Student Number: ____________________

St George Girls High School

Trial Higher School Certificate Examination

2017

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Section I	/10
Section II	
Question 11	/15
Question 12	/15
Question 13	/15
Question 14	/15
Question 15	/15
Question 16	/15
Total	**/100**

Total Marks – 100

Section I

Pages 2 – 6

10 marks

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section
- Answer on the multiple choice answer sheet provided at the back of this paper

Section II

Pages 7 – 12

90 marks

- Attempt Questions 11 – 16
- Allow about 2 hours and 45 minutes for this section
- Begin each question in a new writing booklet

Section I

10 marks
Attempt Questions 1 – 10
Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

Section I

1. What is the value of $\dfrac{10}{i|z|}$, if $z=-1+i$?

(A) $-5i\sqrt{2}$

(B) $2-5i$

(C) $5i\sqrt{2}$

(D) $2+5i$

2. Which of the following are the coordinates of the foci of $9x^2-36y^2=324$?

(A) $(\pm\sqrt{5},0)$

(B) $(0,\ \pm\sqrt{5})$

(C) $(\pm 3\sqrt{5},0)$

(D) $(0,\pm 3\sqrt{5})$

3. Consider the region bounded by the *y*–axis, the line *y* = 4 and the curve $y = x^2$.
If this region is rotated about the line *y* = 4, which expression gives the volume of the solid of revolution?

(A) $V = \pi\int_0^4 x^2\,dy$

(B) $V = 2\pi\int_0^2 (4-y)x\,dy$

(C) $V = \pi\int_0^2 (4-y)^2\,dx$

(D) $V = \pi\int_0^4 (4-y)^2\,dx$

Section I (cont'd)

4. $\int_0^2 \frac{x^2}{\sqrt{x^3+1}}dx$.

(A) $\frac{1}{9}$

(B) $\frac{1}{3}$

(C) $\frac{4}{3}$

(D) 9

5. For the hyperbola $(y+1)^2 - x^2 = 1$, find an expression for $\frac{d^2y}{dx^2}$.

(A) $\frac{x}{(y+1)^3}$

(B) $\frac{1}{(y+1)^3}$

(C) $\frac{x}{y+1}$

(D) $\frac{1}{y+1}$

6. The polynomial $P(x) = 4x^3 + 16x^2 + 11x - 10$ has roots α, β and $\alpha + \beta$. What is the value of $\alpha\beta$?

(A) $\frac{5}{2}$

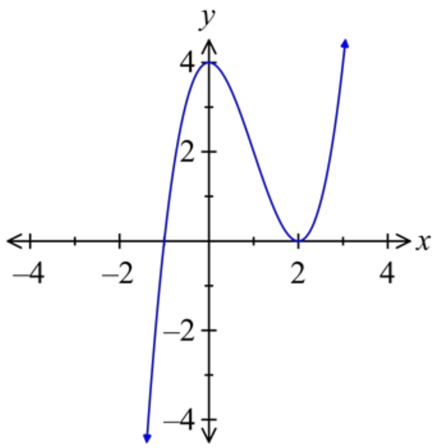
(B) $-\frac{5}{2}$

(C) $\frac{5}{4}$

(D) $-\frac{5}{4}$

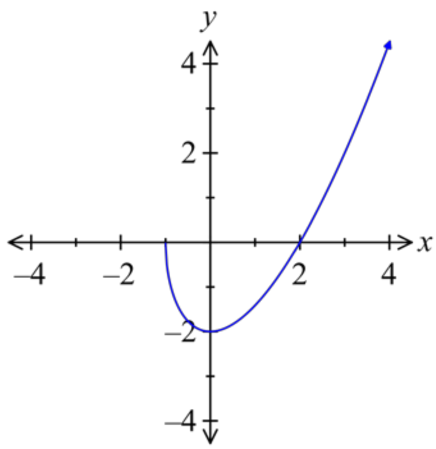
Section I (cont'd)

7. The diagram below shows the graph of the function $y = f(x)$

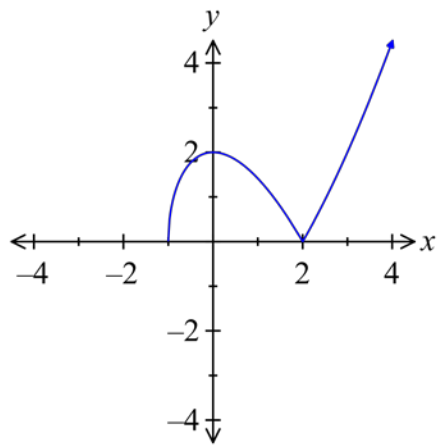


Which of the following is the graph of $y = \dfrac{1}{f(x)}$?

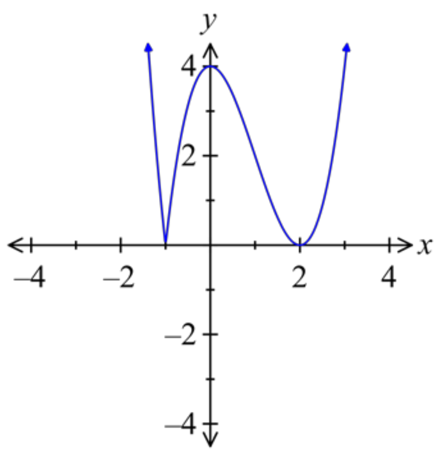
(A)



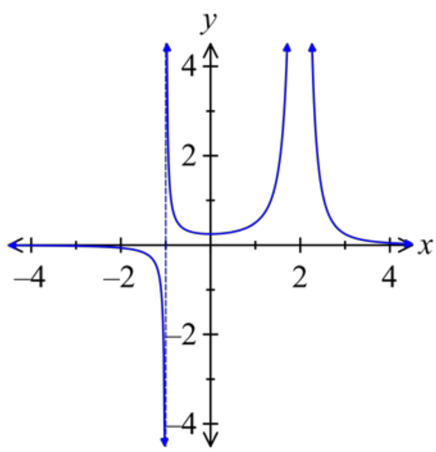
(B)



(C)

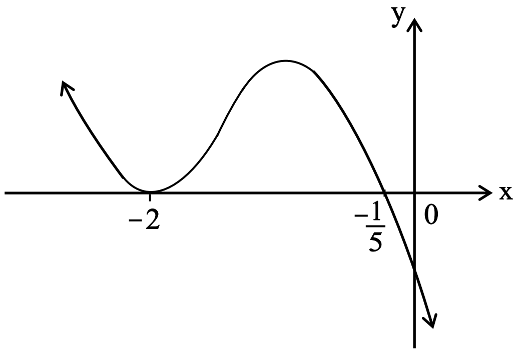


(D)



Section I (cont'd)

8. The diagram shows the graph of $y = P''(x)$ which is the second derivative of a polynomial $P(x)$.



Which of the following expressions could be $P(x)$?

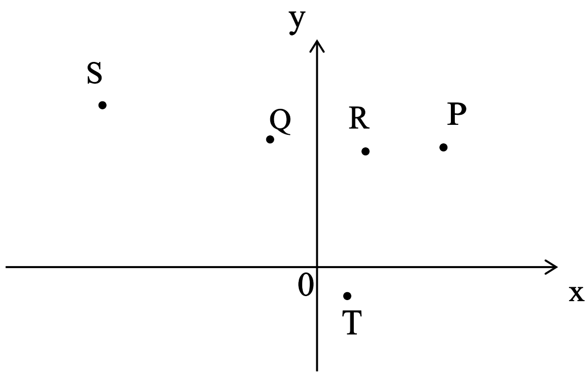
(A) $(x + 2)^2 (x - 1)$

(B) $(x + 2)^4 (x - 1)$

(C) $(x - 2)^4 (x - 1)$

(D) $(x - 2)^4 (x + 1)$

9. In the Argand diagram the point **P** represents the complex number z. When this number is divided by $5i$ it gives a new complex number.



Which one of the points on the diagram above represents the new complex number?

(A) Q

(B) R

(C) S

(D) T

10. The sides of a triangle are the first three terms of an arithmetic progression with the first term 1 and the common difference d. What is the largest set of possible values of d.

(A) $-1 < d < 1$

(B) $-\frac{1}{2} < d < 1$

(C) $-\frac{1}{3} < d < 1$

(D) $-\frac{1}{4} < d < 1$

End of Section

Section II
90 marks
Attempt Questions 11 – 16
Allow about 2 hours 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 – 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet **Marks**

(a) For the complex number $z = \sqrt{2} + \sqrt{2}i$

(i) Express z in modulus-argument form 2

(ii) Find z^{12}. 2

(b) Evaluate $\displaystyle\int_{\frac{1}{2}}^{2} \frac{1}{2x^2 - 2x + 1}\,dx$ to 4 significant figures. 4

(c) Find the square root of $1 + 2\sqrt{2}i$. 3

(d) Reduce the polynomial $x^6 - 9x^3 + 8$ to irreducible factors over the:

(i) real field 2
(ii) complex field 2

End of Question 11

Question 12 (15 marks) Use a SEPARATE writing booklet

Marks

(a) Use the substitution $t = \tan\frac{\theta}{2}$ to find $\int \frac{1}{1+\cos\theta+\sin\theta}\,d\theta$. 3

(b) $P(a\sec\theta, \tan\theta)$ is a point on the hyperbola $\frac{x^2}{a^2} - y^2 = 1$, $a > 1$, with eccentricity e and asymptotes L_1 and L_2. M and N are the feet of the perpendiculars from P to L_1 and L_2 respectively. 3
Show that $PM.PN = \frac{1}{e^2}$.

(c) Use integration by parts to find $\int x\, sec^2 x dx$. 3

(d) The area between the curve $y = 3x - x^2$ and $y = x$, between $x = 1$ and $x = 2$ is rotated about the $y -$ axis. Using the method of cylindrical shells, find the volume of the solid of revolution formed. 3

(e) (i) Given that $\sin x$ can be written as $\sin(2x - x)$ show that
$$\sin x + \sin 3x = 2\sin 2x \cos x.$$ 1

(ii) Hence or otherwise find the general solutions of $\sin x + \sin 2x + \sin 3x = 0$. 2

End of Question 12

Question 13 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) The roots of $x^3 + x^2 + 1 = 0$ are α, β and γ. Find the cubic equation whose roots are 4

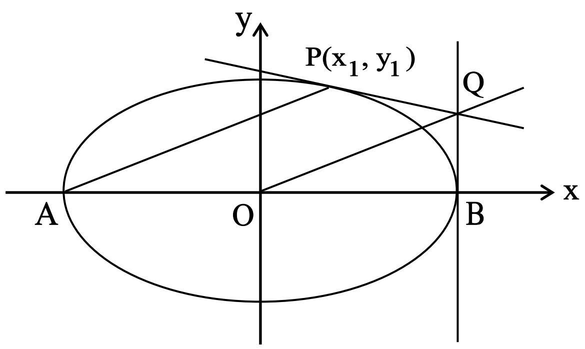
$$\frac{1}{1-\alpha}, \frac{1}{1-\beta}, \frac{1}{1-\gamma}$$

Express your answer in the form $ax^3 + bx^2 + cx + d = 0$.

(b) (i) On the same Argand diagram carefully sketch the region where $|z-1| \le |z-3|$ and $|z-2| \le 1$ hold simultaneously. 3

(ii) Find the greatest possible value for $|z|$ and $\arg z$. 2

(c) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ intersect the x-axis at the points A and B.
The point P (x_1, y_1) lies on the ellipse. The tangent at P intersects the vertical line passing through B at the point Q as shown in the diagram.



(i) Show that the equation of tangent at P is $\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$ 2

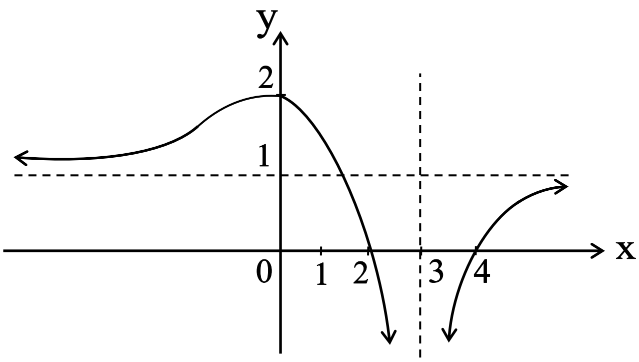
(ii) Show that the coordinates of Q are $\left(a,\ \frac{b^2}{y_1}\left(1 - \frac{x_1}{a}\right)\right)$ 1

(iii) Show that AP is parallel to OQ 3

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A sketch of the function $f(x)$ is shown below.



Draw a separate half-page graph for each of the following functions, showing all asymptotes and intercepts.

(i) $y = |f(x)|$ 2

(ii) $y = [f(x)]^2$ 2

(iii) $y = \ln f(x)$ 2

(b) The point $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$.

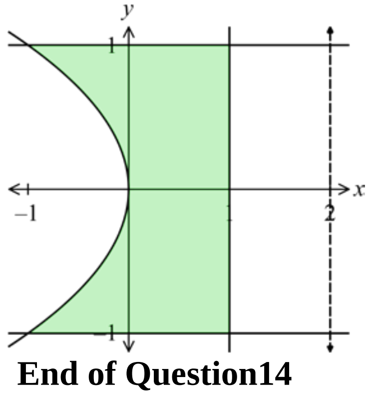
(i) Find the equation of the tangent to the hyperbola at the point *P*. 2

(ii) The tangent at *P* cuts the *x*–axis at A and the *y*–axis at B. Show that the area of the triangle AOB is independent of *t*. 2

(c) Find the values of the real numbers p and q given that

$$x^3 + 2x^2 - 15x - 36 = (x+p)^2(x+q)$$ 2

(d) The region shown below is bounded by the lines $x=1$, $y=1$, $y=-1$ and the curve $x=-y^2$. The region is rotated through 360° about the line $x=2$ to form a solid. Calculate the volume of the solid using the method of slicing? 3



End of Question14

Question 15 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) A particular solid has as its base the region bounded by the hyperbola $\frac{x^2}{4} - \frac{y^2}{5} = 1$ and the line $x = 4$. 4

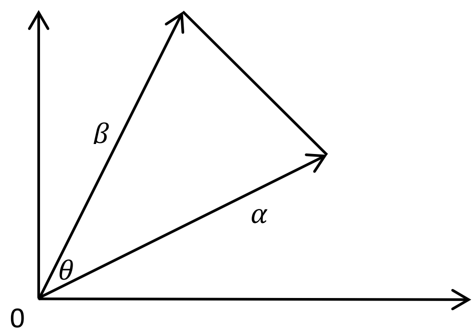
Cross-sections perpendicular to this base and the x −axis are equilateral triangles.
Find the volume of this solid.

(b) Let $I_n = \int_1^4 \left(\sqrt{x} - 1\right)^n dx$, where n = 0, 1, 2.

(i) Show that $(n + 2)I_n = 8 - nI_{n-1}$. 3

(ii) Evaluate I_4. 3

(c) (i) Use the results $z + \bar{z} = 2Re(z)$ and $|z|^2 = z\bar{z}$ for the complex numbers z to show that $|\alpha|^2 + |\beta|^2 - |\alpha - \beta|^2 = 2Re(\alpha\bar{\beta})$. 3



(ii) The diagram shows the angle θ between the complex numbers α and β. 2
Prove that

$$|\alpha||\beta| \cos\theta = Re(\alpha\bar{\beta}).$$

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet. **Marks**

(a) Let $F(x) = e^{x^2}$ for all $x \geq 0$.

(i) Find $F^{-1}(x)$, the inverse function of $F(x)$ 1

(ii) State its domain and range of $F^{-1}(x)$. 2

(iii) On the same set of axes, sketch $F^{-1}(x)$ and $F(x)$ indicate the region represented by 2

$$\int_0^1 F(x)dx \quad \text{and} \quad \int_1^e F^{-1}(x)dx$$

(iv) Evaluate $\int_0^1 F(x)dx + \int_1^e F^{-1}(x)dx$. 2

(b) The cubic equation $x^3 + kx + 1 = 0$, where k is a constant, has roots α, β and γ. For each positive integer n,

$$S_n = \alpha^n + \beta^n + \gamma^n.$$

(i) State the value of S_1. 1

(ii) Express S_2 in terms of k. 2

(iii) Show that for all values of n,

$$S_{n+3} + kS_{n+1} + S_n = 0.$$ 3

(iv) Hence or otherwise express $\alpha^5 + \beta^5 + \gamma^5$ in terms of k. 2

End of Examination

Ext 2 Trial 2017

Solutions

1. $\frac{10}{i|3|} = \frac{10}{i\sqrt{1+1}}$

$= \frac{10}{i\sqrt{2}} \times \frac{i\sqrt{2}}{i\sqrt{2}}$

$= \frac{10\sqrt{2}\,i}{-2}$

$= -5\sqrt{2}\,i$ A

2. $9x^2 - 36y^2 = 324$

$\frac{x^2}{36} - \frac{y^2}{9} = 1$

$b^2 = a^2(e^2-1)$

$\frac{9}{36} = e^2 - 1$

$e^2 = 5/4$

$e = \frac{\sqrt{5}}{2}$ $a = 6$

Foci $(\pm ae, 0) = (\pm 6 \times \frac{\sqrt{5}}{2}, 0)$

$= (\pm 3\sqrt{5}, 0)$ C

3. Using Disc Method – C

4. $\int_0^2 \frac{x^2}{\sqrt{x^3+1}}\,dx = \frac{1}{3}\int_0^2 \frac{3x^2}{\sqrt{x^3+1}}\,dx$

$= \frac{1}{3}\int_1^9 \frac{du}{\sqrt{u}}$

$u = x^3+1$

$\frac{du}{dx} = 3x^2$

$x=2, u=$

$x=0, u=1$

$= \frac{1}{3} \cdot 2 \left[u^{1/2}\right]_1^9$

$= 4/3$ C

5. $(y+1)^2 - x^2 = 1$

$$2(y+1)\frac{dy}{dx} - 2x = 0$$

$$\frac{dy}{dx} = \frac{x}{y+1}$$

$$\frac{d^2y}{dx^2} = \frac{y+1 - x\,y'}{(y+1)^2}$$

$$= \frac{y+1 - \frac{x^2}{y+1}}{(y+1)^2}$$

$$= \frac{(y+1)^2 - x^2}{(y+1)^3}$$

$$= \frac{1}{(y+1)^3}$$ B

6. $P(x) = 4x^3 + 16x^2 + 11x - 10$

$\Sigma\alpha$: $\alpha + \beta + \alpha + \beta = \frac{-16}{4}$

$= -4$

$2(\alpha+\beta) = -4$

$\alpha + \beta = -2$

$\Sigma\alpha\beta$: $\alpha\beta + \alpha(\alpha+\beta) + \beta(\alpha+\beta) = \frac{11}{4}$

$\alpha\beta + (\alpha+\beta)^2 = \frac{11}{4}$

$\alpha\beta + (-2)^2 = \frac{11}{4}$

$\alpha\beta = \frac{11}{4} - 4$ D

$= \frac{-5}{4}$

7. D

8. Second derivative has double root at $x=2$.
So the original function should have a root of multiplicity 4 at $x = -2$

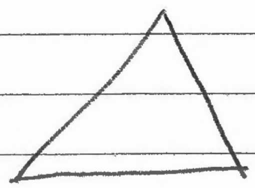
B.

9. A division of $5i$ is equivalent to a multiplication of $\frac{1}{5i} \times \frac{5i}{i} = \frac{i}{-5}$

Multiplyg P by $\frac{i}{-5}$ means

D.

10.



Sides of triangle
$1, \ 1+d, \ 1+2d$

Now $1+d+1 < 1+2d$
$2+d < 1+2d$
$1 < d$

or

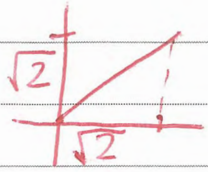
$1+1+2d < 1+d$
$2+2d < 1+d$
$d < -1$

or $1+d+1+2d < 1$
$2+3d < 1$
$3d < -1$
$d < -\frac{1}{3}$

$\therefore -\frac{1}{3} < d < 1$

C.

MATHEMATICS EXTENSION 2 – QUESTION 11

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) i $\lvert z\rvert = \sqrt{2+2}$		
$= 2$	1	
$\arg z = \frac{\pi}{4}$	1	
$\therefore z = 2\,\text{cis}\,\frac{\pi}{4}$		
		
ii $z^{12} = 2^{12}\,\text{cis}\left(\frac{\pi}{4}\times 12\right)$	1, 1	
$= 4096\,(\cos 3\pi + i\sin 3\pi)$		
$= 4096\,(-1+0)$		
$= -4096$		
b) $\displaystyle\int_{\frac{1}{2}}^{2} \frac{1}{2x^2-2x+1}\,dx = \frac{1}{2}\int_{\frac{1}{2}}^{2}\frac{1}{x^2-x+\frac{1}{2}}\,dx$		
$\displaystyle = \frac{1}{2}\int_{\frac{1}{2}}^{2}\frac{1}{\left(x-\frac{1}{2}\right)^2+\left(\frac{1}{2}\right)^2}\,dx$	1	
$\displaystyle = \frac{1}{2}\left[\frac{1}{\frac{1}{2}}\tan^{-1}\frac{\left(x-\frac{1}{2}\right)}{\frac{1}{2}}\right]_{\frac{1}{2}}^{2}$	1	
$\displaystyle = \frac{1}{2}\left[2\tan^{-1}(2x-1)\right]_{\frac{1}{2}}^{2}$		Note that trigonometric calculus is performed in radians.
$= \tan^{-1}3 - \tan^{-1}0$	1	
$= 1.249045\ldots$		
$\doteq 1.249$	1	The equivalent in degrees (71.57°) was awarded $3\frac{1}{2}$
c) Let $x+iy$ be the square root of $1+2\sqrt{2}i$		
$\therefore (x+iy)^2 = 1+2\sqrt{2}\,i \qquad x,y\in\mathbb{R}$		
$x^2-y^2+2ixy = 1+2\sqrt{2}\,i$		
Equating coefficients,		
$x^2-y^2=1 \qquad 2xy = 2\sqrt{2}$	1	
$xy=\sqrt{2}$		
$y = \frac{\sqrt{2}}{x}$		

MATHEMATICS EXTENSION 2 – QUESTION 11

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
sub $y=\frac{\sqrt{2}}{x}$ into $x^2-y^2=1$:		
$x^2-\frac{2}{x^2}=1$		Note that $\pm\sqrt{2}\pm i \neq \pm(\sqrt{2}+i)$, as the LHS suggests 4 solutions.
$x^4-2=x^2$		
$x^4-x^2-2=0$		
$(x^2-2)(x^2+1)=0$		
$(x-\sqrt{2})(x+\sqrt{2})(x^2+1)=0$		
$\therefore x=\pm\sqrt{2}$ $\quad(\because x\in\mathbb{R})$	1	
$\therefore y=\pm 1$		
$\therefore$ the square root of $1+2\sqrt{2}i$ is $\pm(\sqrt{2}+i)$	1	
i.e. $\sqrt{2}+i$ and $-\sqrt{2}-i$		
d) i $x^6-9x^3+8=(x^3)^2-9x^3+8$		This is a degree 6 polynomial; long division was a bad choice.
$=(x^3-1)(x^3-8)$		
$=(x-1)(x^2+x+1)(x-2)(x^2+2x+4)$	1, 1	
irreducible over $\mathbb{R}$		
ii For x^2+x+1,		
$x=\frac{-1\pm\sqrt{3}i}{2}$		
$=-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i$		
for x^2+2x+4,		
$x=\frac{-2\pm 2\sqrt{3}i}{2}$		
$=-1\pm\sqrt{3}i$		
$\therefore x^6-9x^3+8=(x-1)\left(x-\left[-\frac{1}{2}\pm\frac{\sqrt{3}}{2}i\right]\right)(x-2)\left(x-\left[-1\pm\sqrt{3}i\right]\right)$		
$=(x-1)\left(x+\frac{1}{2}+\frac{\sqrt{3}}{2}i\right)\left(x+\frac{1}{2}-\frac{\sqrt{3}}{2}i\right)(x-2)(x+1+\sqrt{3}i)(x+1-\sqrt{3}i)$	1, 1	
		errors of sign were worth $\frac{1}{2}$ a mark.

MATHEMATICS EXTENSION 2 – QUESTION

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
12 a)		

$t = \tan\frac{\theta}{2}$ $\quad$ $\sin^2\theta + \cos^2\theta = 1$

$\frac{dt}{d\theta} = \frac{1}{2}\sec^2\frac{\theta}{2}$ $\quad$ $\tan^2\theta + 1 = \sec^2\theta$

$\frac{dt}{d\theta} = \frac{1}{2}(t^2+1)$

$d\theta = \frac{2\,dt}{t^2+1}$ $\quad$ 1

$\int \frac{1}{1+\cos\theta+\sin\theta}\,d\theta$

$= \int \frac{1}{1+\frac{1-t^2}{1+t^2}+\frac{2t}{1+t^2}}\frac{2}{1+t^2}\,dt$ $\quad$ 1

$\tan\theta = \frac{2t}{1-t^2}$

$= \int \frac{2\,dt}{(1+t^2)+(1-t^2)+2t}$

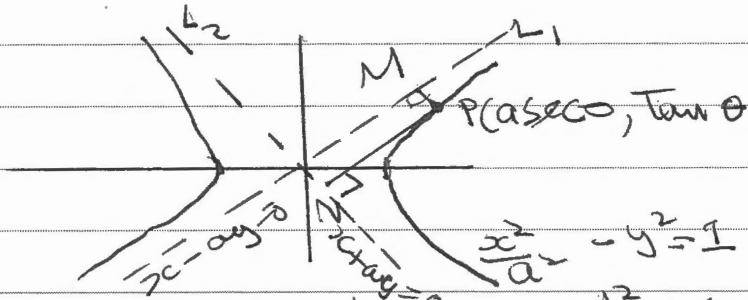
$= \int \frac{2\,dt}{2+2t}$

$= \int \frac{1}{1+t}\,dt$

$= \ln|1+t| + C$ $\quad$ $\cos\theta = \frac{1-t^2}{1+t^2}$

$= \ln\left|1+\tan\frac{\theta}{2}\right| + C$ $\quad$ $\sin\theta = \frac{2t}{1+t^2}$ $\quad$ 1

12 b)



Asymptotes: $y = \pm\frac{b}{a}x$ $\quad$ $b^2 = 1$

$b = 1$

i.e. $y = \pm\frac{x}{a}$

L_1: $ay = x$ $\quad$ L_2: $ay = -x$

i.e. $x - ay = 0$ $\quad$ i.e. $x + ay = 0$ $\quad$ 1

Using perpendicular formula for distance of PM and PN

$PM = \left|\frac{ax_1+by_1+c}{\sqrt{a^2+b^2}}\right|$

$= \left|\frac{a\cdot\sec\theta - a\cdot\tan\theta}{\sqrt{1+a^2}}\right|$

$PN = \left|\frac{a\cdot\sec\theta + a\cdot\tan\theta}{\sqrt{1+a^2}}\right|$

$PM \times PN = \left|\frac{a(\sec\theta - \tan\theta)}{\sqrt{1+a^2}} \times \frac{a(\sec\theta + \tan\theta)}{\sqrt{1+a^2}}\right|$

$= \frac{a^2}{1+a^2}(\sec^2\theta - \tan^2\theta)$

MATHEMATICS EXTENSION 2 – QUESTION

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS

12b) cont $PM \times PN = \frac{a^2}{1+a^2} \times 1$

$= \frac{a^2}{1+a^2}$ $\quad$ but $b^2 = a^2(e^2-1)$

$b=1, \quad 1 = a^2(e^2-1)$

$= \frac{a^2}{a^2e^2}$ $\quad$ $1 = a^2e^2 - a^2$

$1 + a^2 = a^2e^2$

$\therefore PM \times PN = \frac{1}{e^2}$ $\quad$ $e^2 = a$ — 1, 1

12c) $\int x \cdot \sec^2 x \, dx$ $\quad$ $u = x$ $\quad$ $dv = \sec^2 x$

$= uv - \int v \cdot du$ $\quad$ $du = 1$ $\quad$ $v = \tan x$ — 1

$= x \cdot \tan x - \int \tan x \, dx$ — 1

$= x \cdot \tan x - \int \frac{\sin x}{\cos x} dx$

$= x \cdot \tan x + \int \frac{-\sin x}{\cos x} dx$

$= x \cdot \tan x + \ln|\cos x| + c$ — 1

12d) $y = 3x - x^2$

$y = x$

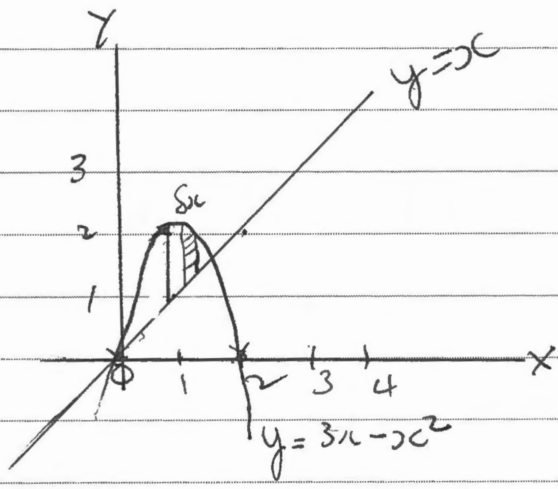
$x = 3x - x^2$

$x^2 + x - 3x = 0$

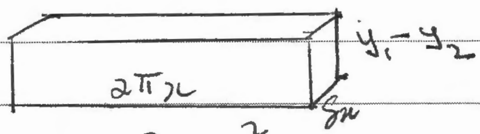
$x^2 - 2x = 0$

$x(x-2) = 0$

$x = 0$ or 2 — 1



$A = 2\pi x \cdot \delta x$



$\delta V = 2\pi x y \cdot \delta x$

$= 2\pi x(2x - x^2) \cdot \delta x$ $\quad$ $y_1 = 3x - x^2$

$V = \lim_{\delta x \to 0} 2\pi x(2x - x^2) \cdot \delta x$ $\quad$ $y_2 = x$ — 1

$y_1 - y_2 = 3x - x^2 - x$

$= \lim_{\delta x \to 0} 2\pi(2x^2 - x^3)$ $\quad$ $= 2x - x^2$

$= 2\pi \int_0^2 (2x^2 - x^3) dx$

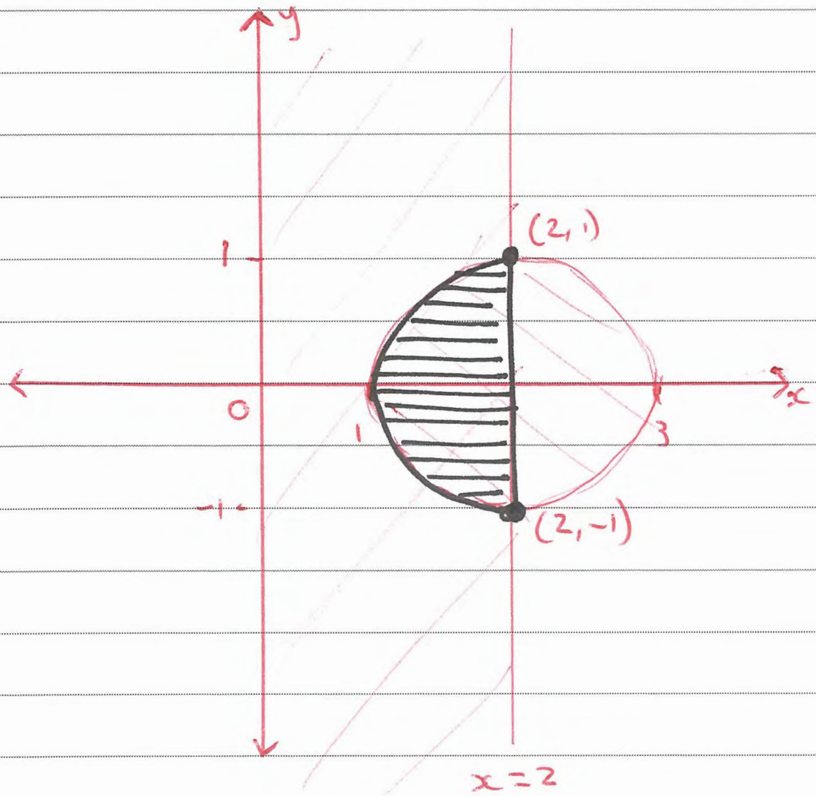
$= 2\pi \left[\frac{2}{3}x^3 - \frac{x^4}{4} + c\right]_0^2$

$= \frac{11\pi}{6}$ cu units — 1

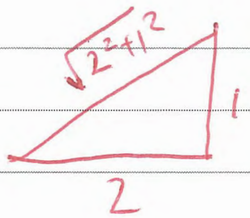
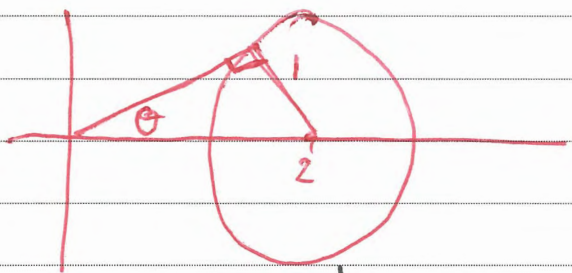
MATHEMATICS EXTENSION 2 – QUESTION

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
12e) i) $\sin x + \sin 3x$		
$= \sin(2x - x) + \sin(2x + x)$		
$= \sin 2x \cdot \cos x - \cancel{\cos 2x \cdot \sin x} + \sin 2x \cdot \cos x + \cancel{\cos 2x \cdot \sin x}$		
$= 2\sin 2x \cdot \cos x$		
$\therefore \sin x + \sin 3x = 2 \cdot \sin 2x \cdot \cos x$	1	
ii) $\sin x + \sin 2x + \sin 3x = 0$		
$2 \cdot \sin 2x \cdot \cos x + \sin 2x = 0$		
$\sin 2x\,(2 \cdot \cos x + 1) = 0$		
$\sin 2x = 0$ or $2 \cdot \cos x + 1 = 0$		
$x = \frac{n\pi}{2}$ $2 \cos x = -1$	1	
$\cos x = -\frac{1}{2}$		
$x = 2n\pi \pm \frac{2\pi}{3}$		
OR $x = (2n+1)\pi \pm \frac{\pi}{3}$		
$\therefore x = \frac{n\pi}{2}$ or $x = 2n\pi \pm \frac{2\pi}{3}$		
(or $(2n+1)\pi \pm \frac{\pi}{3}$)	1	

MATHEMATICS EXTENSION 2 – QUESTION 13

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) Let $x = \frac{1}{1-\alpha}$	1	
$1-\alpha = \frac{1}{x}$		
$\alpha = 1 - \frac{1}{x}$		
$= \frac{x-1}{x}$	1	
$\therefore$ cubic is		
$\left(\frac{x-1}{x}\right)^3 + \left(\frac{x-1}{x}\right)^2 + 1 = 0$	1	
$\frac{x^3-3x^2+3x-1}{x^3} + \frac{x^2-2x+1}{x^2} + 1 = 0$		
$x^3-3x^2+3x-1 + x^3-2x^2+x + x^3 = 0$		
$3x^3 - 5x^2 + 4x - 1 = 0$	1	
b) i		
	1 1 1	for $x=2$ for $(x-2)^2+y^2=1$ for correct region

MATHEMATICS EXTENSION 2 – QUESTION 13

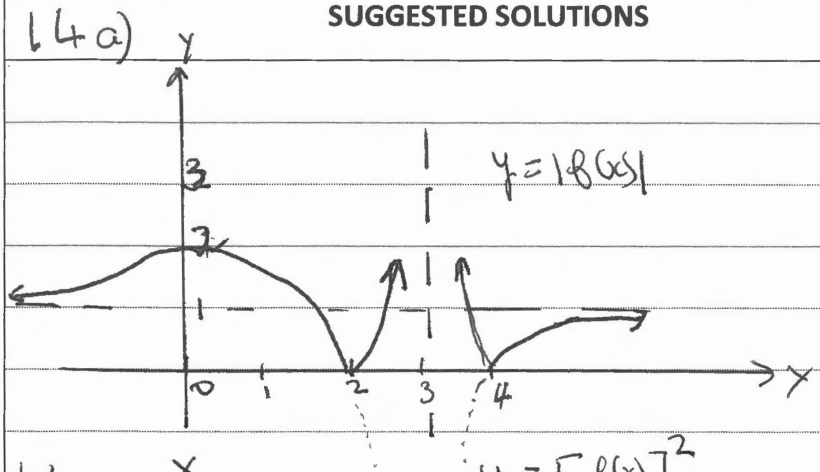
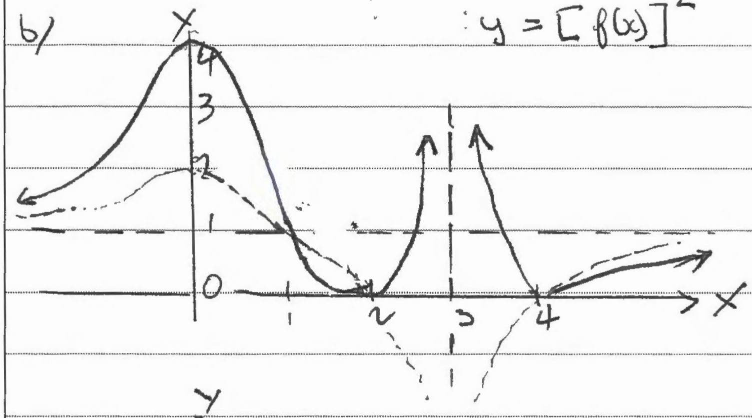
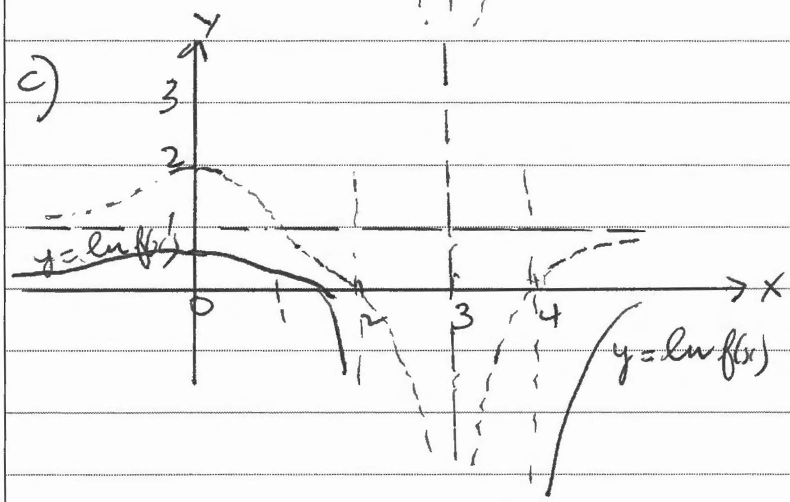
SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
b) ii  Max $\lvert z\rvert$ occurs at $(2,1)$ $[\text{or } (2,-1)]$ $\lvert z\rvert = \sqrt{2^2+1^2}$ $= \sqrt{5}$	1	Best responses came from students who redrew the relevant diagrams. Note that if your answer to b) i was incorrect, it was difficult to demonstrate the required skills for part ii
Max arg z occurs when tangent is perpendicular to radius  $\sin\theta = \frac{1}{2}$ $\theta = \pi/6$	1	Very few students correctly answered this part.
c) i $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ $\frac{d}{dx}\left[\frac{x^2}{a^2} + \frac{y^2}{b^2}\right] = \frac{d}{dx}(1)$ $\frac{2x}{a^2} + \frac{2y}{b^2}\frac{dy}{dx} = 0$ $\frac{dy}{dx} = \frac{-2x/a^2}{2y/b^2}$ $= \frac{-xb^2}{ya^2}$	1	correct differentiation
At $P(x_1,y_1)$, $\frac{dy}{dx} = \frac{-x_1b^2}{y_1a^2}$		

MATHEMATICS EXTENSION 2 – QUESTION 13		
SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
Equation of tangent through (x_1, y_1):		
$y - y_1 = \frac{-x_1 b^2}{y_1 a^2}(x - x_1)$		
$yy_1a^2 - y_1^2a^2 = -xx_1b^2 + x_1^2b^2$		
$xx_1b^2 + yy_1a^2 = x_1^2b^2 + y_1^2a^2$		
$\frac{xx_1}{a^2} + \frac{yy_1}{b^2} = \frac{x_1^2}{a^2} + \frac{y_1^2}{b^2}$		
$= 1$		
[Because $\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$, as $P(x_1, y_1)$ lies on the ellipse $\frac{x^2}{a} + \frac{y^2}{b} = 1$]		
$\therefore \frac{xx_1}{a^2} + \frac{yy_1}{b^2} = 1$	1	correct substitution and algebra
c) ii The coordinates of B are $(a, 0)$ $\therefore$ Q lies on the line $x = a$		
Sub $x = a$ into equation of tangent		
$\frac{ax_1}{a^2} + \frac{yy_1}{b^2} = 1$		
$\frac{x_1}{a} + \frac{yy_1}{b^2} = 1$		
$\frac{yy_1}{b^2} = 1 - \frac{x_1}{a}$		

MATHEMATICS EXTENSION 2 – QUESTION 13

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
$y = \frac{b^2}{y_1}\left(1 - \frac{x_1}{a}\right)$	1	substitution of $x=a$ and correct algebra.
$\therefore Q\left(a, \frac{b^2}{y_1}\left(1-\frac{x_1}{a}\right)\right)$		
c) iii $m_{AP} = \frac{y_1 - 0}{x_1 - -a}$		
$= \frac{y_1}{x_1 + a}$	1	
$m_{OQ} = \frac{\frac{b^2}{y_1}\left(1-\frac{x_1}{a}\right) - 0}{a - 0}$		
$= \frac{b^2}{y_1} \times \frac{a-x_1}{a} \times \frac{1}{a}$		
$= \frac{b^2(a-x_1)}{a^2 y_1}$ (1)	1	simplified expression for m_{OQ}
Since (x_1, y_1) lies on the ellipse,		
$\frac{x_1^2}{a^2} + \frac{y_1^2}{b^2} = 1$		
$b^2x_1^2 + a^2y_1^2 = a^2b^2$		
$a^2y_1^2 = a^2b^2 - x_1^2b^2$		
$= b^2(a^2 - x_1^2)$		
$= b^2(a-x_1)(a+x_1)$		
$\therefore \frac{a^2y_1^2}{a+x_1} = b^2(a-x_1)$ (2)		
sub (2) into (1)		
$m_{OQ} = \frac{a^2y_1^2}{a^2y_1(a+x_1)}$	1	correct algebra
$= \frac{y_1}{a+x_1} = m_{AP}$ $\therefore AP \parallel OQ$		

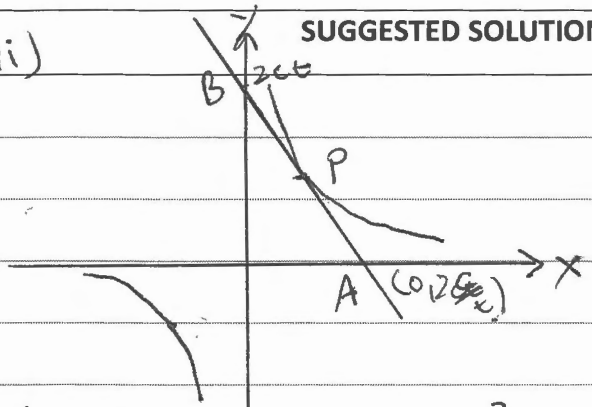
MATHEMATICS EXTENSION 2 – QUESTION

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
14 a)		
	2	
b) 	2	
c) 	2	
b) $P(ct, \frac{c}{t})$ $\quad xy = c^2$		
i) $y = c \cdot x^{-1}$		alterative
$\frac{dy}{dx} = -c \cdot x^{-2}$		implicit differentiation
$= \frac{-c}{x^2}$		
grad at $P(ct, c/t) = \frac{-c}{c^2t^2} = \frac{-1}{t^2}$	1	
$y - \frac{c}{t} = -\frac{1}{t^2}(x - ct)$		
$t^2(y - c/t) = -(x - ct)$		
$t^2y - ct = -x + ct$		
$x + t^2y - 2ct = 0$	1	

MATHEMATICS EXTENSION 2 – QUESTION

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS

14b ii)



At A, $y=0 \Rightarrow x + t^2 y - 2ct = 0$

$x + 0 - 2ct = 0$

$\therefore x = 2ct$

A is $(2ct, 0)$ — 1

At B, $x = 0 \Rightarrow t^2 y = 2ct$

$y = \frac{2ct}{t^2}$

$y = \frac{2c}{t}$

B is $\left(0, \frac{2c}{t}\right)$

Area $\triangle AOB = \frac{1}{2} \times 2ct \times \frac{2c}{t}$ — 1

$= 2c^2$

$\therefore$ Area is independent of t.

14c) $x^3 + 2x^2 - 15x - 36 = (x+p)^2 \cdot (x+q)$

$-p$ is a root of multiplicity 2

$f(x) = x^3 + 2x^2 - 15x - 36$

$f'(x) = 3x^2 + 4x - 15$

$-p$ is a root of $f'(x) = 0$ of mutiplicity 1

$3x^2 + 4x - 15 = 0$

$(3x-5)(x+3) = 0$

$x = \frac{5}{3}$ or $x = -3$ — 1

$f\left(\frac{5}{3}\right) = \left(\frac{5}{3}\right)^3 + 2\times\left(\frac{5}{3}\right)^2 - 15\times\frac{5}{3} - 36 \neq 0$

$f(-3) = (-3)^3 + 2\times(-3)^2 - 15\times -3 - 36 = 0$

$\therefore p = 3$

Let $x = 0$, $-36 = p^2 \cdot q$

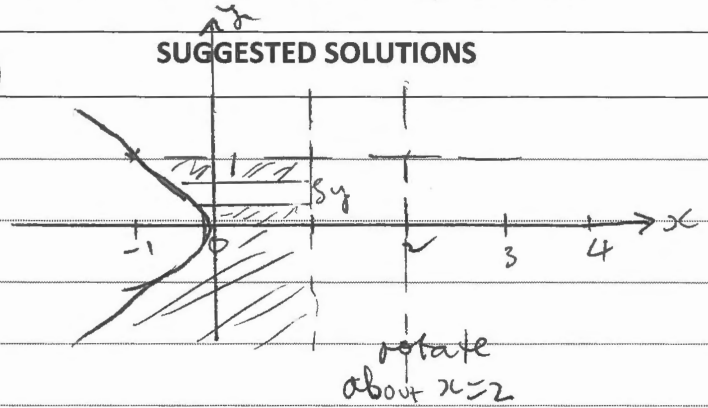
$-36 = 9 \cdot q$

$q = -4$ — 1

$\therefore p = 3,\ q = -4$

MATHEMATICS EXTENSION 2 – QUESTION

14 (d) SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS



$r =$ inner radius $= 1$

$R =$ outer radius $= |x| + 2 = y^2 + 2$

Area of slice $= \pi(R^2 - r^2)$

$= \pi\left((y^2+2)^2 - 1^2\right)$

$= \pi(y^4 + 4y^2 + 4 - 1)$

$= \pi(y^4 + 4y^2 + 3)$ — 1

$\delta V = \pi(y^4 + 4y^2 + 3)\cdot\delta y$

$V = \lim_{\delta y \to 0} \sum_{y=-1}^{1} \pi(y^4 + 4y^2 + 3)\cdot\delta y$ — 1

$= 2\pi \int_0^1 (y^4 + 4y^2 + 3)\,dy$

$= 2\pi\left[\frac{y^5}{5} + \frac{4y^3}{3} + 3y + c\right]_0^1$

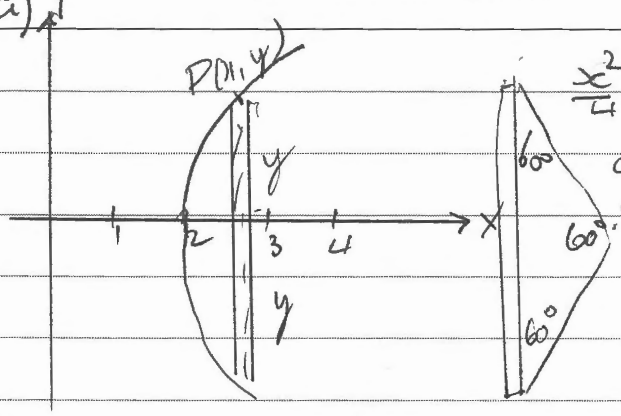
$= 2\pi\left[\left(\frac{1}{5} + \frac{4}{3} + 3 + c\right) - (0 + c)\right]$

$= 2\pi \times \frac{68}{15}$

$= \frac{136}{15}\pi$ cu units. — 1

MATHEMATICS EXTENSION 2 – QUESTION

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
15 a)		



$\frac{x^2}{4} - \frac{y^2}{5} = 1$

$a^2 = 4$

$a = 2$

Take $P(x,y)$ on the curve, $y > 0$

Area of cross-section

$A(x) = \frac{1}{2} ab \cdot \sin C$

$= \frac{1}{2} \cdot 2y \cdot 2y \cdot \sin 60°$

$= 2y^2 \times \frac{\sqrt{3}}{2}$

$A(x) = \sqrt{3} \cdot y^2$ — 1

$\delta V = \sqrt{3} \cdot y^2 \cdot \delta x$

$V = \lim_{\delta x \to 0} \sqrt{3} \cdot y^2 \cdot \delta x$ — 1

Now $\frac{x^2}{4} - \frac{y^2}{5} = 1$

$\frac{x^2}{4} + 1 = \frac{y^2}{5}$

$y^2 = 5\left(\frac{x^2}{4} - 1\right)$ — 1

so $V = \lim_{\delta x \to 0} \sum_{x=2}^{4} \sqrt{3} \cdot 5\left(\frac{x^2}{4} - 1\right) \delta x$

$= 5\sqrt{3} \int_2^4 \left(\frac{x^2}{4} - 1\right) dx$

$= 5\sqrt{3} \left[\frac{x^3}{12} - x + c\right]_2^4$

$= 5\sqrt{3} \left[\frac{64}{12} - 4 - \left(\frac{8}{12} - 2\right)\right]$

$= 5\sqrt{3} \times \frac{8}{3}$

$= \frac{40\sqrt{3}}{3}$ units^3 — 1

15 b) $I_n = \int_1^4 (\sqrt{x} - 1)^n \, dx, \quad n = 0, 1, 2, \dots$

i) $u = (x^{\frac{1}{2}} - 1)^n \qquad dv = 1$

$du = n(x^{\frac{1}{2}} - 1)^{n-1} \times \frac{1}{2} x^{-\frac{1}{2}} \qquad v = x$

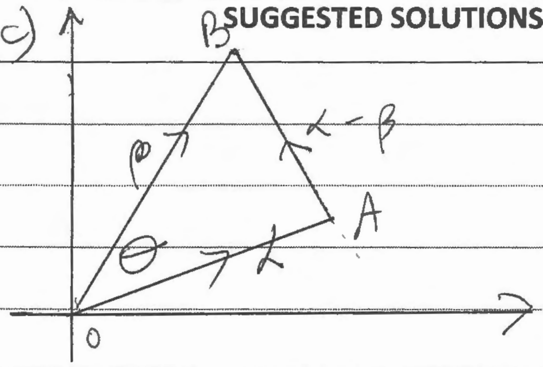
$= \frac{n(x^{\frac{1}{2}} - 1)^{n-1}}{2\sqrt{x}}$ — 1

$I_n = uv - \int v \, du$

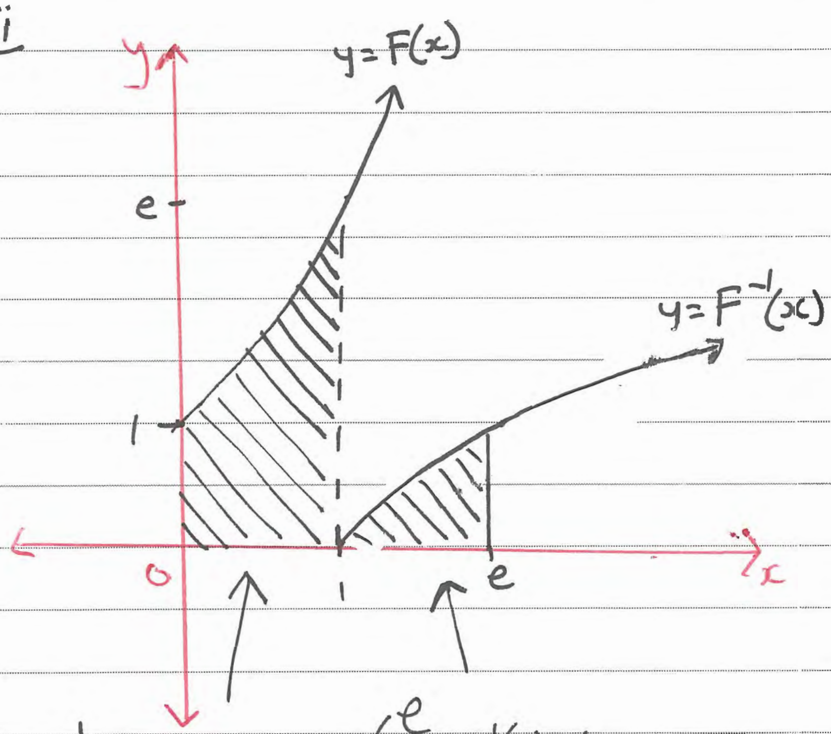
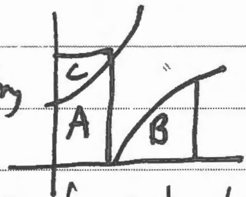
MATHEMATICS EXTENSION 2 – QUESTION

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
$I_n = \left[x(\sqrt{x}-1)^n + c\right]_1^4 - \int x \cdot \frac{n}{2} \frac{(\sqrt{x}-1)}{\sqrt{x}}\, dx$		
$= [4\times(2-1) - 0] - \frac{n}{2}\int \sqrt{x}(\sqrt{x}-1)^{n-1}\, dx$		
$= 4 - \frac{n}{2}\int((\sqrt{x}-1)+1)(\sqrt{x}-1)^{n-1}\, dx$		
$= 4 - \frac{n}{2}\int((\sqrt{x}-1)(\sqrt{x}-1)^{n-1} + 1\times(\sqrt{x}-1)^{n-1})\, dx$		
$= 4 - \frac{n}{2}\int[(\sqrt{x}-1)^n\, dx + (\sqrt{x}-1)^{n-1}]\, dx$		
$= 4 - \frac{n}{2}\int(\sqrt{x}-1)^n - \frac{n}{2}\int(\sqrt{x}-1)^{n-1}\, dx$	1	
$I_n = 4 - \frac{n}{2} I_n - \frac{n}{2} I_{n-1}$		
$2\cdot I_n = 8 - n\cdot I_n - n\cdot I_{n-1}$		
$\therefore (n+2)\cdot I_n = 8 - n\cdot I_{n-1}$	1	
15 b) ii) Put $n=2$, $4I_2 = 8 - 2\cdot I_1$		
$I_2 = 2 - \frac{1}{2} I_1$		
$I_2 = 2 - \frac{1}{2}\int_1^4 (x^{\frac{1}{2}} - 1)$		
$= 2 - \frac{1}{2}\left[\frac{2}{3}x^{3/2} - x + c\right]_1^4$		
$= 2 - \frac{1}{2}\left[\left(\frac{2}{3}\times 4^{3/2} - 4\right) - \left(\frac{2}{3} - 1\right)\right]$		
$= 2 - \frac{5}{6}$		
$= \frac{7}{6}$	1	
Put $n=3$, $I = 8 - 3\cdot I_2$		
$5\cdot I_3 = 8 - 3\times\frac{7}{6}$		
$I_3 = \frac{9}{10}$	1	
Put $n=4$, $6\cdot I_4 = 8 - 4\cdot I_3$		
$= 8 - 4\times\frac{9}{10}$		
$I_4 = \frac{1}{6}\times\frac{22}{5}$		
$\therefore I_4 = \frac{11}{15}$	1	

MATHEMATICS EXTENSION 2 – QUESTION

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
15 c) i) 		
$z+\bar{z} = 2\,\mathrm{Re}(z)$, $\lvert z\rvert = z\bar{z}$		
Prove $\lvert\alpha\rvert^2+\lvert\beta\rvert^2-\lvert\alpha-\beta\rvert^2 = 2\,\mathrm{Re}(\alpha\bar{\beta})$		
LHS $= \alpha\cdot\bar{\alpha} + \beta\bar{\beta} - (\alpha-\beta)(\overline{\alpha-\beta})$	1	
$= \alpha\cdot\bar{\alpha} + \beta\bar{\beta} - (\alpha-\beta)(\bar{\alpha}-\bar{\beta})$		
$= \alpha\cdot\bar{\alpha} + \beta\bar{\beta} - (\alpha\bar{\alpha} - \alpha\bar{\beta} - \beta\bar{\ } + \beta\bar{\beta})$		
$= \cancel{\alpha\cdot\bar{\alpha}} + \cancel{\beta\bar{\beta}} - \cancel{\alpha\bar{\alpha}} + \alpha\bar{\beta} + \beta\bar{\alpha} - \cancel{\beta\bar{\beta}}$		
$= \alpha\bar{\beta} + \bar{\alpha}\beta$	1	
$= \alpha\bar{\beta} + \overline{\alpha\bar{\beta}}$		
$= 2\,\mathrm{Re}(\alpha\bar{\beta})$	1	
ii) $AB = (\alpha-\beta)$		
Using cosine rule		
$\cos\theta = \dfrac{\lvert\alpha\rvert^2+\lvert\beta\rvert^2-\lvert\alpha-\beta\rvert^2}{2\cdot\lvert\alpha\rvert\cdot\lvert\beta\rvert}$	1	
$\cos\theta = \dfrac{2\,\mathrm{Re}(\alpha\bar{\beta})}{2\cdot\lvert\alpha\rvert\cdot\lvert\beta\rvert}$		
$\cos\theta = \dfrac{\mathrm{Re}(\alpha\bar{\beta})}{\lvert\alpha\rvert\cdot\lvert\beta\rvert}$		
$\lvert\alpha\rvert\cdot\lvert\beta\rvert\cdot\cos\theta = \mathrm{Re}(\alpha\bar{\beta})$	1	

MATHEMATICS EXTENSION 2 – QUESTION 16

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
a) i $F(x) = e^{x^2}, \quad x \geq 0$		
let $y = e^{x^2}$,		
For inverse, $x = e^{y^2}, \quad y \geq 0$		
$\ln x = y^2$		Errors of sign
$y = \sqrt{\ln x} \quad (\because y \geq 0)$	1	were worth $\frac{1}{2}$ a mark
ii $\ln x \geq 0$		
$\therefore x \geq 1$		
$\therefore$ domain : all real x, $x \geq 1$	1	No half marks
range : all real y, $y \geq 0$	1	were awarded.
		Note that if you got part i wrong, it was difficult to demonstrate the skills for part ii
iii 		
		1 for curves
		1 for regions
		(or one correct curve and its region = 1 mark)
$\int_0^1 F(x)\,dx \qquad \int_1^e F^{-1}(x)\,dx$		
iv) $\text{Area}_C = \text{Area}_B$ by symmetry 	1	
$\therefore \int_0^1 F(x)\,dx + \int_1^e F^{-1}(x)\,dx = \text{area of rectangle}$		
$= e \times 1$		
$= e \text{ units}^2$	1	

MATHEMATICS EXTENSION 2 – QUESTION 16

SUGGESTED SOLUTIONS	MARKS	MARKER'S COMMENTS
b) i $S_1 = \alpha^1 + \beta^1 + \gamma^1$		The answer to
$= \frac{-b}{a}$		this question is
$= \frac{0}{1}$		a value, not
$= 0$	1	an expression.
ii $S_2 = \alpha^2 + \beta^2 + \gamma^2$		
$= (\alpha + \beta + \gamma)^2 - 2(\alpha\beta + \alpha\gamma + \beta\gamma)$	1	
$= \frac{-b}{a} - 2\left(\frac{c}{a}\right)$		
$= 0 - 2k$		
$= -2k$	1	
iii LHS $= S_{n+3} + kS_{n+1} + S_n$		
$= \alpha^{n+3} + \beta^{n+3} + \gamma^{n+3} + k(\alpha^{n+1} + \beta^{n+1} + \gamma^{n+1}) + \alpha^n + \beta^n + \gamma^n$	1	
$= \alpha^{n+3} + k\alpha^{n+1} + \alpha^n$		
$+ \beta^{n+3} + k\beta^{n+1} + \beta^n + \gamma^{n+3} + k\gamma^{n+1} + \gamma^n$		
$= \alpha^n(\alpha^3 + k\alpha + 1) + \beta^n(\beta^3 + k\beta + 1) + \gamma^n(\gamma^3 + k\gamma + 1)$	1	
$= \alpha^n(0) + \beta^n(0) + \gamma^n(0)$	1	
$= 0$ ($\because \alpha, \beta$ and γ are roots)		
$=$ RHS		Note that α
		being a root does
iv When $n = 0$,		not imply that α^n
$S_3 + kS_1 + S_0 = 0$		is also a root.
$S_3 = -kS_1 - S_0$		
$= -k(0) - (\alpha^0 + \beta^0 + \gamma^0)$		
$= -3$	1	
when $n = 2$		
$S_5 + kS_3 + S_2 = 0$		
$S_5 = -kS_3 - S_2$		
$= -k(-3) - -2k$		
$\therefore S_5 = 5k$		
$\therefore \alpha^5 + \beta^5 + \gamma^5 = 5k$	1	