2017 SYDNEY BOYS HIGH SCHOOL
TRIAL HIGHER SCHOOL CERTIFICATE EXAMINATION

Mathematics Extension 2

General Instructions

- Reading time – 5 minutes
- Working time – 3 hours
- Write using black pen
- Board-approved calculators may be used
- A reference sheet is provided with this paper
- Leave your answers in the simplest exact form, unless otherwise stated
- All necessary working should be shown in every question if full marks are to be awarded
- Marks may **NOT** be awarded for messy or badly arranged work
- In Questions 11–16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I Pages 3–6

10 marks

- Attempt Questions 1–10
- Allow about 15 minutes for this section

Section II Pages 8–15

90 marks

- Attempt Questions 11–16
- Allow about 2 hour and 45 minutes for this section

Examiner: P.B.

Section I

10 marks
Attempt Questions 1–10

Allow about 15 minutes for this section

Use the multiple-choice answer sheet for Questions 1–10.

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1 One root of $z^2 - 3z + (3 + i) = 0$ is $1 + i$.
What is the other root?

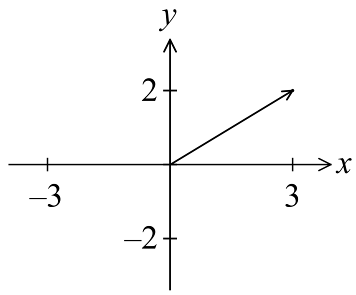
(A) $1 - i$

(B) $-(1 + i)$

(C) $2 - i$

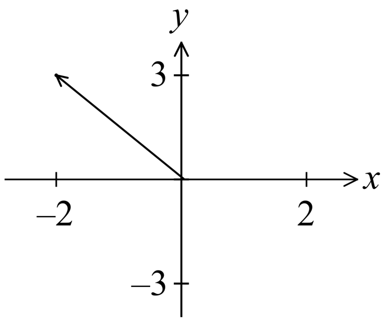
(D) $-2 + i$

2 The Argand diagram shows the complex number z.

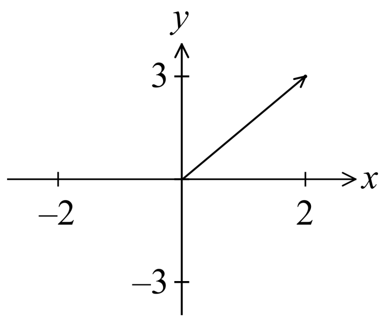


Which of the following represents $i\bar{z}$?

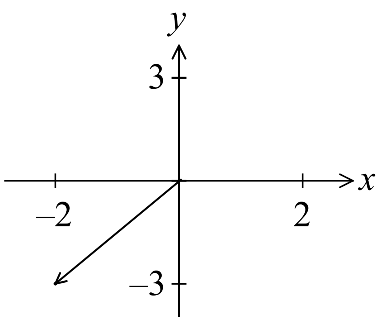
(A)



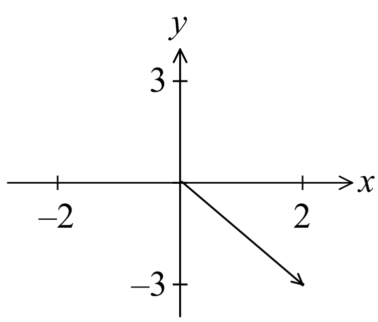
(B)



(C)



(D)



3 What is the value of $\int_0^1 x(1-x)^{99}\,dx$?

(A) $\dfrac{11}{10\,010}$

(B) $\dfrac{11}{10\,100}$

(C) $\dfrac{1}{10\,100}$

(D) $\dfrac{1}{10\,010}$

4 Given that $\omega^3 = 1$, where ω is not real.
What is the value of $(1 - \omega^2 + \omega)^3$?

(A) –8

(B) –1

(C) 1

(D) 8

5 The polynomial $P(x) = x^3 + 3x^2 - 24x + 28$ has a double zero.
What is its value?

(A) –7

(B) –4

(C) 2

(D) 4

6 Without evaluating the integrals, which of the following is false?

(A) $\int_1^2 e^{-x^2}\,dx < \int_0^1 e^{-x^2}\,dx$

(B) $\int_0^{\frac{\pi}{4}} \tan^2 x\,dx < \int_0^{\frac{\pi}{4}} \tan^3 x\,dx$

(C) $\int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{\sin x}{x}\,dx < \int_{\frac{\pi}{6}}^{\frac{\pi}{3}} \frac{1}{x}\,dx$

(D) $\int_1^2 \frac{1}{x+1}\,dx < \int_1^2 \frac{1}{x}\,dx$

7 Consider the graph of $x^3 + y^3 = 8$.
Which of the following is false?

(A) There is a vertical tangent at (2, 0) and a horizontal tangent at (0, 2).

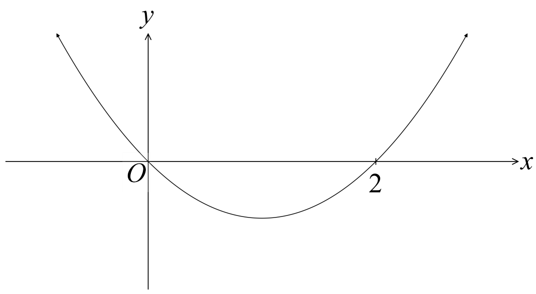
(B) $y = x$ is an oblique asymptote.

(C) $\dfrac{dy}{dx} < 0$ for all values of x and y except for $x = 0$ and $y = 0$.

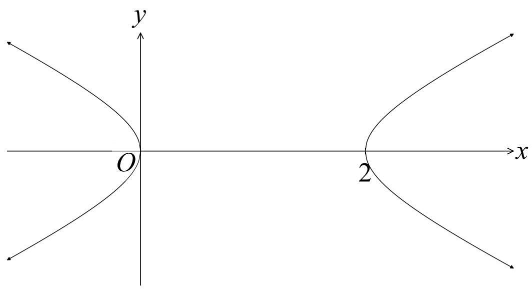
(D) The domain and range are both the set of all real numbers.

8 Which graph best represents $y^2 + 2x = x^2$?

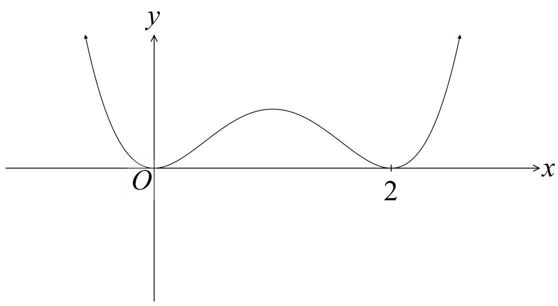
(A)



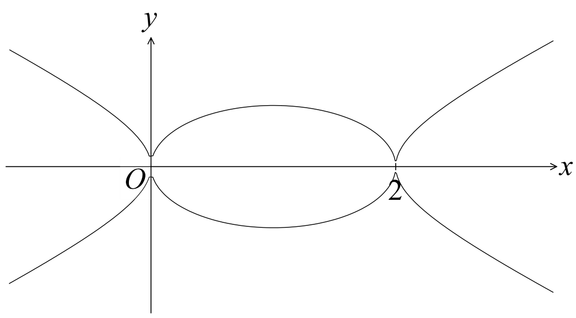
(B)



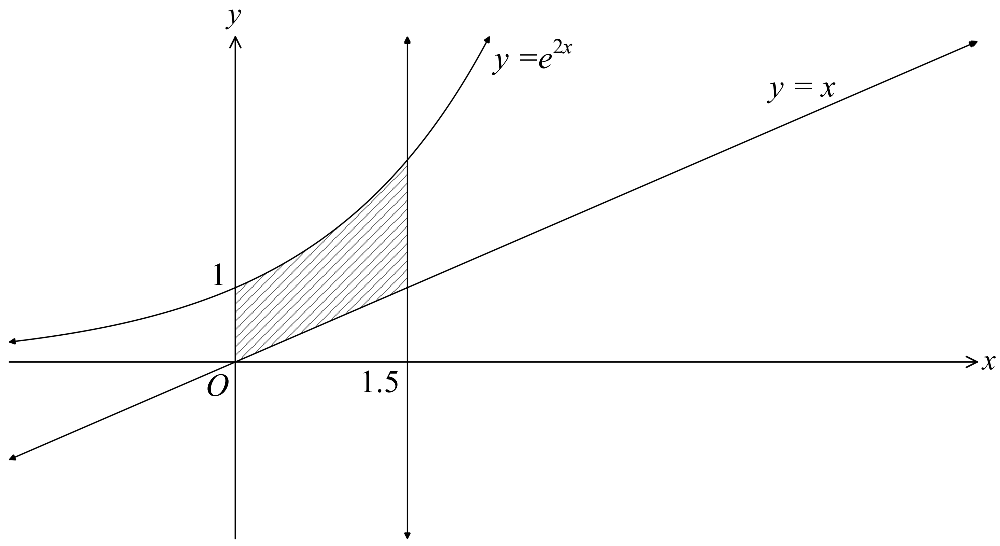
(C)



(D)



9 The region bounded by the curve $y = e^{2x}$, the line $y = x$, the y-axis and the line $x = 1.5$ is rotated about the y-axis to form a solid.



Using cylindrical shells, which integral represents the volume of this solid?

(A) $2\pi \int_0^{1.5} x(x - e^{2x})\, dx$

(B) $2\pi \int_0^{1.5} (e^{2x} - x)\, dx$

(C) $2\pi \int_0^{1.5} (x - e^{2x})\, dx$

(D) $2\pi \int_0^{1.5} x(e^{2x} - x)\, dx$

10 A hotel has three vacant rooms. Each room can accommodate a maximum of three people. In how many ways can five people be accommodated in the three rooms?

(A) 210

(B) 213

(C) 240

(D) 243

Section II

90 marks
Attempt Questions 11–16
Allow about 2 hour and 45 minutes for this section

Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11–16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find $\int \operatorname{cosec} x \, dx$ using $t = \tan \frac{1}{2}x$ **2**

(b) Using $u = x - 1$, find $\int \frac{x}{\sqrt{x-1}} \, dx$ **2**

(c) Find $\int \sin^6 x \cos^3 x \, dx$ **2**

(d) (i) Find A, B and C if $\frac{4x^2 - 2x}{(x+1)(x^2+1)} = \frac{A}{x+1} + \frac{Bx + C}{x^2+1}$ **2**

(ii) Hence find $\int \frac{4x^2 - 2x}{(x+1)(x^2+1)} \, dx$ **2**

(e) The region bounded by $y = x^3$, $0 \leq x \leq 2$ and the x-axis is rotated about the line $x = 4$. **3**
Use the method of cylindrical shells to find the volume generated.

(f) Shade the region given by $2 \leq z + \bar{z} \leq 8$ on an Argand diagram, **2**

where $z = x + iy$ for $x, y \in \mathbb{R}$.

Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) Find the quadratic equation whose roots are **2**

$$2 - i \text{ and } \frac{1}{2-i}.$$

The coefficients need to be in the form $a + ib$, where necessary.

(b) Find the square roots of $2 - 2\sqrt{3}i$. **2**
Express your answers in the form $a + ib$.

(c) (i) Express $1 - i\sqrt{3}$ and $1 + i\sqrt{3}$ in modulus-argument form. **1**

(ii) Hence, using de Moivre's Theorem, evaluate $\left(1 - i\sqrt{3}\right)^{10} + \left(1 + i\sqrt{3}\right)^{10}$. **2**

(d) What is the maximum value of $|z|$ if $|z + 1 + 2i| \leq 1$ **2**

(e) Sketch on separate diagrams

(i) $y = x^2 - x - 6$ and hence $y = |x - 3|(x + 2)$ **2**

(ii) $y = 3\sin 2x + 1$ and hence $|y| - 1 = 3\sin 2x$ for $|x| \leq \pi$. **2**

(iii) $y = \cos(\sin^{-1} x)$ **2**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) The equation $y^3 + 2y - 1 = 0$ has roots α, β and γ.
In each of the following find the polynomial equation which has roots:

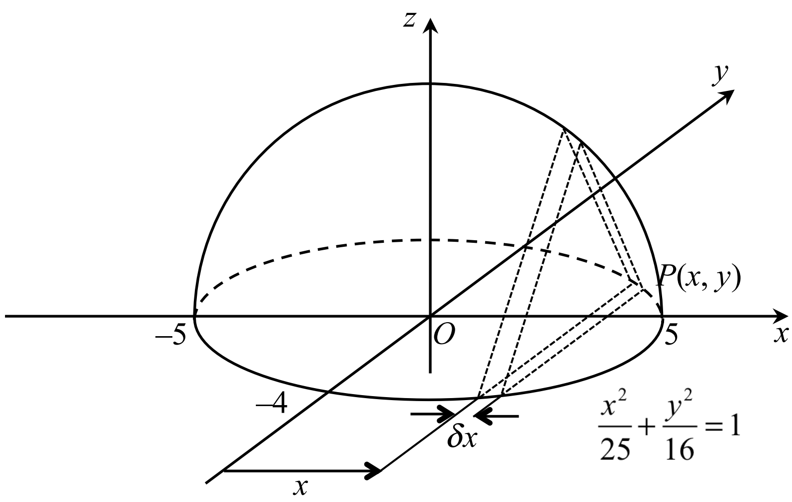
(i) $-\alpha, -\beta$ and $-\gamma$ **2**

(ii) α^2, β^2 and γ^2 **2**

(iii) $\pm\alpha, \pm\beta$ and $\pm\gamma$ **1**

(b) Find the equation of the tangent to the curve $x^3 + y^3 - 3xy - 3 = 0$ at the point (1, 2). **2**

(c) The base of a solid is an ellipse with equation $\dfrac{x^2}{25} + \dfrac{y^2}{16} = 1$.



Every cross section perpendicular to the x-axis is an equilateral triangle, one side of which lies in the base as shown above in the diagram above.

(i) Show that the cross section at $P(x, y)$ has area $y^2\sqrt{3}$. **2**

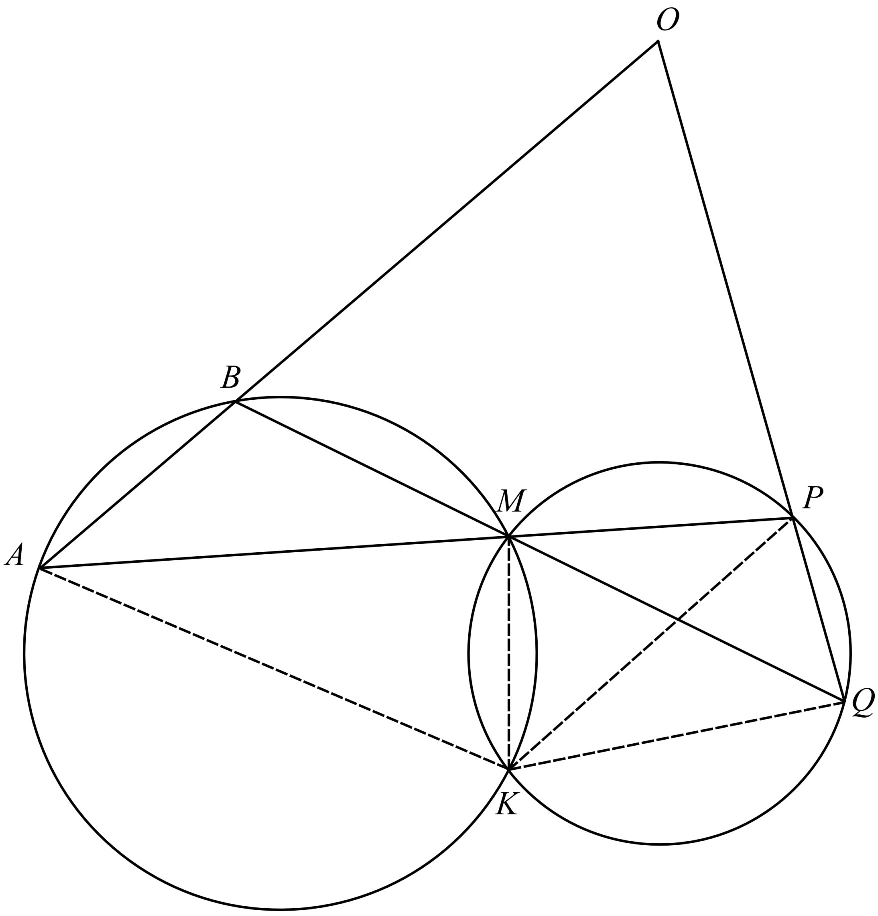
(ii) Hence find the volume of the slice of thickness δx as a function of x. **2**

(iii) Find the volume of the solid. **2**

(d) Find the limiting sum of the series $\dfrac{1}{5} + \dfrac{2}{5^2} + \dfrac{3}{5^3} + \dfrac{4}{5^4} + \ldots$ **2**

Question 14 (15 marks) Use a SEPARATE writing booklet.

(a)



The diagram shows two circles intersecting at K and M.
From points A and B on the arc of the larger circle, lines are drawn through M, to meet the smaller circle at P and Q respectively.
The lines AB and QP meet at O.

Answer on the insert provided.

(i) If $\theta = \angle KAB$ give a reason why $\angle KMQ = \theta$. **1**

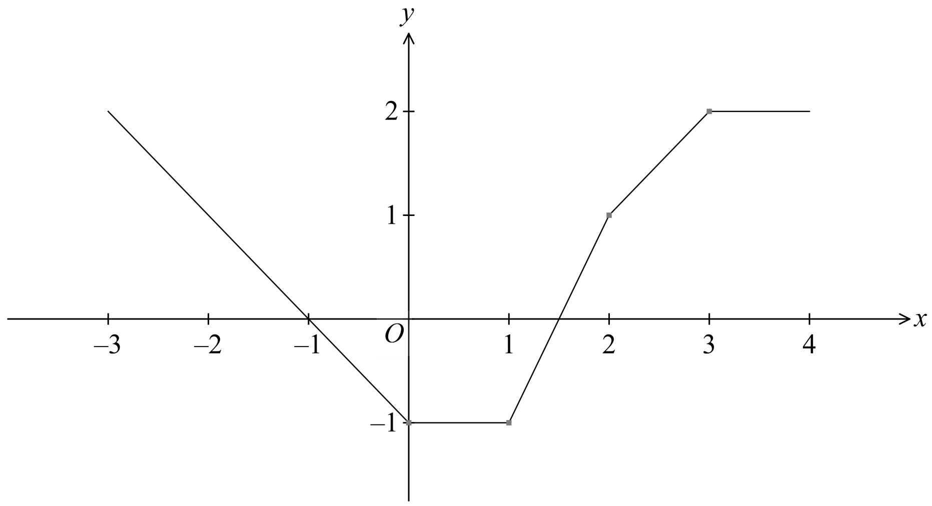
(ii) Prove that $AKPO$ is a cyclic quadrilateral. **1**

(iii) Let $\alpha = \angle AKM$. **3**
Show that if $OBMP$ is a cyclic quadrilateral, then the points A, K and Q are collinear.

Question 14 continues on page 13

Question 14 (continued)

(b)



The diagram above shows the graph of the function $y = f(x)$ for $-3 \leq x \leq 4$.

On the inserts provided sketch the following:

(i) $y \times f(x) = 1$ **2**

(ii) $y = |f(|x|)|$ **2**

(iii) $y = f'(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

(c) Twelve people are to be seated at two circular tables labelled A and B. **2**
In how many ways can this done if there are five people at table A and the remainder at table B?
Leave your answer in terms of combinations and factorials.

End of Question 14

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) The expansion of $\left(1+\frac{1}{n}\right)^n$ is

$$1+\frac{\binom{n}{1}}{n}+\frac{\binom{n}{2}}{n^2}+\ldots+\frac{\binom{n}{r}}{n^r}+\ldots+\frac{\binom{n}{n}}{n^n}$$

(i) Show that the $(r + 1)$th term, T_{r+1} in the expansion can be written as **3**

$$T_{r+1}=\frac{1}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)$$

(ii) Similarly, if U_{r+1} is the $(r + 1)$th term in the expansion of $\left(1+\frac{1}{n}\right)^{n+1}$, **3**

show that $U_{r+1} > T_{r+1}$.

(b) A particle of mass m kg is dropped from rest in a medium which causes a resistance of mkv, where v m/s is the particle's velocity and k is a constant.

(i) Show that the terminal velocity, V_T is given by $V_T = \frac{g}{k}$. **1**

(ii) Find the time taken to reach a velocity of $\frac{1}{2}V_T$. **2**

(iii) Find the distance travelled in this time. **3**

(c) A cube (6 faces) is to be painted using a different colour on each face.
In how many can this be done

(i) using six colours? **2**

(ii) using eight colours? **1**

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) Prove by the principle of mathematical induction that for all integral $n \geq 1$ **3**

$$\int t^n e^t \, dt = n! e^t \left[\frac{t^n}{n!} - \frac{t^{n-1}}{(n-1)!} + \frac{t^{n-2}}{(n-2)!} - \dots + (-1)^n \right]$$

[Ignore any constants of integration]

(b) (i) Find the non-real solutions of $z^7 = 1$ **2**

(ii) Express $z^7 - 1$ as a product of linear and quadratic factors, with real coefficients. **2**

(iii) Prove that $\cos\frac{3\pi}{7} + \cos\frac{\pi}{7} - \cos\frac{2\pi}{7} = \frac{1}{2}$. **2**

(c) (i) Prove that $\cot^{-1}(2x-1) - \cot^{-1}(2x+1) = \tan^{-1}\left(\frac{1}{2x^2}\right)$ **3**

(ii) For a positive integer n, define S as follows: **2**

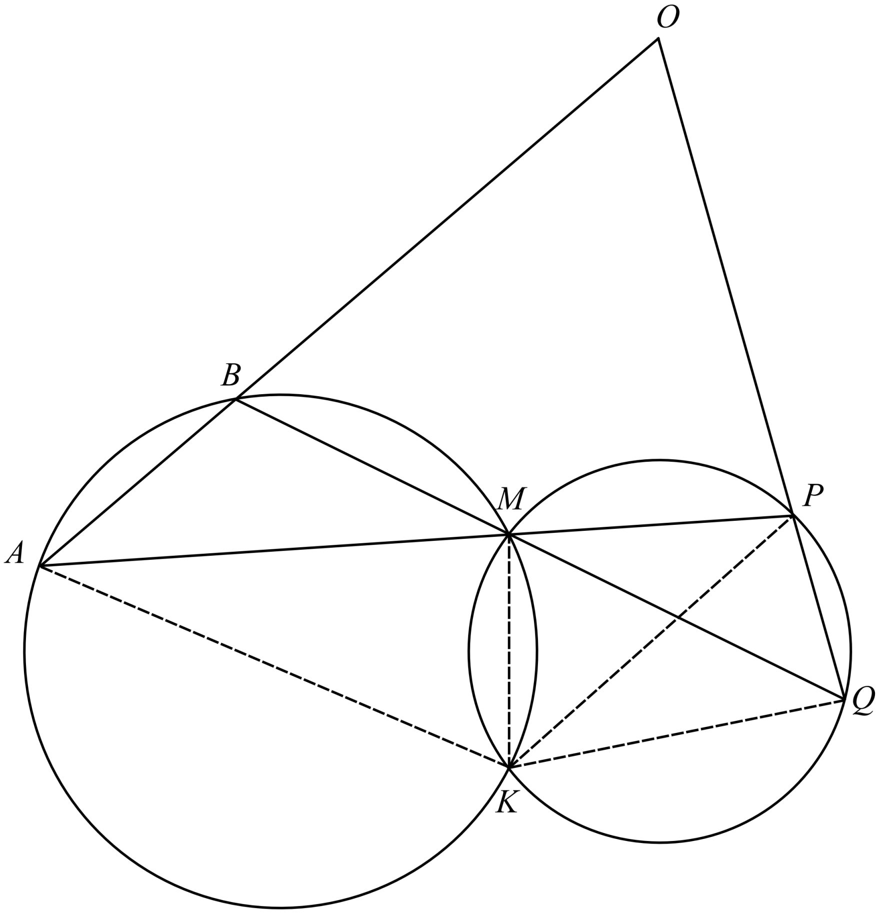
$$S = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18} + \dots + \tan^{-1}\left(\frac{1}{2n^2}\right),$$

Express S in simplest form.

(iii) Show that $\lim_{n \to \infty} S = \frac{\pi}{4}$. **1**

End of paper

(a)



The diagram shows two circles intersecting at K and M.
From points A and B on the arc of the larger circle, lines are drawn through M, to meet the smaller circle at P and Q respectively.
The lines AB and QP meet at O.

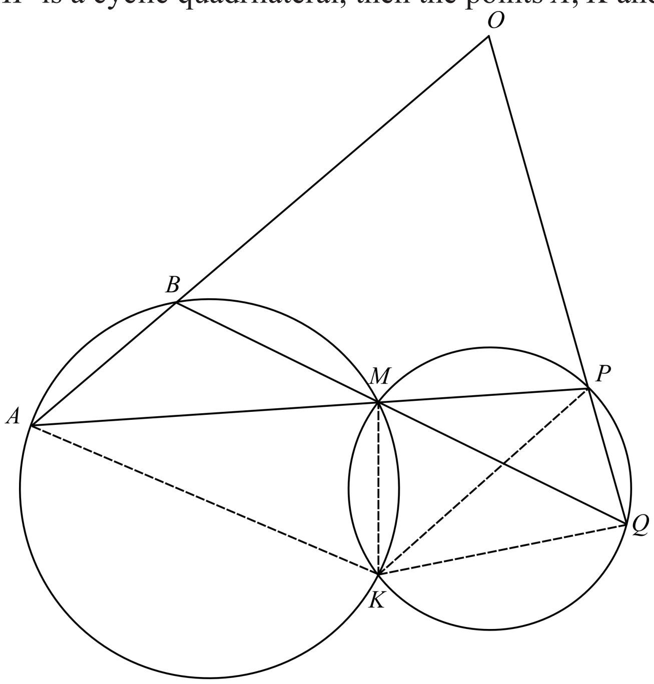
(i) If $\theta = \angle KAB$ give a reason why $\angle KMQ = \theta$. **1**

(ii) Prove that $AKPO$ is a cyclic quadrilateral. **1**

Turn over for part (iii)

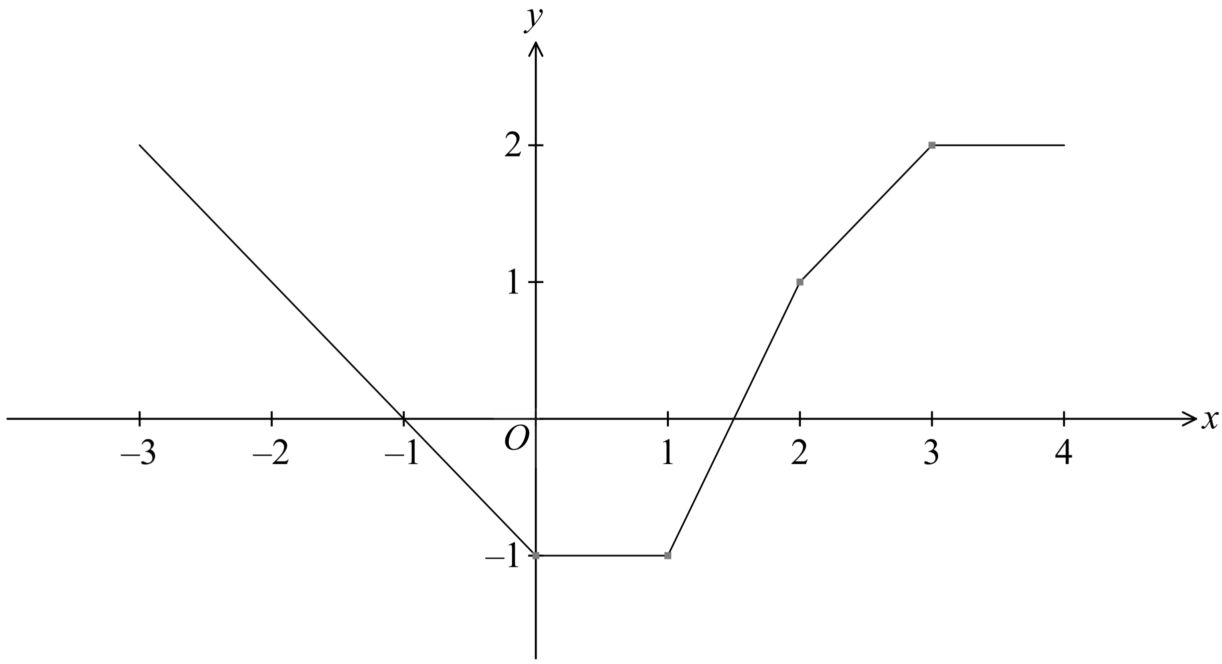
(iii) Let $\alpha = \angle AKM$. **3**

Show that if $OBMP$ is a cyclic quadrilateral, then the points A, K and Q are collinear.

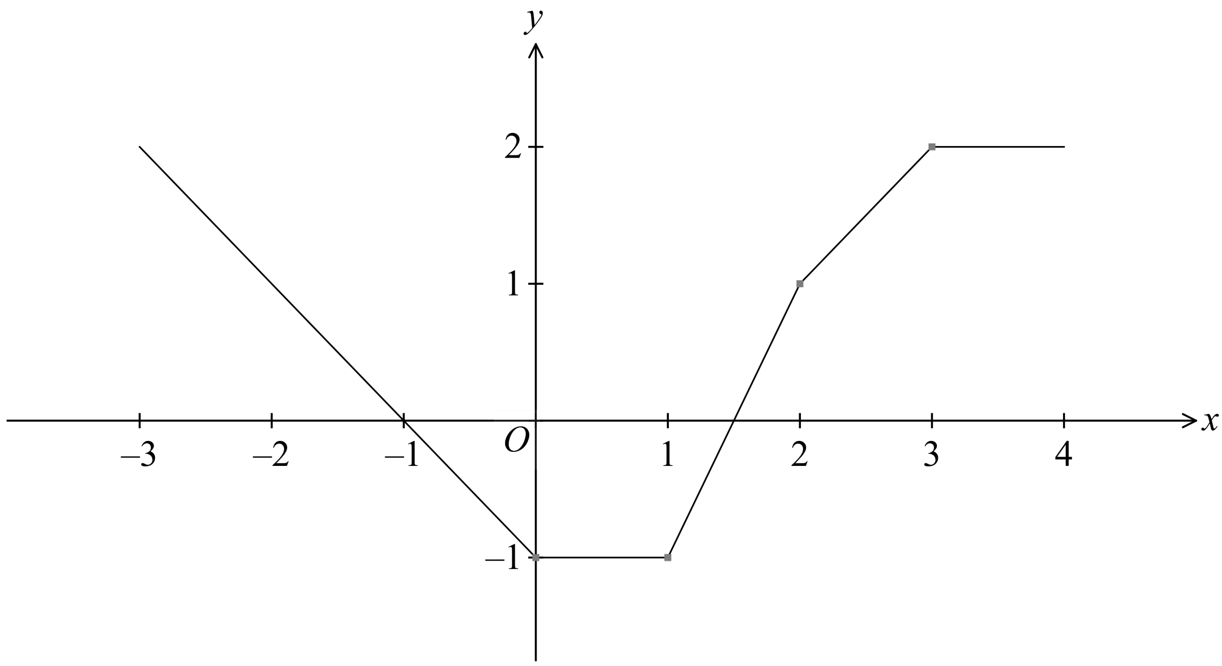


ME2 THSC Question 14 (b) **Student Number** ____________________

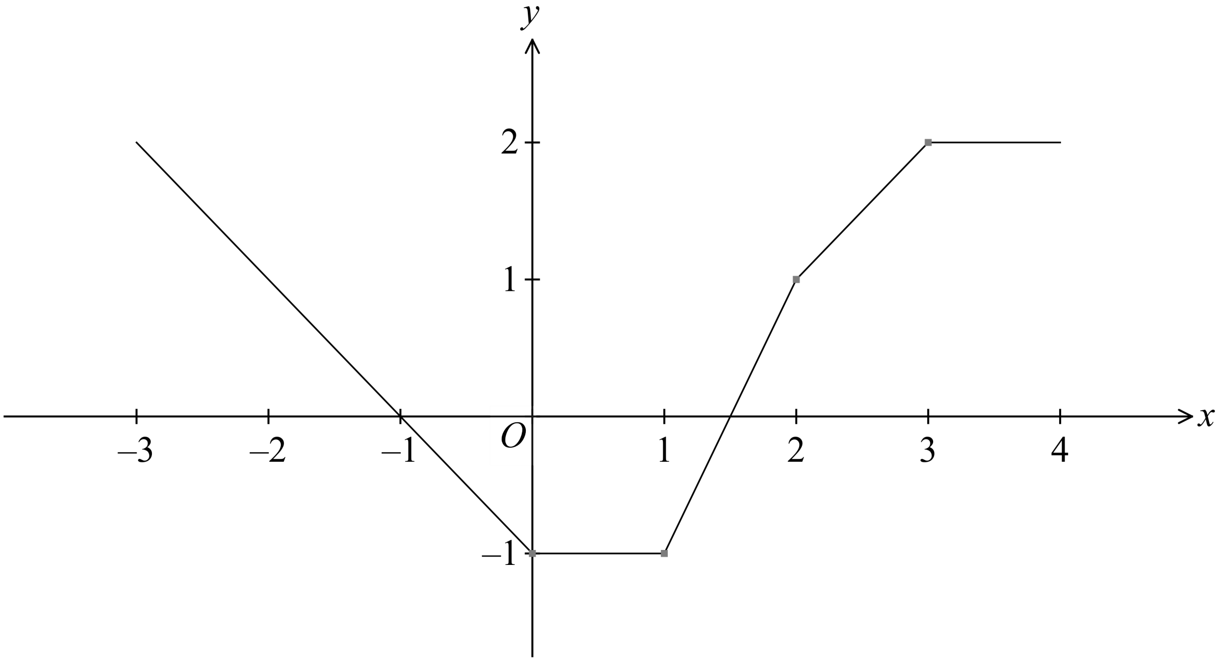
(i) $y \times f(x) = 1$ **2**



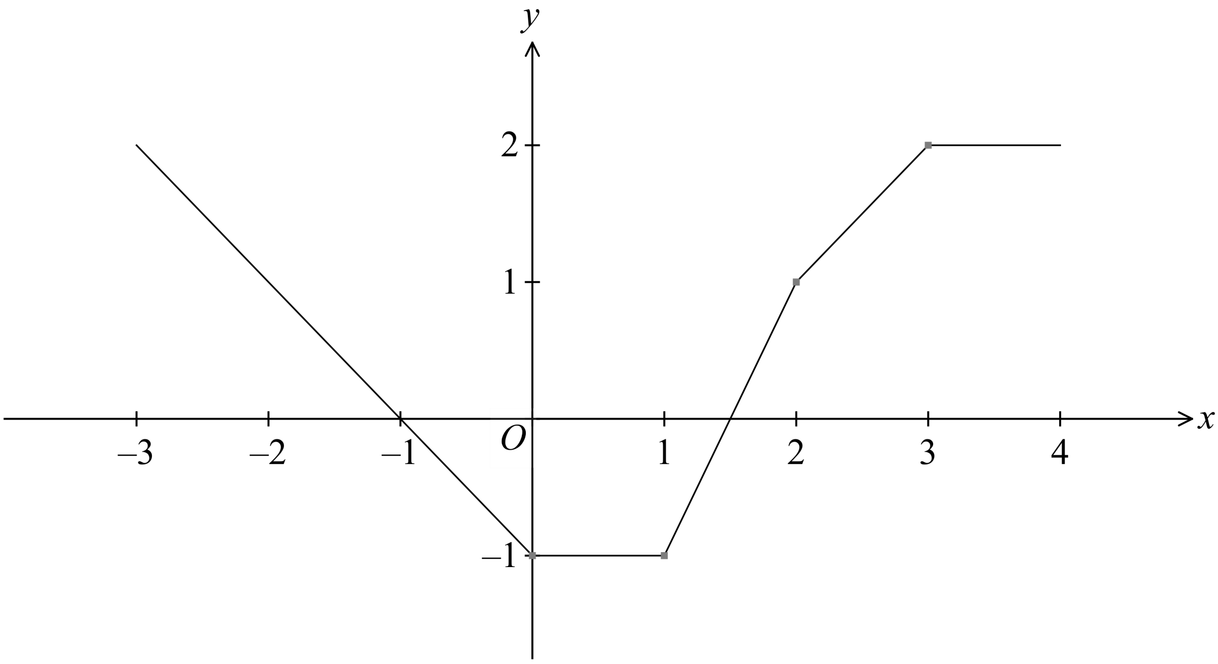
(ii) $y = |f(|x|)|$ **2**



(iii) $y = f'(x)$ **2**



(iv) $y = e^{f(x)}$ **2**



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Mathematics Extension 2

Suggested Solutions

MC Answers

Q1	C
Q2	B
Q3	C
Q4	A
Q5	C
Q6	B
Q7	B
Q8	B
Q9	D
Q10	A

X2 Y12 Assessment THSC 2017 Multiple choice solutions

Mean (out of 10): 8.83

1. Sum of roots = 3

$\therefore$ other root is

$3-(1+i) = 2-i$ (C)

A	6
B	1
C	109
D	2

2. z is $3+2i$

$\therefore \bar{z} = 3-2i$

$\therefore i\bar{z} = 2+3i$ (B)

A	4
B	109
C	5
D	0

3. $\int_0^1 x(1-x)^{99}\,dx$ Let $u=1-x$ $\therefore du = -dx$

$= \int_1^0 (1-u)u^{99}(-du)$

$= \int_0^1 (u^{99} - u^{100})\,du$

$= \left[\frac{1}{100}u^{100} - \frac{1}{101}u^{101}\right]_0^1$

$= \left[\frac{1}{100} - \frac{1}{101}\right] - [0]$

$= \frac{101-100}{10100}$

$= \frac{1}{10100}$ (C)

A	1
B	3
C	114
D	0

4. $\omega^3 = 1$

$1+\omega+\omega^2 = 0$

$\therefore -\omega^2 = 1+\omega$

$\therefore (1-\omega^2+\omega)^3 = (2(1+\omega))^3$

$= 8(1+3\omega+3\omega^2+\omega^3)$

$= 8(2+3\omega+3\omega^2)$

$= 8(3(1+\omega+\omega^2)-1)$

$= -8$ (A)

A	100
B	8
C	3
D	7

5. $P(x) = x^3+3x^2-24x+28$

$P'(x) = 3x^2+6x-24$

$= 3(x^2+2x-8)$

$= 3(x+4)(x-2)$

$\therefore$ Possible double zeros are $-4, 2$

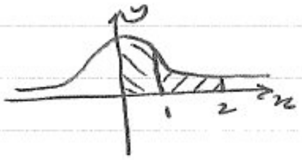
$P(2) = 8+12-48+28$

$= 0$

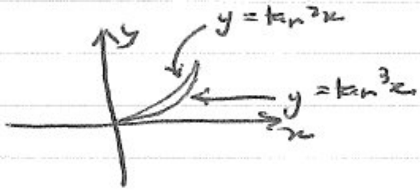
$\therefore 2$ is a double zero (C)

A	1
B	1
C	116
D	0

6.



A is true



∴ B is false

$\sin x < 1$ for $\frac{\pi}{6} < x < \frac{\pi}{3}$

∴ C is true

Larger denominator

⇒ smaller number.

∴ D is true

(B)

A	2
B	91
C	20
D	5

7. $x^3 + y^3 = 8$

$3x^2 + 3y^2 y' = 0$

At $(2,0)$ y' is undefined

At $(0,2)$ $y' = 0$

∴ A is true

$1 + \frac{y^3}{x^3} = \frac{8}{x^3}$

As $x \to \infty$ $\frac{y^3}{x^3} \to -1$

∴ $y = -x$ is an oblique asymptote

∴ B is false

$3y^2 y' = -3x^2$

∴ $y' = -\frac{x^2}{y^2} < 0$

(If $x = 0$ $y' = 0$

If $y = 0$, y undefined)

∴ C is true

$y^3 = 8 - x^3$ ∴ All values of x can be used

$x^3 = 8 - y^3$ ∴ All values of y can be used.

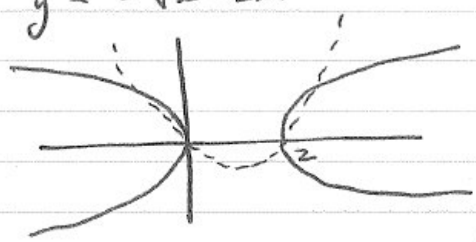
(B)

A	8
B	88
C	9
D	13

8. $y^2 + 2x = x^2$

$y^2 = x^2 - 2x$

$y = \pm\sqrt{x^2 - 2x}$



(B)

— OR —

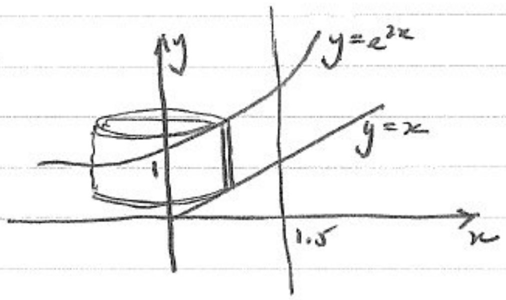
$x^2 - 2x + 1 = y^2 + 1$

$\therefore (x-1)^2 - y^2 = 1$

which is an hyperbola (B)

A	2
B	108
C	2
D	6

9.



Volume of a typical shell

$= \pi\left((x+\delta x)^2 - x^2\right) f(x)$

$= 2\pi x f(x)\, \delta x$

$\therefore$ Volume $= 2\pi \int_0^{1.5} x f(x)\, dx$

$= 2\pi \int_0^{1.5} x\left(e^{2x} - x\right) dx$

(D)

A	0
B	9
C	0
D	109

10. 3 1 1 $\quad {}^5C_3 \times 1 \times 6 = 60$

3 2 0 $\quad {}^5C_3 \times 1 \times 6 = 60$

2 2 1 $\quad \dfrac{{}^5C_2 \times {}^3C_2 \times {}^1C_1}{2} \times 6 = 90$

$\therefore$ 210 arrangements

— OR —

5 in a room $\quad 3$

4 in a room, 1 in another $\quad {}^5C_4 \times 6 = 30$

Total possible arrangements

$= 3^5$

$= 243$

$\therefore$ Required arrangements

$= 243 - 30 - 3 = 210$

A	62
B	4
C	43
D	8

11) a) $\int \operatorname{cosec} x \, dx$

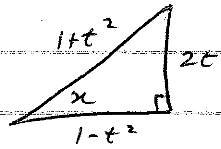
$t = \tan \frac{x}{2}$

$\frac{x}{2} = \tan^{-1} t$

$x = 2\tan^{-1} t$

$\frac{dx}{dt} = \frac{2}{1+t^2}$

$dx = \frac{2\,dt}{1+t^2}$



$= \int \frac{1+t^2}{2t} \times \frac{2\,dt}{1+t^2}$

$= \int \frac{dt}{t}$

$= \ln|t| + C$

$= \ln\left|\tan \frac{x}{2}\right| + C$

COMMENT: Students should note that the t-results are on the reference sheet.

b) $\int \frac{x}{\sqrt{x-1}} \, dx$

$u = x - 1$

$\frac{du}{dx} = 1$

$dx = du$

$= \int \frac{u+1}{\sqrt{u}} \, du$

$= \int \left(u^{\frac{1}{2}} + u^{-\frac{1}{2}}\right) du$

$= \frac{2}{3} u^{3/2} + 2u^{\frac{1}{2}} + C$

$= \frac{2}{3}\sqrt{(x-1)^3} + 2\sqrt{x-1} + C$

OR

$= \frac{2}{3}(x-1)\sqrt{x-1} + 2\sqrt{x-1} + C$

OR

$= \frac{2}{3}(x+2)\sqrt{x-1} + C$

COMMENT: Some students forgot to write their answer in terms of x.

c) $\int \sin^6 x \cos^3 x \, dx$

$= \int \cos x (1 - \sin^2 x) . \sin^6 x \, dx$

$= \int \cos x (\sin^6 x - \sin^8 x) dx$

$= \int (\cos x \sin^6 x - \cos x \sin^8 x) \, dx$

$= \frac{1}{7} \sin^7 x - \frac{1}{9} \sin^9 x + C$

COMMENT: Students needed to recognise that $\cos x$ allows for a substitution or the reverse chain rule.

d) i) $4x^2 - 2x = A(x^2+1) + (Bx+C)(x+1)$

let $x = -1$

$4(-1)^2 - 2(-1) = A((-1)^2 + 1)$

$2A = 6$

$A = 3$

let $x = 0$

$0 = A + C$

$C = -3$

equate coeff. of x^2

$4 = A + B$

$B = 1$

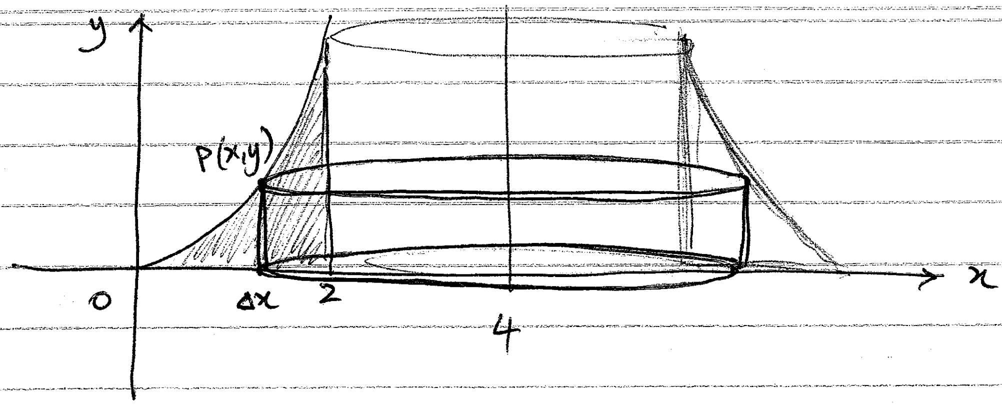
$\therefore A = 3, B = 1, C = -3$

ii) $\int \frac{4x^2 - 2x}{(x+1)(x^2+1)} dx = \int \left(3 \cdot \frac{1}{x+1} + \frac{1}{2} \cdot \frac{2x}{x^2+1} - 3 \cdot \frac{1}{x^2+1} \right) dx$

$= 3 \ln|x+1| + \frac{1}{2} \ln|x^2+1| - 3\tan^{-1} x + C$

COMMENT: This question was generally done well.

e)



$$\Delta V = 2\pi r h \Delta x$$
$$= 2\pi(4-x)y\Delta x$$
$$= 2\pi(4-x)x^3\Delta x$$
$$V = \lim_{\Delta x \to 0} \sum_{x=0}^{2} 2\pi(4x^3 - x^4)\Delta x$$
$$V = 2\pi \int_0^2 (4x^3 - x^4)\,dx$$
$$= 2\pi \left[x^4 - \frac{x^5}{5} \right]_0^2$$
$$= 2\pi \left[(2)^4 - \frac{(2)^5}{5} - (0) \right]$$
$$= \frac{96\pi}{5} \text{ cubic units}$$

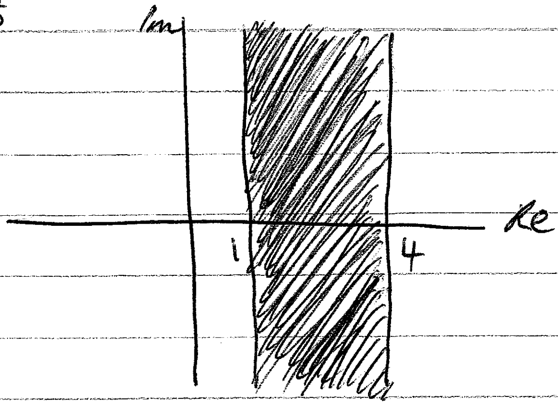
COMMENT: Very few students displayed any good habits in answering this question. For instance in order for x & y to mean anything there should be a variable point $P(x,y)$ on the graph. Also, in order to use the method of cylindrical shells students should not jump straight to the definite integral without considering ΔV.

f) $2 \le z + \bar{z} \le 8$

$2 \le 2\text{Re}(z) \le 8$

$2 \le 2x \le 8$

$1 \le x \le 4$



COMMENT: This question was done well.

Ext 2 Y12 THSC 2017 Q12 solutions

Mean (out of 15): 13.06

(a) Sum of roots

$= 2 - i + \frac{1}{2-i}$

$= 2 - i + \frac{2+i}{5}$

$= \frac{10 - 5i + 2 + i}{5}$

$= \frac{12 - 4i}{5}$

Product of roots $= 1$

$\therefore$ Required equation is

$z^2 - \frac{12-4i}{5} z + 1 = 0$

0	0.5	1	1.5	2	Mean
0	2	5	23	88	1.83

(b) $(a + ib)^2 = 2 - 2\sqrt{3}\, i$

$\therefore a^2 - b^2 + i.2ab = 2 - 2\sqrt{3}\, i$

$\therefore a^2 - b^2 = 2$

$2ab = -2\sqrt{3}$

$\therefore (a^2 - b^2)^2 = 4$

$\therefore a^4 - 2a^2b^2 + b^4 = 4$

$\therefore a^4 + 2a^2b^2 + b^4 = 4 + 12$

$= 16$

$\therefore a^2 + b^2 = 4$

$\therefore 2a^2 = 6$

$a^2 = 3$

$a = \pm\sqrt{3}$

$\therefore 2(\pm\sqrt{3})b = -2\sqrt{3}$

$\therefore b = \mp 1$

$\therefore$ Square roots are $\sqrt{3} - i$ and $-\sqrt{3} + i$

0	0.5	1	1.5	2	Mean
0	1	2	17	98	1.90

(c) (i) $1 - i\sqrt{3}$

$= 2 \operatorname{cis}\left(-\frac{\pi}{3}\right)$

$1 + i\sqrt{3}$

$= 2 \operatorname{cis}\left(\frac{\pi}{3}\right)$

0	0.5	1	Mean
0	4	114	0.98

(ii) $(1 - i\sqrt{3})^{10} + (1 + i\sqrt{3})^{10}$

$= 2^{10} \operatorname{cis}\left(-\frac{10\pi}{3}\right) + 2^{10} \operatorname{cis}\left(\frac{10\pi}{3}\right)$

$= 2^{10}\left(\operatorname{cis}\left(-\frac{4\pi}{3}\right) + \operatorname{cis}\left(\frac{4\pi}{3}\right)\right)$

$= 2^{10} . 2\cos\left(\frac{4\pi}{3}\right)$

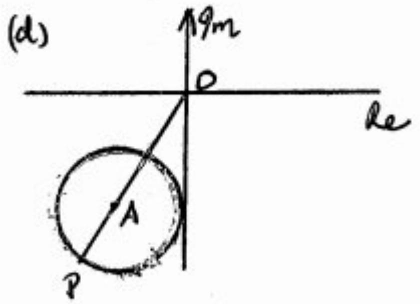
$= -2^{10} . 1$

$= -1024$

Some left there answer in terms of cos.

0	0.5	1	1.5	2	Mean
0	0	3	15	100	1.91

(d)



Max value of $|z| = OA + AP$

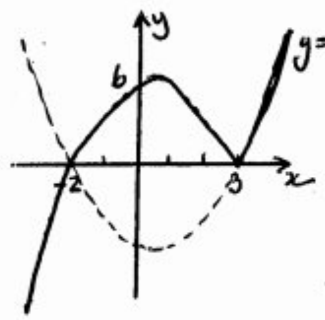
$= \sqrt{1+4} + 1$

$= \sqrt{5} + 1$

Recognising that the maximum value was achieved by passing through the centre of the circle was the key to success.

0	0.5	1	1.5	2	Mean
12	5	4	10	87	1.66

(e)



$y=|x-3|(x+2)$

Graph will be negative if $x+2$ is negative

$\therefore x+2<0$

$\therefore x<-2$

Some care should be taken to ensure that the required graph is obvious eg by dotting the original curve.

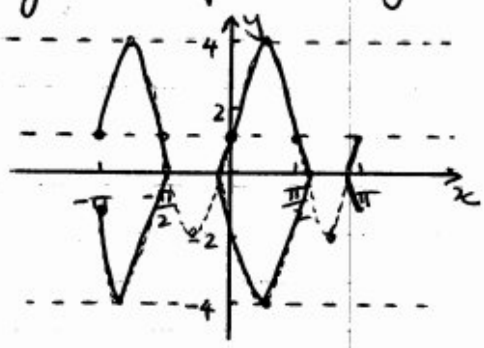
0	0.5	1	1.5	2	Mean
0	8	11	21	78	1.72

(ii) $|y|-1 = 3\sin 2x$

$\therefore |y| = 3\sin 2x + 1$

$\therefore 3\sin 2x+1 \geq 0$

y can be positive or negative



A common error was graphing $y=|3\sin 2x+1|$

0	0.5	1	1.5	2	Mean
3	11	31	27	46	1.43

(iii)

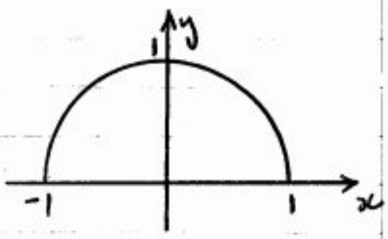
$y = \cos(\sin^{-1}x)$

$= \sqrt{1-x^2}$

$-1 \leq x \leq 1$

$-\frac{\pi}{2} \leq \sin^{-1}x \leq \frac{\pi}{2}$

$0 \leq \cos(\sin^{-1}x) \leq 1$



Recognising the graph was a semicircle was not done well. Some drew straight lines joining the intercepts.

0	0.5	1	1.5	2	Mean
9	1	15	18	75	1.63

Ext 2 Y12 THSC 2017 Q13 solutions

Mean (out of 15): 12.63

(a) (i) Let $u = -y$

$\therefore y = -u$

$\therefore (-u)^3 + 2(-u) - 1 = 0$

$\therefore -u^3 - 2u - 1 = 0$

$\therefore$ Eqn is $y^3 + 2y + 1 = 0$

Some students did not write an equation (leaving out = 0)

0	0.5	1	1.5	2	Mean
1	0	0	14	106	1.94

(ii) Let $u = y^2$

$\therefore y = \pm\sqrt{u}$

$\therefore (\pm\sqrt{u})^3 + 2(\pm\sqrt{u}) - 1 = 0$

$\therefore \pm u^{3/2} \pm 2u^{1/2} = 1$

$\therefore \pm\sqrt{u}(u+2) = 1$

$\therefore u(u^2 + 4u + 4) = 1$

$\therefore u^3 + 4u^2 + 4u - 1 = 0$

$\therefore$ Eqn is $y^3 + 4y^2 + 4y - 1 = 0$

Some students left square root signs in their equation => not a polynomial equation.

0	0.5	1	1.5	2	Mean
5	6	7	11	89	1.73

(iii) Let $u = \pm y$

$\therefore y = \mp u$

$\therefore (\mp u)^3 + 2(\mp u) - 1 = 0$

$\therefore \mp u^3 \mp 2u - 1 = 0$

$\therefore$ Eqn is $y^3 + 2y - 1 = 0$

OR $y^3 + 2y + 1 = 0$

— OR —

Another interpretation was that the question referred to 6 roots: $\alpha, -\alpha, \beta, -\beta, \gamma, -\gamma$

(NOTE: $\pm\alpha$ means $+\alpha$ OR $-\alpha$, not $+\alpha$ AND $-\alpha$

This interpretation was not penalised. Techniques used were:

$(y^3 + 2y - 1)(y^3 + 2y + 1) = 0$

$\therefore (y^3 + 2y)^2 - 1 = 0$

$\therefore y^6 + 4y^4 + 4y^2 - 1 = 0$

— OR —

$(x+\alpha)(x-\alpha)(x+\beta)(x-\beta)(x+\gamma)(x-\gamma) = 0$

$(x^2 - \alpha^2)(x^2 - \beta^2)(x^2 - \gamma^2) = 0$

$\therefore (x^2)^3 + 4(x^2)^2 + 4(x^2) - 1 = 0$ from (ii)

$\therefore x^6 + 4x^4 + 4x^2 - 1 = 0$

0	0.5	1	Mean
43	11	64	0.59

(b) $\frac{d}{dx}(x^3 + y^3 - 3xy - 3)$

$= 3x^2 + 3y^2y' - (3y + 3xy') = 0$

$\therefore y'(3y^2 - 3x) = 3y - 3x^2$

$\therefore y' = \dfrac{y - x^2}{y^2 - x}$

At $(1,2)$ $y' = \dfrac{2-1}{4-1}$

$= \frac{1}{3}$.

$\therefore$ Tangent is

$y - 2 = \frac{1}{3}(x - 1)$

$\therefore 3y - 6 = x - 1$

$\therefore x - 3y + 5 = 0$

0	0.5	1	1.5	2	Mean
1	4	20	13	80	1.71

(c) Area of equilateral triangle

$= \frac{1}{2} a^2 \sin 60^\circ$

$= \frac{1}{2} a^2 . \frac{\sqrt{3}}{2}$

$= \frac{\sqrt{3} a^2}{4}$

$\therefore$ Area of cross section

$= \frac{\sqrt{3} (2y)^2}{4}$

$= \sqrt{3} y^2$.

0	0.5	1	1.5	2	Mean
2	0	7	1	108	1.90

(ii) Volume of a typical slice

$= \sqrt{3} . 16 \left(1 - \frac{x^2}{25}\right) \delta x$

$= 16\sqrt{3} \left(1 - \frac{x^2}{25}\right) \delta x$

Some students did not write an expression for the volume of a slice.

0	0.5	1	1.5	2	Mean
0	2	13	9	94	1.83

(iii) Volume

$= \lim_{\delta x \to 0} 16\sqrt{3} \sum_{-5}^{5} \left(1 - \frac{x^2}{25}\right) \delta x$

$= 16\sqrt{3} \int_{-5}^{5} \left(1 - \frac{x^2}{25}\right) dx$

$= 16\sqrt{3} \left[x - \frac{x^3}{75}\right]_{-5}^{5}$

$= 16\sqrt{3} \left\{\left[5 - \frac{125}{75}\right] - \left[-5 - \frac{-125}{75}\right]\right\}$

$= 16\sqrt{3} \left\{2 \times \left(5 - \frac{5}{3}\right)\right\}$

$= 32\sqrt{3} \times \frac{10}{3}$

$= \frac{320\sqrt{3}}{3}$ units^3

0	0.5	1	1.5	2	Mean
1	1	7	11	98	1.86

(d) $\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \dots$

Consider $\frac{d}{dx}(x + x^2 + x^3 + \dots)$

$= 1 + 2x + 3x^2 + 4x^3 + \dots$

Let $x = \frac{1}{5}$:

$= 1 + \frac{2}{5} + \frac{3}{5^2} + \frac{4}{5^3} + \dots$

$= 5\left(\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \frac{4}{5^4} + \dots\right)$

$x + x^2 + x^3 + \dots$

$= \frac{x}{1-x}$ (for $|x| < 1$)

$\frac{d}{dx}\left(\frac{x}{1-x}\right) = \frac{(1-x).1 - x.(-1)}{(1-x)^2}$

$= \frac{1 - x + x}{(1-x)^2}$

$= \frac{1}{(1-x)^2}$

If $x = \frac{1}{5}$

$\frac{1}{(1-x)^2} = \frac{1}{\left(\frac{4}{5}\right)^2} = \frac{25}{16}$

$\therefore 5\left(\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \dots\right) = \frac{25}{16}$

$\therefore \frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \dots = \frac{5}{16}$.

— OR —

$S = \frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \dots$

$5.S = 1 + \frac{2}{5} + \frac{3}{5^2} + \dots$

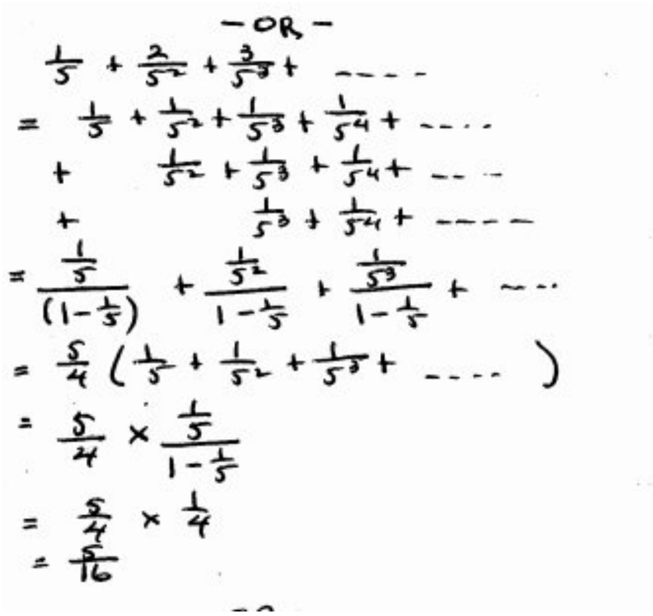
$\therefore 4S = 1 + \left(\frac{2}{5} - \frac{1}{5}\right) + \left(\frac{3}{5^2} - \frac{2}{5^2}\right) + \dots$

$= 1 + \frac{1}{5} + \frac{1}{5^2} + \dots$

$= \frac{1}{1 - \frac{1}{5}}$

$= \frac{5}{4}$

$\therefore S = \frac{5}{16}$


– OR –

$$\frac{1}{5} + \frac{2}{5^2} + \frac{3}{5^3} + \ldots$$

$$= \frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \ldots$$

$$+ \quad \frac{1}{5^2} + \frac{1}{5^3} + \frac{1}{5^4} + \ldots$$

$$+ \quad \frac{1}{5^3} + \frac{1}{5^4} + \ldots$$

$$= \frac{\frac{1}{5}}{(1-\frac{1}{5})} + \frac{\frac{1}{5^2}}{1-\frac{1}{5}} + \frac{\frac{1}{5^3}}{1-\frac{1}{5}} + \ldots$$

$$= \frac{5}{4}\left(\frac{1}{5} + \frac{1}{5^2} + \frac{1}{5^3} + \ldots\right)$$

$$= \frac{5}{4} \times \frac{\frac{1}{5}}{1-\frac{1}{5}}$$

$$= \frac{5}{4} \times \frac{1}{4}$$

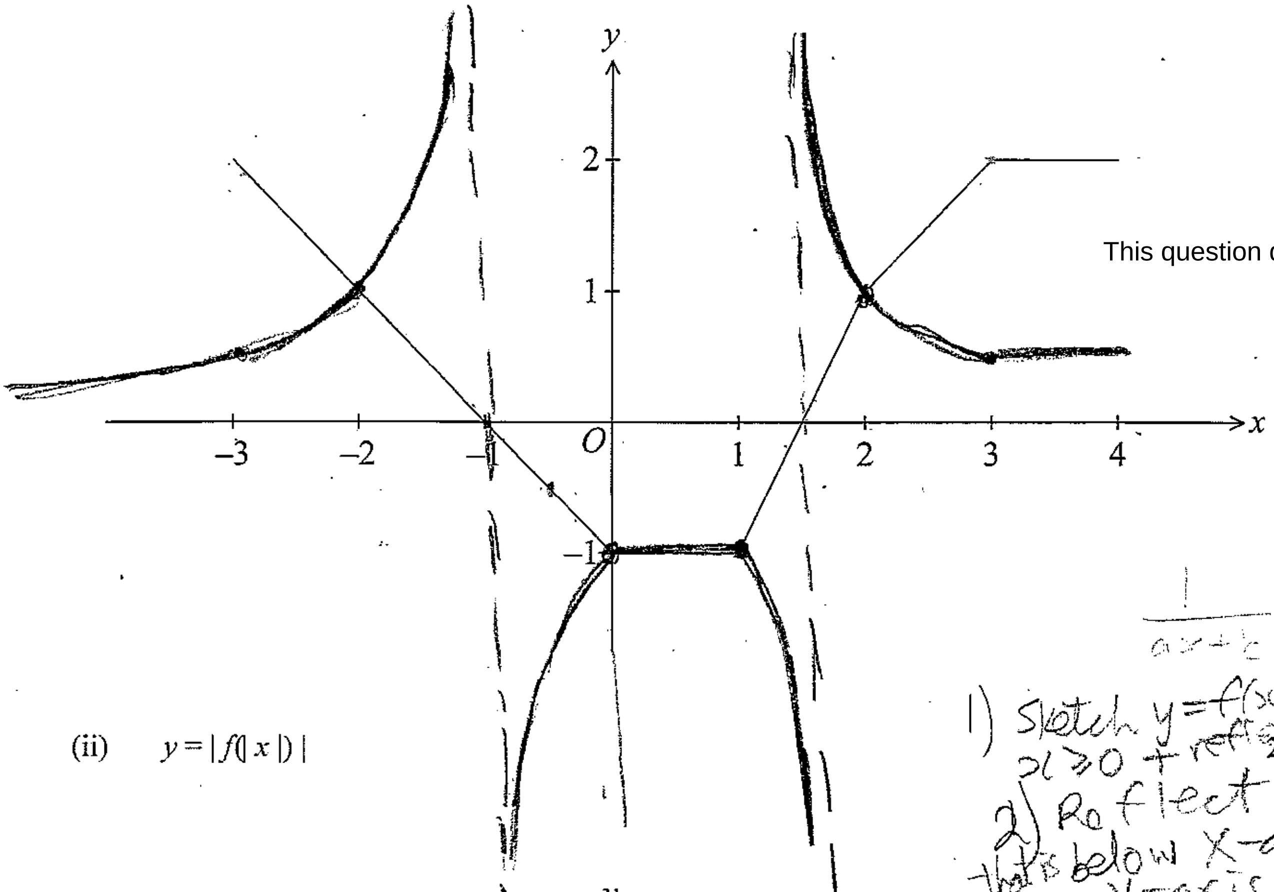
$$= \frac{5}{16}$$

Some students just wrote the answer or did not have working that justified their answer.

0	0.5	1	1.5	2	Mean
45	5	11	3	54	1.07

ME2 THSC Question 14 (b) **Student Number** ____________________

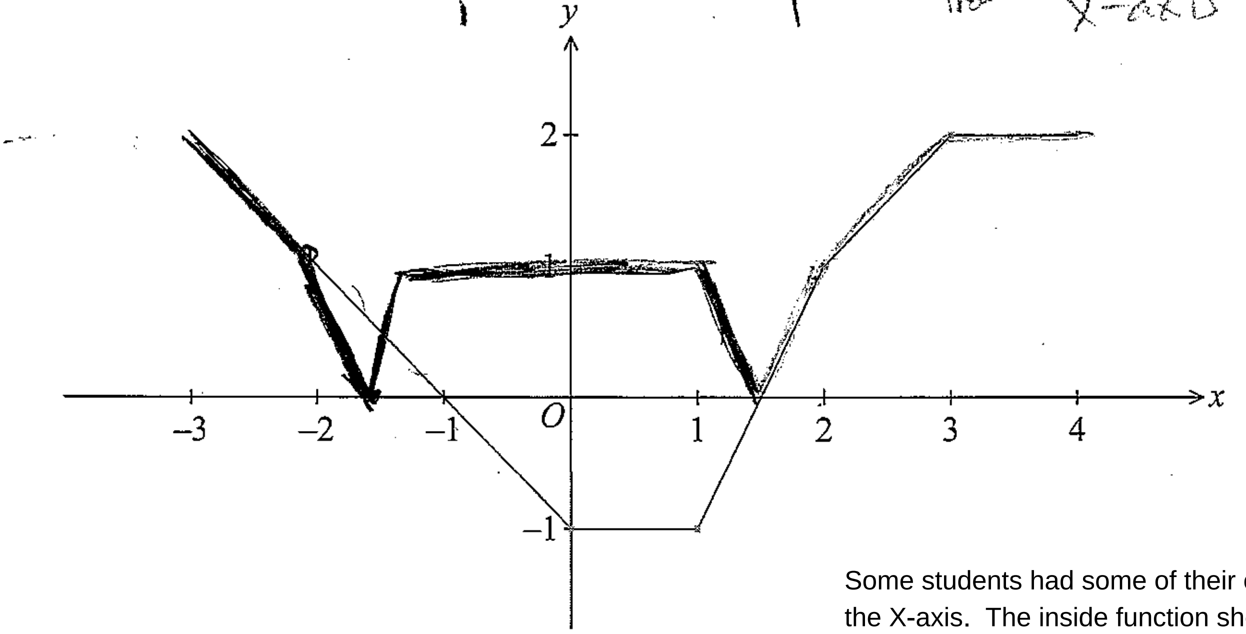
(i) $y \times f(x) = 1$ $\quad y = \frac{1}{f(x)}$ **2**



This question done very well.

(ii) $y = |f(|x|)|$ **2**

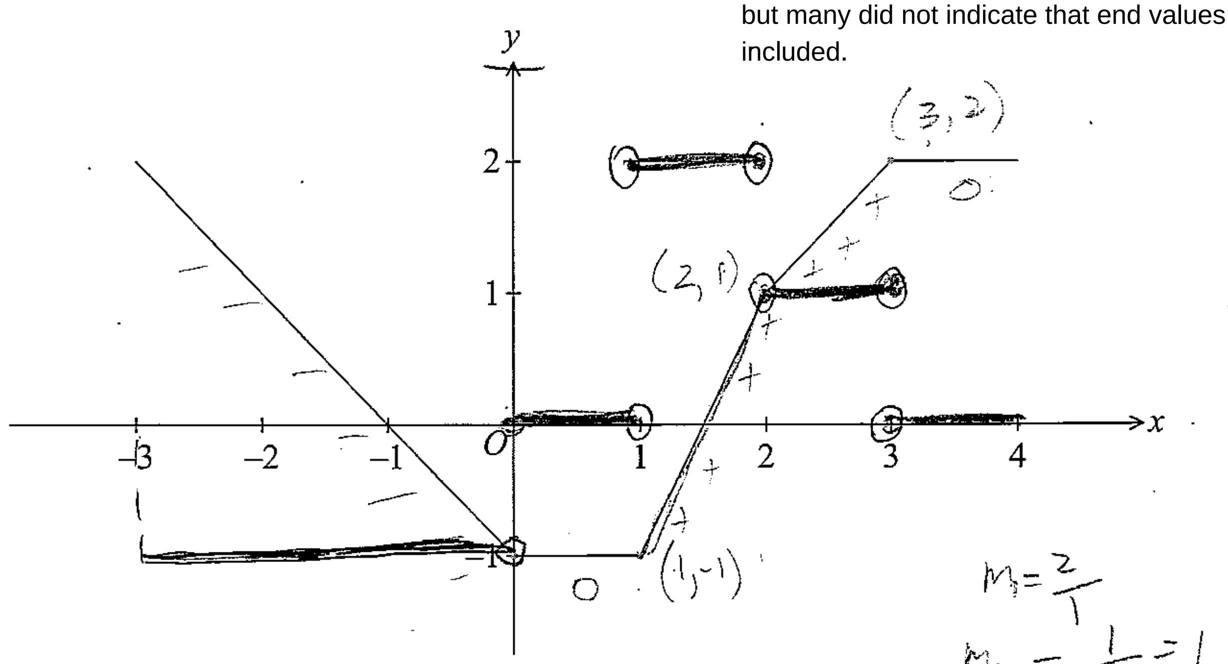
1) Sketch $y=f(x)$ for $x \geq 0$ + reflect in y-axis
2) Reflect graph that is below X-axis in X-axis



Some students had some of their curve below the X-axis. The inside function should be applied first.

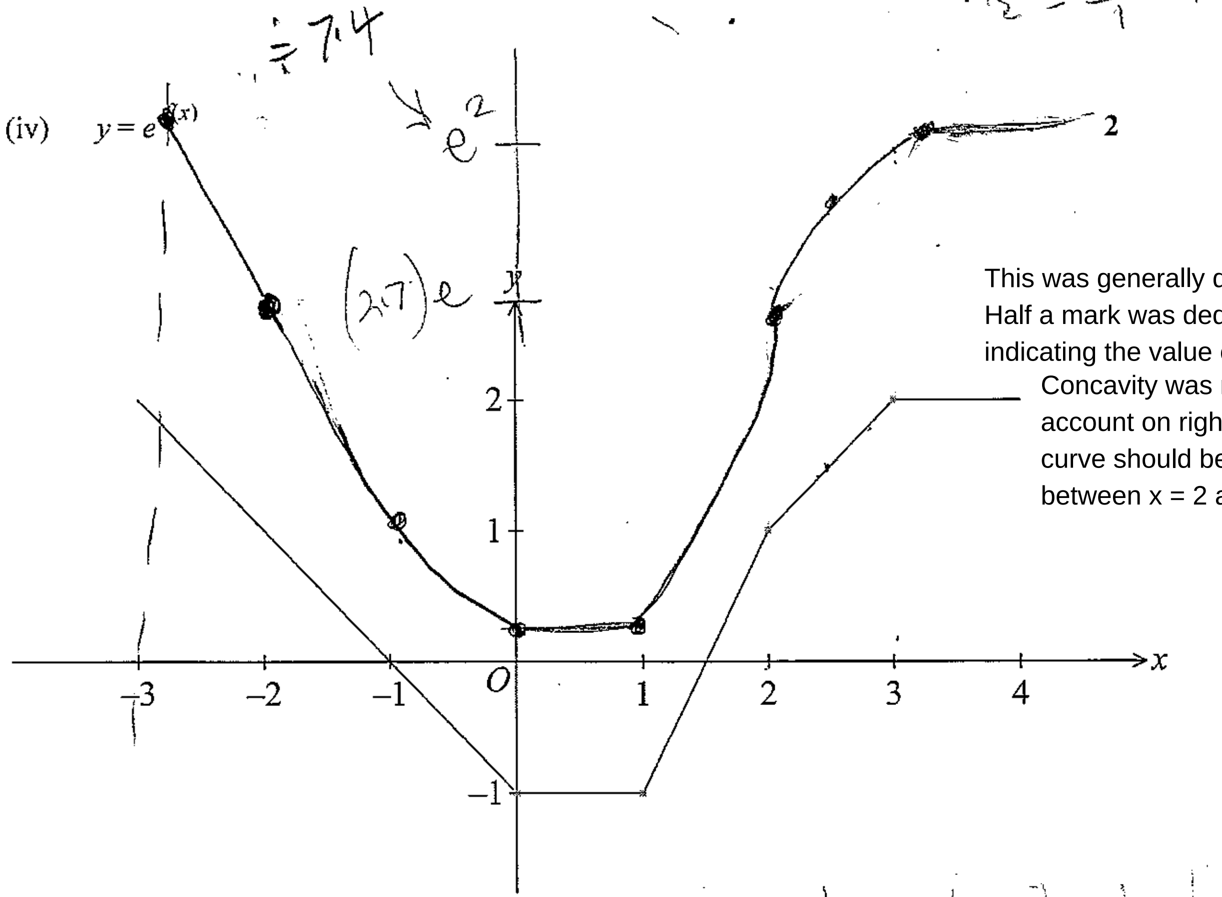
(iii) $y = f'(x)$ **2**

Students generally drew the intervals correctly but many did not indicate that end values are not included.



(iv) $y = e^{f(x)}$ **2**

This was generally done very well. Half a mark was deducted for not indicating the value of e^2 on y-axis. Concavity was not taken into account on right hand side but curve should be more steep between x = 2 and 3.

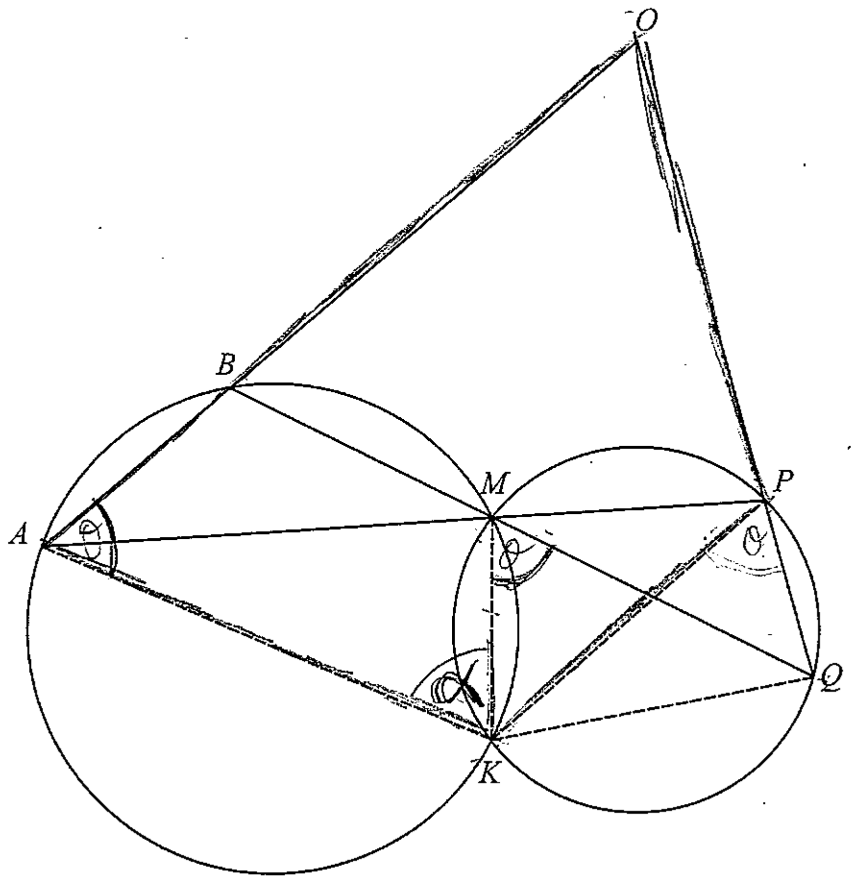


x	−3	−2	−1	0	1	2	3	4
$f(x)$	2	1	0	−1	−1	1	2	2
$e^{f(x)}$	e^2	e ≑2.7	1	$\frac{1}{e}$	$\frac{1}{e}$	e	e^2	e

$x = 2.5$

$e^{f(x)} = e^{1.5} ≑ 4.5$

(a)



The diagram shows two circles intersecting at K and M.
From points A and B on the arc of the larger circle, lines are drawn through M, to meet the smaller circle at P and Q respectively.
The lines AB and QP meet at O.

(i) If $\theta = \angle KAB$ give a reason why $\angle KMQ = \theta$. **1**

$\angle KMQ$ is the exterior $\angle$ of a cyclic quadrilateral which is equal to the interior opposite angle

This question was done well.

(ii) Prove that $AKPO$ is a cyclic quadrilateral. **1**

$\angle KPQ = \angle KMQ$ (Angles in same segment)

$\therefore \angle KPQ = \theta$

$\Rightarrow \angle OAK = \angle KPQ = \theta \Rightarrow$ exterior $\angle$ of quad = interior opp $\angle$

$\therefore AKPO$ is cyclic

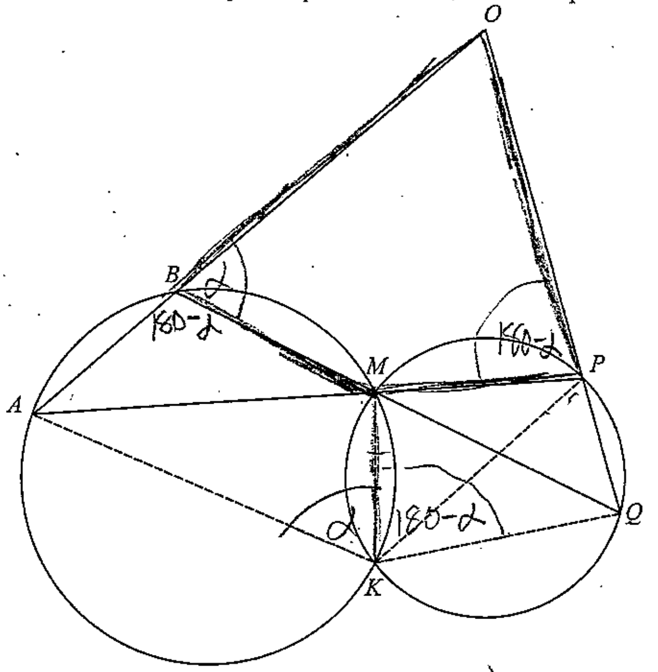
Turn over for part (iii)

This question was also done well.
Some reasons were a bit sloppy.

soln s

(iii) Let $\alpha = \angle AKM$. **3**

Show that if *OBMP* is a cyclic quadrilateral, then the points *A*, *K* and *Q* are collinear.



Prove $\angle AKM = \angle MKQ$.

$\angle AKM = \alpha$ (given)

Then $\angle ABM = 180 - \alpha$ (opp $\angle$s in cyclic quad. are supplementary)

Then $\angle OBM = \alpha$ (Angles on straight line)

$\Rightarrow \angle OPM = 180 - \alpha$ (assuming OBMP is cyclic $\Rightarrow$ opp $\angle$'s are supplementary)

Then $\angle MKQ = 180 - \alpha$ (ext $\angle$ of cyclic quad. are $=$ int. opp angle)

$\therefore \angle AKM + \angle MKQ = \alpha + 180 - \alpha = 180°$

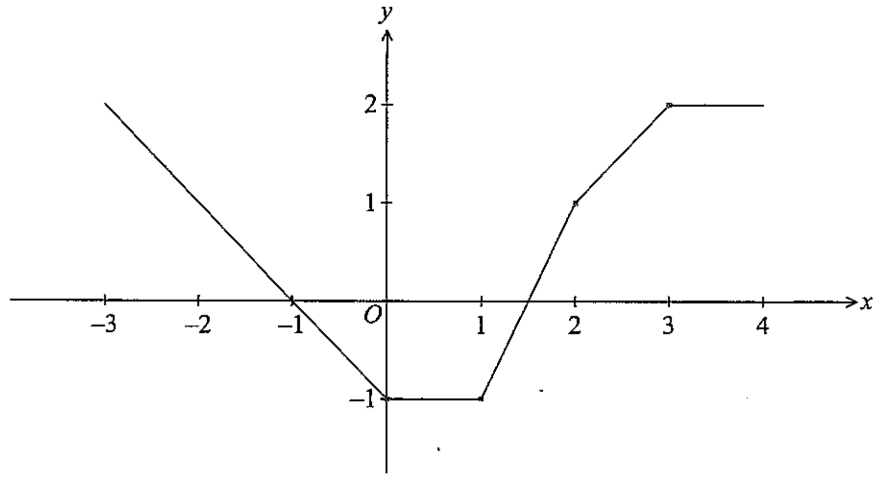
$\therefore \angle AKM$ and $\angle MKQ$ are on a straight line

$\Rightarrow$ A, K, Q are collinear

Most students got this question out. A few did not attempt it.

Question 14 (continued)

(b)



The diagram above shows the graph of the function $y = f(x)$ for $-3 \le x \le 4$.

On separate diagrams sketch

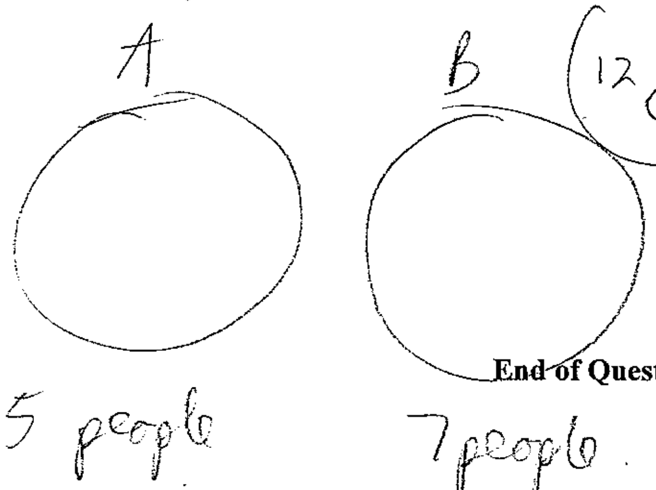
(i) $y \times f(x) = 1$ $\Rightarrow y = \frac{1}{f(x)}$ **2**

(ii) $y = |f(|x|)|$ **2**

(iii) $y = f'(x)$ **2**

(iv) $y = e^{f(x)}$ **2**

(c) Twelve people are to be seated at two circular tables labelled A and B. **2**
In how many ways can this done if there are five people at table A and the remainder at table B?
Leave your answer in terms of combinations and factorials.



$({}^{12}C_5 \times 4! \times 1 \times 6!)$

${}^{12}C_5 \times 4! \times 6!$

End of Question 14

This question was done well. 1 mark given for 12C5 and 1 mark for the factorials.

(a) The expansion of $\left(1+\frac{1}{n}\right)^n$ is

$$1+\frac{\binom{n}{1}}{n}+\frac{\binom{n}{2}}{n^2}+\ldots+\frac{\binom{n}{r}}{n^r}+\ldots+\frac{\binom{n}{n}}{n^n}$$

(i) Show that the $(r+1)$th term, T_{r+1} in the expansion can be written as **3**

$$T_{r+1}=\frac{1}{r!}\left(1-\frac{1}{n}\right)\left(1-\frac{2}{n}\right)\cdots\left(1-\frac{r-1}{n}\right)$$

$$\left(1+\frac{1}{n}\right)^n=\underbrace{1}_{T_0}+\underbrace{\frac{{}^nC_1}{n}}_{T_1}+\underbrace{\frac{{}^nC_2}{n^2}}_{T_2}+\ldots+\underbrace{\frac{{}^nC_r}{n^r}}_{T_{r+1}}+\ldots+\underbrace{\frac{{}^nC_n}{n^n}}_{T_{n+1}}$$

$$\begin{aligned}
T_{r+1}&=\frac{{}^nC_r}{n^r}\\
&=\frac{n!}{(n-r)!r!n^r}\\
&=\frac{1}{r!}\times\frac{n!}{(n-r)!}\times\frac{1}{n^r}\\
&=\frac{1}{r!}\times\frac{n\times(n-1)\times(n-2)\times\ldots\times(n-r+1)(n-r)!}{(n-r)!}\times\frac{1}{n^r}\\
&=\frac{1}{r!}\times\frac{n\times(n-1)\times(n-2)\times\ldots\times[n-(r-1)]}{n^r}\\
&=\frac{1}{r!}\times\frac{n}{n}\times\frac{n-1}{n}\times\frac{n-2}{n}\times\ldots\times\frac{n-(r-1)}{n}\\
&=\frac{1}{r!}\times\left(1-\frac{1}{n}\right)\times\left(1-\frac{2}{n}\right)\times\ldots\times\left(1-\frac{r-2}{n}\right)\times\left(1-\frac{r-1}{n}\right)
\end{aligned}$$

Comment:

There was a lot of confusion about what T_{r+1} meant.
Some took it as referring to a power of n, and hence there was a lot of fudging.
If a student got T_{r+1} wrong it was very hard for the student to get any marks.

A lot of students just ignored the information in the question and started the problem they wanted to and so they shouldn't be surprised if they lost marks.

(a) (ii) Similarly, if U_{r+1} is the $(r+1)$th term in the expansion of $\left(1+\frac{1}{n}\right)^{n+1}$, **3**

show that $U_{r+1} > T_{r+1}$.

Similarly, $\left(1+\frac{1}{n}\right)^{n+1} = 1+\frac{{}^{n+1}C_1}{n}+\frac{{}^{n+1}C_2}{n^2}+\ldots+\frac{{}^{n+1}C_r}{n^r}+\ldots+\frac{{}^{n+1}C_{n+1}}{n^{n+1}}$

$$U_{r+1} = \frac{{}^{n+1}C_r}{n^r}$$
$$= \frac{(n+1)!}{n^r r!(n-r+1)!}$$
$$= \frac{1}{r!}\times\frac{1}{n^r}\times\frac{(n+1)\times\ldots\times(n-r)\times(n-r+1)!}{(n-r+1)!}$$
$$= \frac{1}{r!}\times\frac{(n+1)\times\ldots\times(n-r)}{n^r}$$
$$= \frac{1}{r!}\times\left(1+\frac{1}{n}\right)\times(1)\times\left(1-\frac{1}{n}\right)\times\left(1-\frac{2}{n}\right)\times\ldots\times\left(1-\frac{r-2}{n}\right)$$

$$\frac{U_{r+1}}{T_{r+1}} = \frac{1+\frac{1}{n}}{1-\frac{r-1}{n}}$$
$$= \frac{1+\frac{1}{n}}{1+\frac{1}{n}-\frac{r}{n}}$$
$$> 1 \qquad \left[r \ge 0 \Rightarrow 1+\tfrac{1}{n} > 1+\tfrac{1}{n}-\tfrac{r}{n}\right]$$

$$\therefore U_{r+1} > T_{r+1}$$

Alternative 1:

$$U_{r+1} = \frac{{}^{n+1}C_r}{n^r}$$
$$= \frac{1}{n^r}\left({}^nC_r + {}^nC_{r-1}\right)$$
$$= T_{r+1} + \frac{{}^{n+1}C_{r-1}}{n^r}$$
$$> T_{r+1}$$

Alternative 2:

$$U_{r+1} = \frac{1}{r!}\times\left(1+\frac{1}{n}\right)\times\left(1-\frac{1}{n}\right)\times\left(1-\frac{2}{n}\right)\times\ldots\times\left(1-\frac{r-2}{n}\right)$$
$$= \frac{1+\frac{1}{n}}{1-\frac{r-1}{n}}\times T_{r+1}$$
$$= \frac{n+1}{(n+1)-r}\times T_{r+1}$$
$$> T_{r+1}$$

Comment:

Students who didn't write out a similar expansion to part (i), were more likely to get this question completely wrong.

It was very hard for students who carried the incorrect logic from part (i) into part(ii) to get any marks.

There were many students who lost 6 marks, because they didn't read the question.

Question 15 (continued)

(b) A particle of mass m kg is dropped from rest in a medium which causes a resistance of mkv, where v m/s is the particle's velocity and k is a constant.

(i) Show that the terminal velocity, V_T is given by $V_T = \frac{g}{k}$. **1**

Let $y = 0$ when $t = 0$.
Let $\dot{y} = v$
$t = 0$, $\dot{y} = 0$ and take $\dot{y} > 0$ as it falls.

$\therefore m\ddot{y} = mg - mkv$
$\therefore \ddot{y} = g - kv$

The terminal velocity is when the particle is travelling at a constant velocity i.e. $\dot{y} = V_T$, $\ddot{y} = 0$

$\therefore g - kV_T = 0$

$\therefore V_T = \frac{g}{k}$

Comment:
Students who just started from $\ddot{y} = g - kv$ with no explanation were penalised.

Students who chose to maintain their preference for downwards being the negative direction, found it hard to get many marks in all of part (b).

(ii) Find the time taken to reach a velocity of $\frac{1}{2}V_T$. **2**

Let $t = T$, $\dot{y} = \frac{1}{2}V_T$

$$\frac{dv}{dt} = g - kv$$

$$\therefore \frac{dv}{g - kv} = dt$$

$$\therefore -\frac{1}{k} \times \frac{-kdv}{g - kv} = dt$$

$$\therefore -\frac{1}{k}\int_0^{\frac{1}{2}V_T} \frac{-kdv}{g - kv} = \int_0^T dt$$

$$\therefore -\frac{1}{k}\left[\ln(g - kv)\right]_0^{\frac{1}{2}V_T} = T$$

$$\therefore T = -\frac{1}{k}\left[\ln(g - k \times \tfrac{1}{2}V_T) - \ln g\right]$$

$$= -\frac{1}{k}\left[\ln(g - k \times \tfrac{1}{2}\tfrac{g}{k}) - \ln g\right] = -\frac{1}{k}\left[\ln(g - \tfrac{g}{2}) - \ln g\right]$$

$$= -\frac{1}{k}\left[\ln(\tfrac{g}{2}) - \ln g\right] = -\frac{1}{k}\ln(\tfrac{1}{2})$$

$$= \frac{\ln 2}{k}$$

Question 15 (continued)

(b) (ii) (continued)

Alternative

Let $t = T, \dot{y} = \frac{1}{2}V_T$

$$\frac{dv}{dt} = g - kv$$

$$\therefore \frac{dv}{g - kv} = dt$$

$$\therefore -\frac{1}{k} \times \frac{-kdv}{g - kv} = dt$$

$$\therefore -\frac{1}{k}\int \frac{-kdv}{g - kv} = \int dt$$

$$\therefore t = -\frac{1}{k}\ln(g - kv) + C$$

$$\therefore 0 = -\frac{1}{k}\ln g + C \Rightarrow C = \frac{1}{k}\ln g$$

$$\therefore t = -\frac{1}{k}\ln(g - kv) + \frac{1}{k}\ln g$$

$$= -\frac{1}{k}\ln\left(\frac{g - kv}{g}\right)$$

$$\therefore T = -\frac{1}{k}\ln\left(\frac{g - k \times \frac{1}{2}V_T}{g}\right)$$

$$= -\frac{1}{k}\ln\left(\frac{g - \frac{1}{2}g}{g}\right)$$

$$= -\frac{1}{k}\ln(\tfrac{1}{2}g)$$

$$= \frac{1}{k}\ln 2$$

Comment:

Some students chose to prove the formula $v = \frac{g}{k}(1 - e^{-kt})$ in part (i).

The main problems of concern in this question involved handling logarithms and index rules.

As well, some students chose to ignore the constant of integration to their peril, when not using the definite integral approach.

Question 15 (continued)

(b)	(iii)	Find the distance travelled in this time.	**3**

Let $y = D$ when $t = T$ and $\dot{y} = \tfrac{1}{2}V_T$

$$\ddot{y} = v\frac{dv}{dy} = g - kv$$

$$\therefore \frac{vdv}{g-kv} = dy$$

$$\therefore \frac{1}{k}\times\frac{g-(g-kv)}{g-kv}dv = dy$$

$$\therefore \frac{1}{k}\times\left(\frac{g}{g-kv}-1\right)dv = dy$$

$$\therefore \frac{1}{k}\times\left(-\frac{g}{k}\times\frac{-k}{g-kv}-1\right)dv = dy$$

$$\therefore \frac{1}{k}\int_0^{\frac{1}{2}V_T}\left(-\frac{g}{k}\times\frac{-k}{g-kv}-1\right)dv = \int_0^D dy$$

$$\therefore D = \frac{1}{k}\int_0^{\frac{1}{2}V_T}\left(-\frac{g}{k}\times\frac{-k}{g-kv}-1\right)dv$$

$$= \frac{1}{k}\left[-\frac{g}{k}\ln(g-kv)-v\right]_0^{\frac{1}{2}V_T}$$

$$= \frac{1}{k}\left[-\frac{g}{k}\ln(g-k\times\tfrac{1}{2}V_T)-\tfrac{1}{2}V_T-\left(-\frac{g}{k}\ln g\right)\right]$$

$$= \frac{1}{k}\left[\frac{g}{k}\ln g-\frac{g}{k}\ln\frac{g}{2}-\frac{g}{2k}\right]$$

$$= \frac{g}{k^2}\left(\ln\frac{g}{\frac{g}{2}}-\frac{1}{2}\right)$$

$$= \frac{g}{k^2}\left(\ln 2-\frac{1}{2}\right)$$

Question 15 (continued)

(b) (iii) (continued)

Alternative

$$t = -\frac{1}{k}\ln\left(\frac{g-kv}{g}\right) \qquad \left[\text{From b (ii)}\right]$$

$$\therefore -tk = \ln\left(\frac{g-kv}{g}\right) \Rightarrow e^{-tk} = \frac{g-kv}{g}$$

$$\therefore g - kv = ge^{-tk} \Rightarrow kv = g(1 - ge^{-tk})$$

$$\therefore v = \frac{g}{k}(1 - ge^{-tk})$$

$$\therefore \frac{dx}{dt} = \frac{g}{k}(1 - ge^{-tk})$$

$$\therefore x = \frac{g}{k}\left(1 + \frac{g}{k}e^{-tk}\right) + C$$

At $t = 0$, $y = 0$

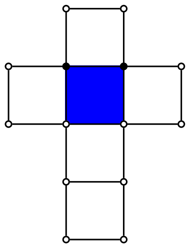
Comment:

Generally done well by most students.

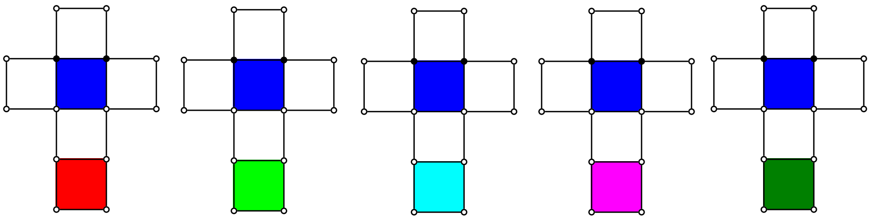
(c) A cube (6 faces) is to be painted using a different colour on each face.
In how many can this be done?

(i) using six colours? **2**

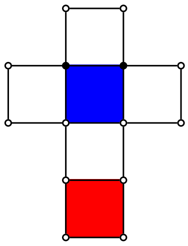
Consider the net of a cube.
Fix one of the faces with a colour i.e. paint it.



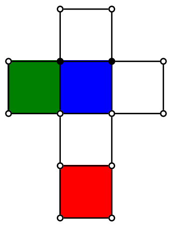
So there are 5 ways the face opposite can be painted



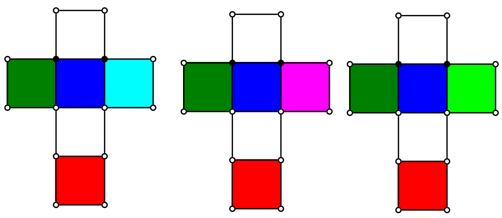
Now consider one of these 5 “cubes”.



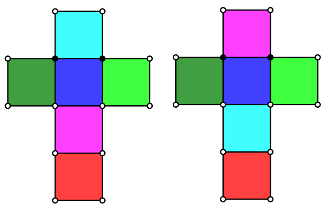
Paint one of the remaining faces



There are now 3 ways its opposite face can be painted.



There are now 2 ways to paint the remaining faces and still end up with different cubes, due to the fact that they can’t be rotated and get the same cube.
i.e. the two “cubes” below are different.



Total number of cubes = $5 \times 3 \times 2 = 30$

(c) (i) **Alternative**

Place the die on a surface. There are 5 possible numbers for the top face.
Now there is a ring (circle) of 4 faces which can be arranged in 3! ways.

$\therefore$ there are $5 \times 3! = \dfrac{6!}{6 \times 4} = 30$ ways.

(ii)	using eight colours?	**1**

There are $\binom{8}{6} = 28$ to choose the six colours to paint the cube.

From (i), having got 6 colours then there are 30 ways to paint the cube.

$\therefore$ there are $28 \times 30 = 840$ ways to do this.

Comment:

Most students were awarded a mark in part (i) if they provided some logic or their calculation showed some discernible logic.

It was surprising though that most students couldn't see that the best way to do part (ii).

16)a) Prove by induction for all integral $n \geq 1$

$$\int t^n e^t \, dt = n! e^t \left[\frac{t^n}{n!} - \frac{t^{n-1}}{(n-1)!} + \frac{t^{n-2}}{(n-2)!} - \ldots + (-1)^n \right]$$

Prove true for $n=1$

$$LHS = \int t e^t \, dt \qquad u = t \quad v' = e^t, \quad u' = 1 \quad v = e^t \qquad RHS = 1! e^t \left[\frac{t^1}{1!} - \frac{t^0}{0!} \right]$$

$$= te^t - \int e^t \, dt \qquad\qquad = e^t(t-1)$$

$$= te^t - e^t$$

$$= e^t(t-1)$$

$LHS = RHS$

$\therefore$ true for $n=1$

Assume true for $n=k$ where $k \in \mathbb{N}$

$$\int t^k e^t \, dt = k! e^t \left[\frac{t^k}{k!} - \frac{t^{k-1}}{(k-1)!} + \frac{t^{k-2}}{(k-2)!} - \ldots + (-1)^k \right]$$

Prove true for $n = k+1$

ie $$\int t^{k+1} e^t \, dt = (k+1)! e^t \left[\frac{t^{k+1}}{(k+1)!} - \frac{t^k}{k!} + \frac{t^{k-1}}{(k-1)!} - \ldots + (-1)^{k+1} \right]$$

$$LHS = \int t^{k+1} e^t \, dt \qquad u = t^{k+1} \quad v' = e^t, \quad u' = (k+1)t^k \quad v = e^t$$

$$= t^{k+1} e^t - (k+1) \int t^k e^t \, dt$$

$$= t^{k+1} e^t - (k+1) . k! e^t \left[\frac{t^k}{k!} - \frac{t^{k-1}}{(k-1)!} + \frac{t^{k-2}}{(k-2)!} - \ldots + (-1)^k \right]$$

$$= \frac{(k+1)! t^{k+1} e^t}{(k+1)!} - (k+1)! e^t \left[\frac{t^k}{k!} - \frac{t^{k-1}}{(k-1)!} + \frac{t^{k-2}}{(k-2)!} - \ldots + (-1)^k \right]$$

$$= (k+1)! e^t \left[\frac{t^{k+1}}{(k+1)!} - \frac{t^k}{k!} + \frac{t^{k-1}}{(k-1)!} - \ldots + (-1)^{k+1} \right]$$

$= RHS$

$\therefore$ true for $n = k+1$

$\therefore$ true by induction for all integral $n \geq 1$

COMMENT: Care needed to be taken with integration by parts so that the assumption could be used.

b) i) let $z = \text{cis}\,\theta$

$$(\text{cis}\,\theta)^7 = \text{cis}\,0$$

$$\text{cis}\,7\theta = \text{cis}\,0$$

$$7\theta = 2k\pi$$

$$\theta = \frac{2k\pi}{7}$$

$$z_1 = \text{cis}\,\frac{2\pi}{7}$$

$$z_2 = \text{cis}\,\frac{4\pi}{7}$$

$$z_3 = \text{cis}\,\frac{6\pi}{7}$$

$$z_4 = \text{cis}\,\frac{8\pi}{7}$$

$$z_5 = \text{cis}\,\frac{10\pi}{7}$$

$$z_6 = \text{cis}\,\frac{12\pi}{7}$$

$$z_7 = \text{cis}\,0 = 1$$

Non-real solutions of $z^7 = 1$ are $\text{cis}\frac{2\pi}{7}, \text{cis}\frac{4\pi}{7}, \text{cis}\frac{6\pi}{7}, \text{cis}\left(-\frac{6\pi}{7}\right), \text{cis}\left(-\frac{4\pi}{7}\right), \text{cis}\left(\frac{2\pi}{7}\right)$

ii) $$z^7 - 1 = (z-1)\underbrace{\left(z - \text{cis}\frac{2\pi}{7}\right)\left(z - \text{cis}\left(-\frac{2\pi}{7}\right)\right)}\underbrace{\left(z - \text{cis}\frac{4\pi}{7}\right)\left(z - \text{cis}\left(-\frac{4\pi}{7}\right)\right)}\underbrace{\left(z - \text{cis}\frac{6\pi}{7}\right)\left(z - \text{cis}\left(-\frac{6\pi}{7}\right)\right)}$$

$$= (z-1)\left(z^2 - 2\cos\frac{2\pi}{7}z + 1\right)\left(z^2 - 2\cos\frac{4\pi}{7}z + 1\right)\left(z^2 - 2\cos\frac{6\pi}{7}z + 1\right)$$

iii) sum of the roots

$$1 + \text{cis}\frac{2\pi}{7} + \text{cis}\left(-\frac{2\pi}{7}\right) + \text{cis}\frac{4\pi}{7} + \text{cis}\left(-\frac{4\pi}{7}\right) + \text{cis}\frac{6\pi}{7} + \text{cis}\left(-\frac{6\pi}{7}\right) = 0$$

$$1 + 2\cos\frac{2\pi}{7} + 2\cos\frac{4\pi}{7} + 2\cos\frac{6\pi}{7} = 0$$

$$\cos\frac{2\pi}{7} + \cos\frac{4\pi}{7} + \cos\frac{6\pi}{7} = -\frac{1}{2}$$

$$\cos\frac{2\pi}{7} - \cos\left(\pi - \frac{4\pi}{7}\right) - \cos\left(\pi - \frac{6\pi}{7}\right) = -\frac{1}{2}$$

$$\cos\frac{2\pi}{7} - \cos\frac{3\pi}{7} - \cos\frac{\pi}{7} = -\frac{1}{2}$$

$$\cos\frac{3\pi}{7} + \cos\frac{\pi}{7} - \cos\frac{2\pi}{7} = \frac{1}{2}$$

COMMENT: The most common mistake students made was leaving the z term out of the quadratic factors.

c) i) let $\cot^{-1}(2x-1) = \alpha$ $\qquad$ $\cot^{-1}(2x+1) = \beta$

$\cot\alpha = 2x-1$ $\qquad$ $\cot\beta = 2x+1$

$\tan\alpha = \frac{1}{2x-1}$ $\qquad$ $\tan\beta = \frac{1}{2x+1}$

$$\tan(\alpha-\beta) = \frac{\tan\alpha - \tan\beta}{1+\tan\alpha\tan\beta}$$

$$= \frac{\frac{1}{2x-1} - \frac{1}{2x+1}}{1+\frac{1}{2x-1}\cdot\frac{1}{2x+1}} \times \frac{(2x-1)(2x+1)}{(2x-1)(2x+1)}$$

$$= \frac{2x+1-(2x-1)}{4x^2-1+1}$$

$$= \frac{2}{4x^2}$$

$$= \frac{1}{2x^2}$$

$$\alpha - \beta = \tan^{-1}\left(\frac{1}{2x^2}\right)$$

$$\therefore \cot^{-1}(2x-1) - \cot^{-1}(2x+1) = \tan^{-1}\left(\frac{1}{2x^2}\right)$$

ii) $S = \tan^{-1}\frac{1}{2} + \tan^{-1}\frac{1}{8} + \tan^{-1}\frac{1}{18} + \dots + \tan^{-1}\left(\frac{1}{2n^2}\right)$

$= \cot^{-1}1 - \cot^{-1}3 + \cot^{-1}3 - \cot^{-1}5 + \cot^{-1}5 - \cot^{-1}7 + \dots + \cot^{-1}(2n-1) - \cot^{-1}(2n+1)$

$= \cot^{-1}1 - \cot^{-1}(2n+1)$

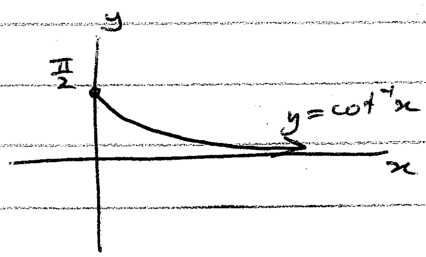
$= \frac{\pi}{4} - \cot^{-1}(2n+1)$

iii) as $n \to \infty$

$2n+1 \to \infty$

$\cot^{-1}(2n+1) \to 0^+$

$\therefore \lim_{n\to\infty} S = \frac{\pi}{4}^-$



COMMENT: Part (i) & (ii) were done reasonably well. Given the result in (iii) was given some work needed to be done either by graph or in terms of $\tan^{-1}\left(\frac{1}{2n+1}\right)$.