

Q4.



NOT TO SCALE

DE is a tangent to the circle with centre O. Given that $\angle BAC = 51°$, the size of $\angle EDC$ is:

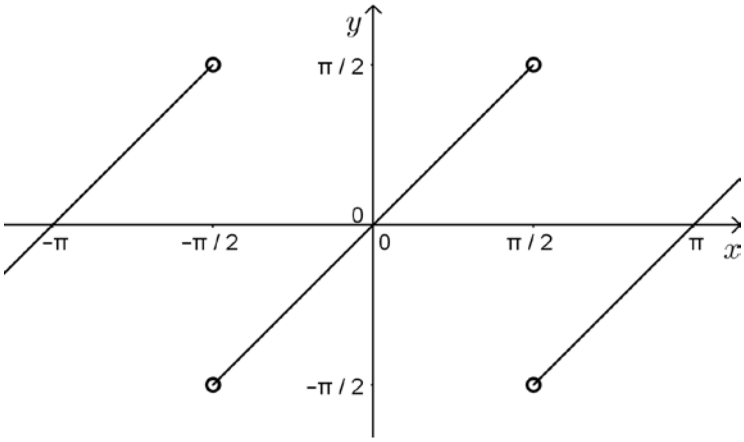
(A) $28°$

(B) $39°$

(C) $40°$

(D) $51°$

Q5.



The possible equation of the graph shown above is:

(A) $y = \tan^{-1}(\tan x)$

(B) $y = \tan(\tan^{-1} x)$

(C) $y = \sin(\sin^{-1} x)$

(D) $y = \sin^{-1}(\sin x)$

Q6. A parabola has the parametric equations $x = -8t^2$, $y = 16t$. The coordinates of the focus for this parabola is:

(A) $(-8, 0)$

(B) $(8, 0)$

(C) $(0, -8)$

(D) $(0, 8)$

Q7. The lock for a gate opens with a five-digit code. Each wheel rotates through the digits 0 to 9. The percentage of five-digit codes that have no repeated digits is closest to:

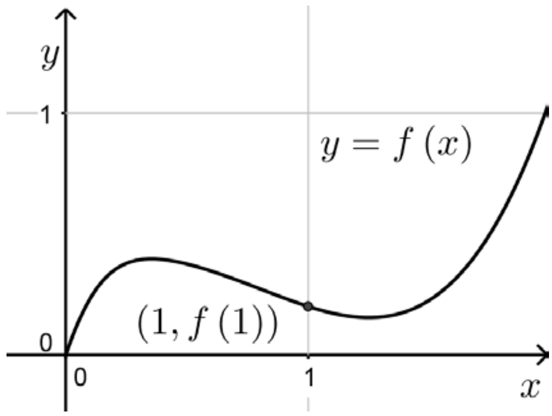
(A) 17%

(B) 30%

(C) 50%

(D) 63%

Q8. The diagram shows $y = f(x)$.



Which of the following is a correct statement?

(A) $f(1) < f'(1) < f''(1) < 1$

(B) $1 < f(1) < f'(1) < f''(1)$

(C) $f''(1) < f(1) < 1 < f'(1)$

(D) $f'(1) < f(1) < f''(1) < 1$

Q9. The polynomial $P(x)$ has degree 6 and the polynomial $Q(x)$ has degree 3. If you divide $P(x)$ by $Q(x)$, the remainder may have degree:

(A) 0

(B) 0 or 1

(C) $0, 1$ or 2

(D) $0, 1, 2$ or 3

Q10. An object moves in a straight line so that at time t its displacement from a fixed origin is x and its velocity is v. The acceleration is $7 - 3x^2$. Which of the following is the correct equation for the velocity given that $v = 4$ when $x = 2$?

(A) $v = \sqrt{14x - 2x^3 + 4}$

(B) $v = \sqrt{14x - x^3 - 4}$

(C) $v = \sqrt{7x - x^3 + 10}$

(D) $v = 7x - x^3 - 2$

Section II

60 Marks
Attempt Questions 11–14
Allow about 1 hour and 45 minutes for this section

Answer each question in a SEPARATE booklet. Extra writing booklets are available.

In Questions 11–14, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

a) Solve $\dfrac{1}{3x+2} \geq 2$ **2**

b) Using the substitution $u = \cos x$, evaluate **3**

$$\int_0^{\frac{\pi}{2}} 3 \sin x \cos^5 x \, dx$$

c) Two school captains, two vice captains and three other students are sitting around a circular table. What is the probability that the two school captains sit next to each other? **2**

d) Find the constant term in the expansion of $\left(5x^4 - \dfrac{1}{2x}\right)^{10}$ **3**

e) Show that $\dfrac{d}{dx}\left(xe^{\tan^{-1} x}\right) = \left(\dfrac{x^2 + x + 1}{x^2 + 1}\right)e^{\tan^{-1} x}$ **2**

f) Given that $\ln|2x - 1| = \tan x$ has a root close to 4.2, use one application of Newton's method to obtain a better approximation of the root. Round your answer to 3 decimal places. **3**

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet.

a) If α, β and γ are the roots of $3x^3 - 4x + 6 = 0$. Evaluate:

i) $\alpha^{-1} + \beta^{-1} + \gamma^{-1}$ **2**

ii) $\alpha^3\beta\gamma + \alpha\beta^3\gamma + \alpha\beta\gamma^3$ **2**

b) N is the number of songbirds in a certain population at time t years. The population size N satisfies the equation $\frac{dN}{dt} = k(N - 750)$, for some constant k.

i) Verify by differentiation that $N = 750 + Ae^{kt}$ is a solution of the equation $\frac{dN}{dt} = k(N - 750)$, where A is a constant. **1**

ii) Initially there are 3500 songbirds but after 3 years there are only 2400 left. Find the exact values of A and k. **2**

iii) Find the population after 7 years, round your answer to the nearest whole number. **1**

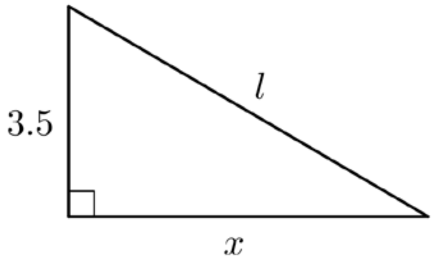
iv) If this trend continues, what would the value of the population eventually become? **1**

v) Hence, sketch the graph of population size against time. **1**

Question 12 continues on page 8

c) A boat is attached by a rope to a jetty 3.5 metres above the front end of the boat. The boat is being pulled in towards the jetty by the rope at a rate of 1 m/s.

Let x be the horizontal distance between the boat and the jetty, and l be the length of the rope as shown below.



Not to scale

i) Show that $2l = \sqrt{49 + 4x^2}$ **1**

ii) Show that the change in the position of the boat is given by **2**

$$\frac{dx}{dt} = -\frac{\sqrt{49 + 4x^2}}{2x}$$

iii) At what rate is the boat approaching the jetty when 4 metres of rope still remains to be pulled in? Leave your answer in exact form. **2**

End of Question 12

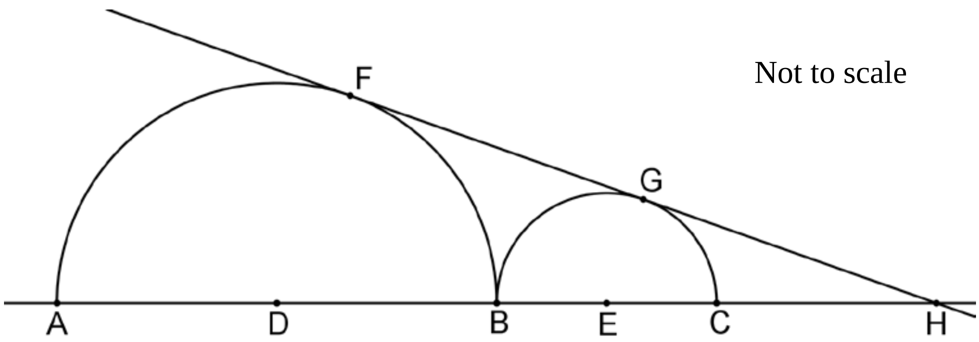
Question 13 (15 marks) Use a SEPARATE writing booklet.

a) Prove by mathematical induction that **3**

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots\cdots+\frac{n}{(n+1)!}=\frac{(n+1)!-1}{(n+1)!}$$

n is any positive integer, where $n \geq 1$.

b) The diagram below shows two semicircles centred at D and E with radii 5 cm and 3 cm respectively. FGH is a common tangent to both semicircles.



i) Prove that $\Delta GEH|||\Delta FDH$. **2**

ii) Show that EH is 12 cm. **1**

iii) Show that FBG lie on another circle with FG as its diameter. **3**

Question 13 continues on page 11

c) $P(2ap, ap^2)$ and $Q(2aq, aq^2)$ are extremities of a focal chord for the parabola $x^2 = 4ay$.

i) Given that the equation of the chord PQ is $2y = (p + q)x - 2apq$ [DO NOT PROVE THIS], show that $pq = -1$. **1**

ii) Given that the equation of the tangent at P is $y = px - ap^2$ and the tangent at Q is $y = qx - aq^2$[DO NOT PROVE THESE]. **2**

Show that the tangents at P and Q meet at $R(a(p + q), -a)$.

iii) Hence state the locus of R. **1**

iv) Show that the chord PQ has length **2**

$$a\left(p + \frac{1}{p}\right)^2$$

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet.

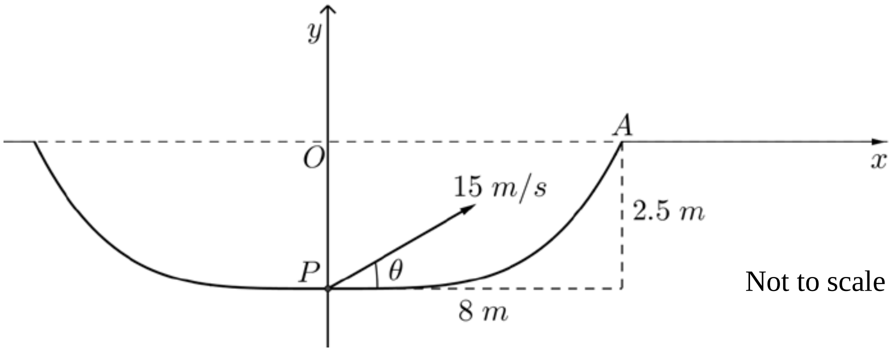
a) Find the exact value of $\sin\left[\cos^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(-\frac{2}{7}\right)\right]$ **3**

b) Given that $(1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k$, show that:

i) $\sum_{k=0}^{2n} \binom{2n}{k} = 4^n$ **1**

ii) $\sum_{k=0}^{2n} \binom{2n}{k} \frac{1}{k+1} = \frac{2^{2n+1}-1}{2n+1}$ **3**

c) A golf ball is lying at P, at the middle of the bottom of a sand bunker which is surrounded by level ground. The point A is at the edge of the bunker 8 m from O. The golf ball is hit with initial velocity of 15 m/s and P is 2.5 m below O. Ignoring air resistance.



i) Using $g = -10\ m/s^2$, show that the golf ball's trajectory at time t seconds after being hit may be defined by the equations: **2**

$x = 15t\cos\theta$ and $y = -5t^2 + 15t\sin\theta - 2.5$

where x and y are the horizontal and vertical displacements, in metres, of the ball from the origin O shown in the diagram, and θ is the angle of projection.

ii) Given $\theta = 30°$, how far to the right of A will the ball land? Round your answer to 2 decimal places. **3**

iii) Find the range of values of θ, to the nearest degree, at which the ball must be hit so that it will land to the right of A. **3**

End of Paper

2017 Mathematics Extension 1 Trial Solution

Section 1:

Q1. D Q2. B Q3. A Q4. B Q5. A Q6. A Q7. B Q8. D Q9. C Q10. A

Q1. D

$x = \frac{4 \times (-5) + 3 \times 2}{4 + 3}$

$x = \frac{-14}{7}$

$x = -2$

$y = \frac{4 \times 11 + 3 \times (-3)}{4 + 3}$

$y = \frac{35}{7}$

$y = 5$

$(-2, 5)$

Q2. B

$f(x) = 2\sqrt{x+1} - \sqrt{x-2}$

For $2\sqrt{x+1}$,

$x + 1 \geq 0$

$x \geq -1$

For $\sqrt{x-2}$,

$x - 2 \geq 0$

$x \geq 2$

For $2\sqrt{x+1} - \sqrt{x-2}$, domain must satisfy both $2\sqrt{x+1}$ and $\sqrt{x-2}$

$\therefore x \geq 2$

Q3. A

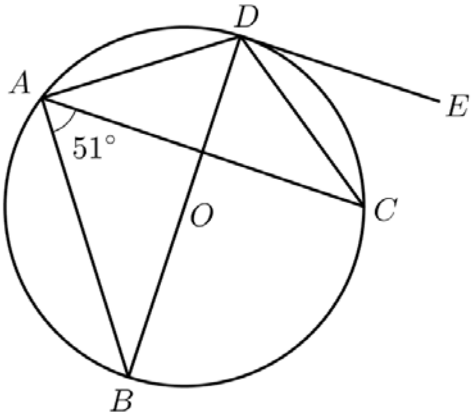
$y = e^{3x}, y' = 3e^{3x}$

At $x = 0, m_1 = 3$; At $x = 1, m_2 = 3e^3$

$\tan\theta = \frac{3e^3 - 3}{1 + 9e^3}$

$\theta = 17°$

Q4. B



$\angle BDC = \angle BAC = 51°$ (angles in the same segment)

$\angle ODE = 90°$ (radius $OD \perp$ tangent DE)

$\angle EDC = \angle ODE - \angle BDC$

$\angle EDC = 90° - 51°$

$\angle EDC = 39°$

Q5. A

Q6. A

$x = -8t^2, y = 16t$

$y^2 = -32x$

$a = 8$

Focus is $(-8, 0)$

Q7. B

$$\frac{10 \times 9 \times 8 \times 7 \times 6}{10^5} \times 100\% = 30\%$$

Q8. D

$0 < f(1) < 1, f'(1) < 0, f''(x) > 0$

$\therefore f'(1) < f(1) < f''(1) < 1$

Q9. C

Q10. A

$$\ddot{x} = 7 - 3x^2$$
$$\frac{d}{dx}\left(\frac{1}{2}v^2\right) = 7 - 3x^2$$
$$\frac{1}{2}v^2 = 7x - x^3 + C$$
$$v^2 = 14x - 2x^3 + C$$

At $x = 2, v = 4$

$16 = 14 \times 2 - 2 \times 2^3 + C$
$C = 4$

$v^2 = 14x - 2x^3 + 4$
$v = \pm\sqrt{14x - 2x^3 + 4}$

Given condition that $x = 2, v = 4$
$v > 0$

$\therefore v = \sqrt{14x - 2x^3 + 4}$

Section 2

Q11.

a)

$$\frac{1}{3x+2} \geq 2 \qquad\qquad x \neq -\frac{2}{3}$$
$$(3x+2) \geq 2(3x+2)^2$$
$$2(3x+2)^2 - (3x+2) \leq 0$$
$$(3x+2)(6x+4-1) \leq 0$$
$$(3x+2)(6x+3) \leq 0$$

$$\therefore -\frac{2}{3} < x \leq -\frac{1}{2}$$

b)

$$I = \int_0^{\frac{\pi}{2}} 3\sin x \cos^5 x \, dx$$

Let $u = \cos x$
$du = -\sin x \, dx$
$x = \frac{\pi}{2}, \quad u = 0$
$x = 0, \quad u = 1$

$$I = \int_1^0 -3u^5 du$$
$$= \int_0^1 3u^5 du$$
$$= \left[\frac{3u^6}{6}\right]_0^1$$
$$= \frac{1}{2}$$

c)

7 people sitting around a table is 6!

Two school captains sitting next to each other is $2! \times 5!$

$$P = \frac{2! \times 5!}{6!} = \frac{1}{3}$$

d)

$$ {}^{10}C_r(5x^4)^{10-r} \times \left(-\frac{1}{2}x^{-1}\right)^r = {}^{10}C_r 5^{10-r} \times \left(-\frac{1}{2}\right)^r \times x^{40-4r} \times x^{-r} $$
$$ = {}^{10}C_r 5^{10-r} \times \left(-\frac{1}{2}\right)^r \times x^{40-5r} $$

For the constant term,

$$ 40 - 5r = 0 $$
$$ r = 8 $$

$$ {}^{10}C_8 5^{10-8} \times \left(-\frac{1}{2}\right)^8 = \frac{1125}{256} $$

e)

$$ \frac{d}{dx}\left(xe^{\tan^{-1}x}\right) = \frac{xe^{\tan^{-1}x}}{x^2+1} + e^{\tan^{-1}z} $$
$$ = \left(\frac{x}{x^2+1} + 1\right)e^{\tan^{-1}x} $$
$$ = \left(\frac{x^2+x+1}{x^2+1}\right)e^{\tan^{-1}x} $$

f)

$$ \ln|2x-1| = \tan x $$
$$ f'(x) = \frac{2}{2x-1} - \sec^2 x $$
$$ x_1 = 4.2 - \frac{\ln(2\times 4.2 - 1) - \tan 4.2}{\frac{2}{2\times 4.2 - 1} - \sec^2 4.2} $$
$$ x_1 = 4.258 \text{ (3 d.p.)} $$

Question 12

a) i)

$3x^3 - 4x + 6 = 0$

$\alpha + \beta + \gamma = 0$

$\alpha\beta + \beta\gamma + \alpha\gamma = -\frac{4}{3}$

$\alpha\beta\gamma = -\frac{6}{3} = -2$

$$\begin{aligned}\alpha^{-1} + \beta^{-1} + \gamma^{-1} &= \frac{1}{\alpha} + \frac{1}{\beta} + \frac{1}{\gamma} \\ &= \frac{\alpha\beta + \beta\gamma + \alpha\gamma}{\alpha\beta\gamma} \\ &= \frac{-\frac{4}{3}}{-2} \\ &= \frac{2}{3}\end{aligned}$$

ii)

$$\begin{aligned}\alpha^3\beta\gamma + \alpha\beta^3\gamma + \alpha\beta\gamma^3 &= \alpha\beta\gamma(\alpha^2 + \beta^2 + \gamma^2) \\ &= \alpha\beta\gamma[(\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \beta\gamma + \alpha\gamma)] \\ &= -2\left[0 - 2 \times \left(-\frac{4}{3}\right)\right] \\ &= -\frac{16}{3}\end{aligned}$$

b) i)

$$N = 750 + Ae^{kt}$$
$$\frac{dN}{dt} = kAe^{kt}$$
$$\frac{dN}{dt} = k(N - 750)$$

ii)

When $t = 0, N = 3500$

$3500 = 750 + Ae^0$

$\therefore A = 2750$

When $t = 3, N = 2400$

$$2400 = 750 + 2750e^{3k}$$
$$1650 = 2750e^{3k}$$
$$\frac{1650}{2750} = e^{3k}$$
$$\ln\left(\frac{3}{5}\right) = 3k$$
$$k = \frac{\ln 0.6}{3}$$

iii)

$N = 750 + 2750e^{7\times\frac{\ln 0.6}{3}}$
$N \approx 1585$

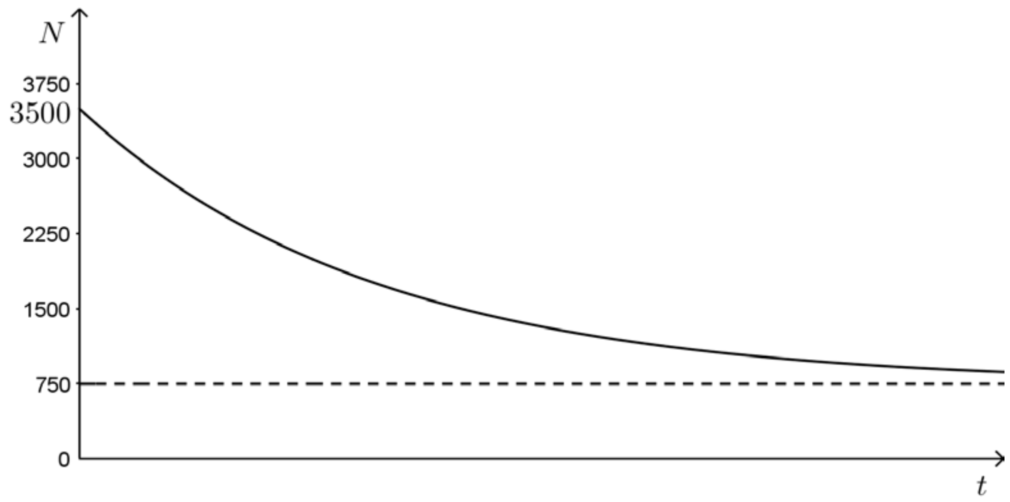
iv)

Since $k = -0.170 \ldots$

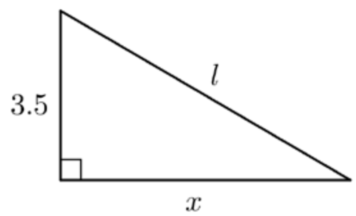
As $t \to +\infty$, $e^{kt} \to 0$

$\therefore N \to 750$

v)



c) i)



$l^2 = 3.5^2 + x^2$ (by Pythagoras' theorem)

$l = \sqrt{\dfrac{49 + 4x^2}{4}}$

$l = \dfrac{1}{2}\sqrt{49 + 4x^2}$

$2l = \sqrt{49 + 4x^2}$

ii)

$$\frac{dl}{dx} = \frac{1}{2} \times \frac{1}{2} \times 8x \times \frac{1}{\sqrt{49 + 4x^2}}$$
$$\frac{dl}{dx} = \frac{2x}{\sqrt{49 + 4x^2}}$$

$$\frac{dl}{dt} = \frac{dl}{dx} \times \frac{dx}{dt}$$
$$-1 = \frac{2x}{\sqrt{49 + 4x^2}} \times \frac{dx}{dt}$$
$$\frac{dx}{dt} = -\frac{\sqrt{49 + 4x^2}}{2x}$$

iii)

At $l = 4$

$$16 = 3.5^2 + x^2$$
$$x = \sqrt{\frac{15}{4}}$$
$$x = \frac{\sqrt{15}}{2}$$

$$\frac{dx}{dt} = -\frac{\sqrt{49 + 4 \times \frac{15}{4}}}{2 \times \frac{\sqrt{15}}{2}}$$
$$\frac{dx}{dt} = -\frac{\sqrt{49 + 4 \times \frac{15}{4}}}{2 \times \frac{\sqrt{15}}{2}}$$
$$\frac{dx}{dt} = -\frac{8}{\sqrt{15}} \text{ m/s}$$

The boat is approaching the jetty at a rate of $\frac{8}{\sqrt{15}}$ m/s

Q13

a)

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots\cdots+\frac{n}{(n+1)!}=\frac{(n+1)!-1}{(n+1)!}$$

1. Prove statement is true for $n=1$

$$LHS=\frac{1}{2!}=\frac{1}{2}$$

$$RHS=\frac{(1+1)!-1}{(1+1)!}=\frac{1}{2}$$

$$LHS=RHS$$

∴ Statement is true for $n=1$

2. Assume statement is true for $n=k$ where k is some positive integer

$$\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots\cdots\frac{k}{(k+1)!}=\frac{(k+1)!-1}{(k+1)!}$$

3. Prove statement is true for $n=k+1$

$$i.e.\ \frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots\cdots\frac{k}{(k+1)!}+\frac{k+1}{(k+2)!}=\frac{(k+2)!-1}{(k+2)!}$$

$$LHS=\frac{1}{2!}+\frac{2}{3!}+\frac{3}{4!}+\cdots\cdots\frac{k}{(k+1)!}+\frac{k+1}{(k+2)!}$$

$$LHS=\frac{(k+1)!-1}{(k+1)!}+\frac{k+1}{(k+2)!}\quad\text{(from step 2)}$$

$$LHS=\frac{(k+1)!-1}{(k+1)!}+\frac{k+1}{(k+2)!}$$

$$LHS=\frac{(k+1)!-1}{(k+1)!}\times\frac{(k+2)}{(k+2)}+\frac{k+1}{(k+2)!}$$

$$LHS=\frac{(k+2)!-(k+2)}{(k+2)!}+\frac{k+1}{(k+2)!}$$

$$LHS=\frac{(k+2)!-(k+2)+k+1}{(k+2)!}$$

$$LHS=\frac{(k+2)!-k-2+k+1}{(k+2)!}$$

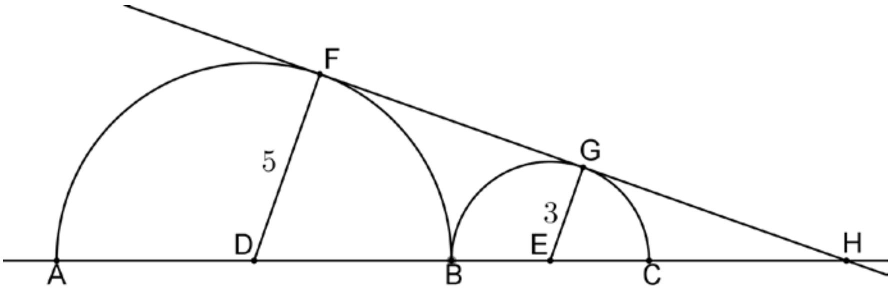
$$LHS=\frac{(k+2)!-1}{(k+2)!}$$

$$LHS=RHS$$

∴ This statement is true by mathematical induction for all positive integers n.

b)

i)



Join FD and GE

In ΔGEH and ΔFDH

$\angle H\ is\ common$

$\angle DFH = 90°$ ($DF \perp FH$, tangent perpendicular to radius)

$\angle EGH = 90°$ ($EG \perp FH$, tangent perpendicular to radius)

$\angle DFH = \angle EGH$

$\therefore \Delta GEH|||\Delta FDH$ (equiangular)

ii)

$\frac{HE}{HD} = \frac{GE}{FD}$ (corresponding sides of similar triangles are in proportion)

$$\frac{HE}{HE + DB + BE} = \frac{3}{5}$$

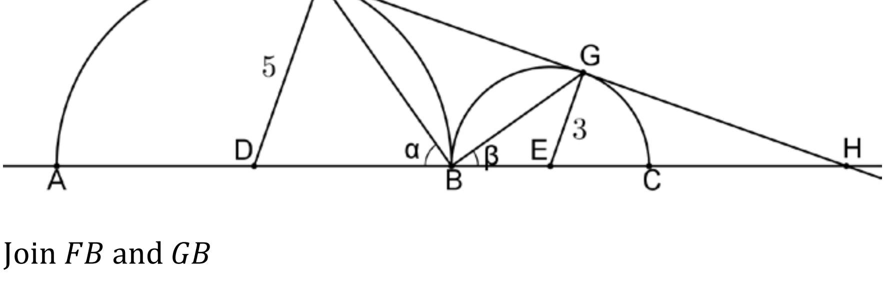
$$\frac{HE}{HE + 8} = \frac{3}{5}$$

$5HE = 3(HE + 8)$

$2HE = 24$

$HE = 12$ cm

iii)



Join FB and GB

Let $\angle FBD = \alpha$ and $\angle GBE = \beta$

$DF = DB$ (equal radii)

ΔDBF is an isosceles triangle (two equal sides)

$\angle FBD = \angle DFB = \alpha$ (equal base angles of isosceles ΔDBF)

$\angle DFB + \angle BFG = \angle DFH$ (complementary angles)

$\angle BFG = 90° - \alpha$

$BE = GE$ (equal radii)

ΔBEG is an isosceles triangle (two equal sides)

$\angle EBG = \angle EGB = \beta$ (equal base angles of isosceles ΔBEG)

$\angle EGB + \angle BGF = \angle EGF$ (complementary angles)

$\angle BGF = 90° - \beta$

$\angle BGF + \angle BFG + \angle FBG = 180°$ (angle sum of ΔFBG)

$\angle FBG = 180° - (90° - \alpha) - (90° - \beta)$

$\angle FBG = \alpha + \beta$

$\angle FBG + \angle FBD + \angle GBE = 180°$ (angle sum on a straight line)

$\alpha + \beta + \alpha + \beta = 180°$

$2(\alpha + \beta) = 180°$

$\alpha + \beta = 90°$

$\angle FBG = 90°$

Since angle in a semicircle is 90°

$\therefore$ FBG lie on another circle with FG as its diameter.

c) i)

Since PQ is a focal chord, it must passes through the focus $(0, a)$
$(p + q) \times 0 - 2a - 2apq = 0$
$-2apq = 2a$
$\therefore pq = -1$

ii)

Tangent at P $\quad y = px - ap^2 \quad$ (1)
Tangent at Q $\quad y = qx - aq^2 \quad$ (2)
Solve (1) and (2) simultaneously to find the point of intersection.
$px - ap^2 = qx - aq^2$
$x = a(p + q)$

Substitute $x = a(p+q)$ into (1)
$y = p \times a(p+q) - ap^2$
$y = ap^2 + apq - ap^2$
$y = apq$
$y = -a \qquad (pq = -1)$
$\therefore (a(p+q), -a)$

iii)

The locus of R is a horizontal line with equation $y = -a$.

iv)

$PQ = \sqrt{(2ap - 2aq)^2 + (ap^2 - aq^2)^2}$
$PQ = \sqrt{4a^2(p-q)^2 + a^2(p^2 - q^2)^2}$
$PQ = a\sqrt{4(p-q)^2 + (p^2 - q^2)^2}$
$PQ = a\sqrt{4(p-q)^2 + (p^2 - q^2)(p^2 - q^2)}$
$PQ = a\sqrt{4(p-q)^2 + (p+q)^2(p-q)^2}$
$PQ = a(p-q)\sqrt{4 + (p+q)^2}$
$PQ = a(p-q)\sqrt{-4pq + p^2 + 2pq + q^2} \qquad (pq = -1)$
$PQ = a(p-q)\sqrt{p^2 - 2pq + q^2}$
$PQ = a(p-q)\sqrt{(p-q)^2}$
$PQ = a(p-q)^2$
$\therefore PQ = a\left(p + \frac{1}{p}\right)^2 \qquad \left(q = -\frac{1}{p}\right)$

Q14.

a)

$$\sin\left[\cos^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(-\frac{2}{7}\right)\right]$$

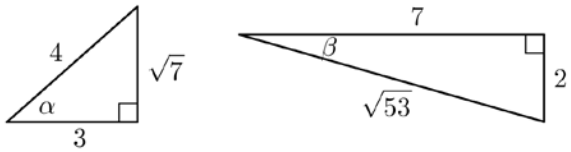
Let $\alpha = \cos^{-1}\left(\frac{3}{4}\right), 0 \le \alpha \le \pi$

$\cos\alpha = \frac{3}{4}$

Let $\beta = \tan^{-1}\left(-\frac{2}{7}\right), -\frac{\pi}{2} \le \beta \le \frac{\pi}{2}$

$\tan\beta = -\frac{2}{7}$

α represents an angle in the first quadrant and β represents an angle in the fourth quadrant.



$$\sin\left[\cos^{-1}\left(\frac{3}{4}\right) - \tan^{-1}\left(-\frac{2}{7}\right)\right]$$
$= \sin[\alpha - \beta]$
$= \sin\alpha\cos\beta - \cos\alpha\sin\beta$
$= \frac{\sqrt{7}}{4} \times \frac{7}{\sqrt{53}} - \frac{3}{4} \times \frac{-2}{\sqrt{53}}$
$= \frac{7\sqrt{7}}{4\sqrt{53}} + \frac{6}{4\sqrt{53}}$
$= \frac{7\sqrt{371} + 6\sqrt{53}}{212}$

b) i)

$$(1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k$$

Substitute $x = 1$

$$\sum_{k=0}^{2n} \binom{2n}{k} 1^k = (1+1)^{2n}$$
$$\sum_{k=0}^{2n} \binom{2n}{k} = 2^{2n}$$
$$\sum_{k=0}^{2n} \binom{2n}{k} = 4^n$$

ii)

$$(1+x)^{2n} = \sum_{k=0}^{2n} \binom{2n}{k} x^k$$

Integrate both sides of:

$$\frac{(1+x)^{2n+1}}{2n+1} + C = \sum_{k=0}^{2n} \binom{2n}{k} \frac{x^{k+1}}{k+1}$$

Subsitute $x = 0$

$$\frac{(1+0)^{2n+1}}{2n+1} + C = \sum_{k=0}^{2n} \binom{2n}{k} \frac{0^{k+1}}{k+1}$$

$$\frac{1}{2n+1} + C = 0$$

$$c = -\frac{1}{2n+1}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} \frac{x^{k+1}}{k+1} = \frac{(1+x)^{2n+1}}{2n+1} - \frac{1}{2n+1}$$

Substitute $x = 1$

$$\sum_{k=0}^{2n} \binom{2n}{k} \frac{1^{k+1}}{k+1} = \frac{(1+1)^{2n+1}}{2n+1} - \frac{1}{2n+1}$$

$$\sum_{k=0}^{2n} \binom{2n}{k} \frac{1}{k+1} = \frac{2^{2n+1}}{2n+1} - \frac{1}{2n+1}$$

$$= \frac{2^{2n+1} - 1}{2n+1}$$

c) i)

Horizontal:

$\ddot{x} = 0$

$\dot{x} = C_1$

When $t = 0, \dot{x} = 15\cos\theta, \therefore C_1 = 12\cos\theta$

$\dot{x} = 15\cos\theta$

$x = 15t\cos\theta + C_3$

When $t = 0, x = 0, \therefore C_3 = 0$

$x = 15t\cos\theta$

Vertical:

$\ddot{y} = -10$

$\dot{y} = -10t + C_2$

When $t = 0, \dot{y} = 15\sin\theta, \therefore C_2 = 15\sin\theta$

$\dot{y} = -10t + 15\sin\theta$

$y = -5t^2 + 15t\sin\theta + C_4$

When $t = 0, y = -2.5, \therefore C_4 = -2.5$

$y = -5t^2 + 15t\sin\theta - 2.5$

ii)

Given that $\theta = 30°$, ball will hit ground when $y = 0$

$y = -5t^2 + 15t\sin\theta - 2.5$

$0 = -5t^2 + 15t\sin 30° - 2.5$

$$0 = -5t^2 + 15t \times \frac{1}{2} - 2.5$$
$$0 = -10t^2 + 15t - 5$$
$$0 = 10t^2 - 15t + 5$$
$$0 = 10t^2 - 10t - 5t + 5$$
$$0 = 10t(t-1) - 5(t-1)$$
$$0 = (10t-5)(t-1)$$
$$t = \frac{1}{2}, \qquad t = 1$$

$t = \frac{1}{2}$ gives first time ball crosses x axis which is not on the ground (it is left of A)

$t = 1$ gives the time the ball hits the ground to the right of A.

When $t = 1$,

$$x = 15 \times 1 \times \cos 30°$$
$$x = \frac{15\sqrt{3}}{2}$$

$$OA = 8$$

$$\frac{15\sqrt{3}}{2} - 8 \approx 4.99$$

∴ The ball will land 4.99 metres to the right of A

iii)

For the ball to land to the right of A, we need the angle necessary to go through A.

$A(8, 0)$

$$8 = 15t\cos\theta$$
$$t = \frac{8}{15\cos\theta}$$

Substitute t into $y = -5t^2 + 15t\sin\theta - 2.5$

$$y = -5\left(\frac{8}{15\cos\theta}\right)^2 + 15\left(\frac{8}{15\cos\theta}\right)\sin\theta - 2.5$$
$$y = -\frac{64}{45\cos^2\theta} + \frac{8\sin\theta}{\cos\theta} - 2.5$$

When $y = 0$

$$0 = -\frac{64}{45\cos^2\theta} + \frac{8\sin\theta}{\cos\theta} - 2.5$$
$$0 = -128\sec^2\theta + 720\tan\theta - 225$$
$$0 = 128(1 + \tan^2\theta) - 720\tan\theta + 225$$
$$0 = 128\tan^2\theta - 720\tan\theta + 353$$
$$\tan\theta = \frac{720 \pm \sqrt{(-720)^2 - 4 \times 128 \times 353}}{2 \times 128}$$
$$\tan\theta = \frac{720 \pm 16\sqrt{1319}}{256}$$

$$\tan\theta = \frac{45 \pm \sqrt{1319}}{16}$$

$$\theta = 28°29' \quad \text{or} \quad 78°52'$$

Anything less than $28°29'$ or greater than $78°52'$ will hit the bank of the sand bunker. So to land to the right of A, $29° \leq \theta \leq 78°$.