Sydney Technical High School

Name: ....

Maths Class: ....

Year 12

Mathematics Extension 1

Trial HSC

August 2017

Time allowed: 120 minutes (plus 5 minutes reading time)

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided at the rear of this Question Booklet, and may be removed at any time.

Section 1 Multiple Choice

Questions 1-10

10 Marks

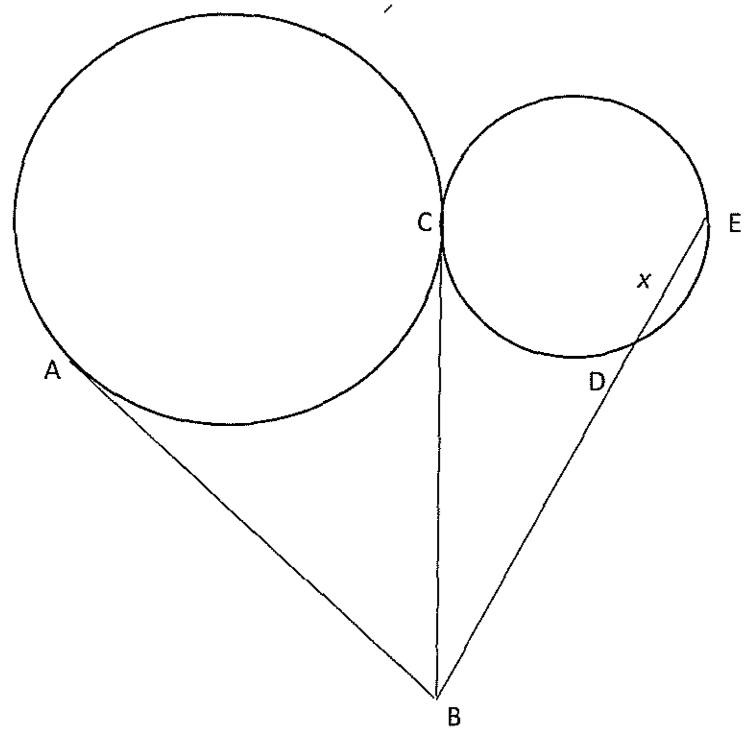
Section II Questions 11-14

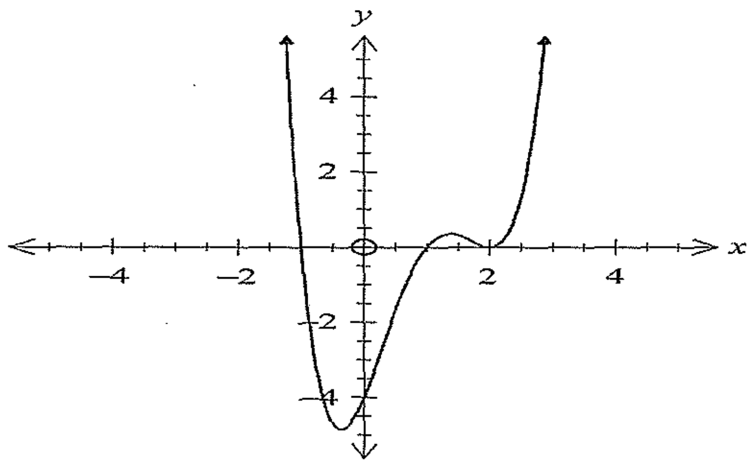
60 Marks

Total = 70 marks

SECTION I – Multiple choice – 10 marks

Fill in the circle on your Multiple Choice answer sheet which corresponds to the correct answer

1.	 AB and BC are tangents to the larger circle, which touches the smaller circle at C. DE = x BD = 4DE The length of AB is A. $2\sqrt{5}x$ B. $3x$ C. $2x$ D. $x\sqrt{5}$
2	The line joining A(2, 3) to B(5, -1) is divided **externally** by the point M in the ratio 2:3. The point M has coordinates: A. (-4, 11) B. (14, 11) C. $(\frac{16}{5}, \frac{23}{5})$ D. $(\frac{31}{2}, 5)$
3	$\frac{d}{dx}\ln\left(\frac{2x+1}{3x+2}\right) =$ A. $\frac{2}{3}$ B. $\ln\left(\frac{2}{3}\right)$ C. $\frac{-1}{(2x+1)(3x+2)}$ D. $\frac{1}{(2x+1)(3x+2)}$

4	The acute angle between the line $x = 2$ and the line $\sqrt{3}y - x + 1 = 0$ is A. $\frac{\pi}{6}$ B. $\frac{\pi}{3}$ C. π D. 0^c
5.	If $\tan x = \frac{-1}{k}$ and $0 \leq x \leq \pi$ then $\sec x =$ A. $\frac{\sqrt{1+k^2}}{k}$ B. $\frac{-\sqrt{1+k^2}}{k}$ C. $\frac{k}{\sqrt{1+k^2}}$ D. $\frac{-k}{\sqrt{1+k^2}}$
6.	 The Polynomial graphed above could be: A. $P(x) = (2 - x)(x^2 - 1)$ B. $P(x) = (x - 2)(x^2 - 1)$ C. $P(x) = (2 - x)^2(x^2 - 1)$ D. $P(x) = (x^2 - 2)(x^2 - 1)$

7	$\int \frac{dx}{\sqrt{9-25x^2}} =$ A. $\frac{1}{3} sin^{-1}(5x)$ B. $\frac{1}{5} sin^{-1}(3x)$ C. $\frac{1}{5} sin^{-1}\left(\frac{5x}{3}\right)$ D. $\frac{1}{5} sin^{-1}\left(\frac{3x}{5}\right)$	
8	$\int_{\frac{\pi}{3}}^{\frac{\pi}{2}} \sin x \cos^2 x \, dx =$ A. 0 B. $\frac{-1}{24}$ C. $\frac{1}{24}$ D. $\frac{8-9\sqrt{3}}{24}$	
9	Simplify $\frac{1-\cos 2\theta}{1+cos 2\theta}$ A. $tan^2\theta$ B. $cot^2\theta$ C. $1 - tan^2\theta$ D. $1 - cot^2\theta$	
10	For what value of x is the ratio of its natural logarithm (lnx) to the number itself (x) a maximum? A. 0 B. 1 C. e D. $ln\, x$	

End of Section I

SECTION II - 60 Marks

Complete all answers in your answer booklets

Begin a new page for each new question.

QUESTION 11: (15 Marks)

Marks

(a) Find **4**

(i) $\int \frac{1}{(2x+3)\sqrt{2x+3}}\,dx$ (ii) $\int sin^2 4x\,dx$

(b) Find the value(s) of k for which the quadratic equation $x^2 - 3kx + (k+3) = 0$ has one root twice the other. **2**

(c) For $y = e^{x^3}$ find $\frac{d^2y}{dx^2}$, in factored form **2**

(d) Using the substitution $u = 1 - x^3$, or otherwise, find **2**

$$\int x^2(1 - x^3)^4\,dx$$

(e) If α and β are the roots of the quadratic equation $3x^2 - x + 2 = 0$, **2**

find the quadratic equation with roots $1 - \alpha$ and $1 - \beta$

QUESTION 11 continues overleaf.......

QUESTION 11 continued……

(f) A body in a room with a constant temperature of 25^oC cools from 150^oC to 100^oC in 30 minutes.

(i) Given that the rate of cooling of the temperature (T) of this body is given by $\frac{dT}{dt} = -k(T - 25)$, show that $T = 25 + 125e^{-kt}$ is a solution to this expression, where t is the time taken in minutes. **1**

(ii) Show that $k = \frac{1}{30}\ln\left(\frac{5}{3}\right)$ **1**

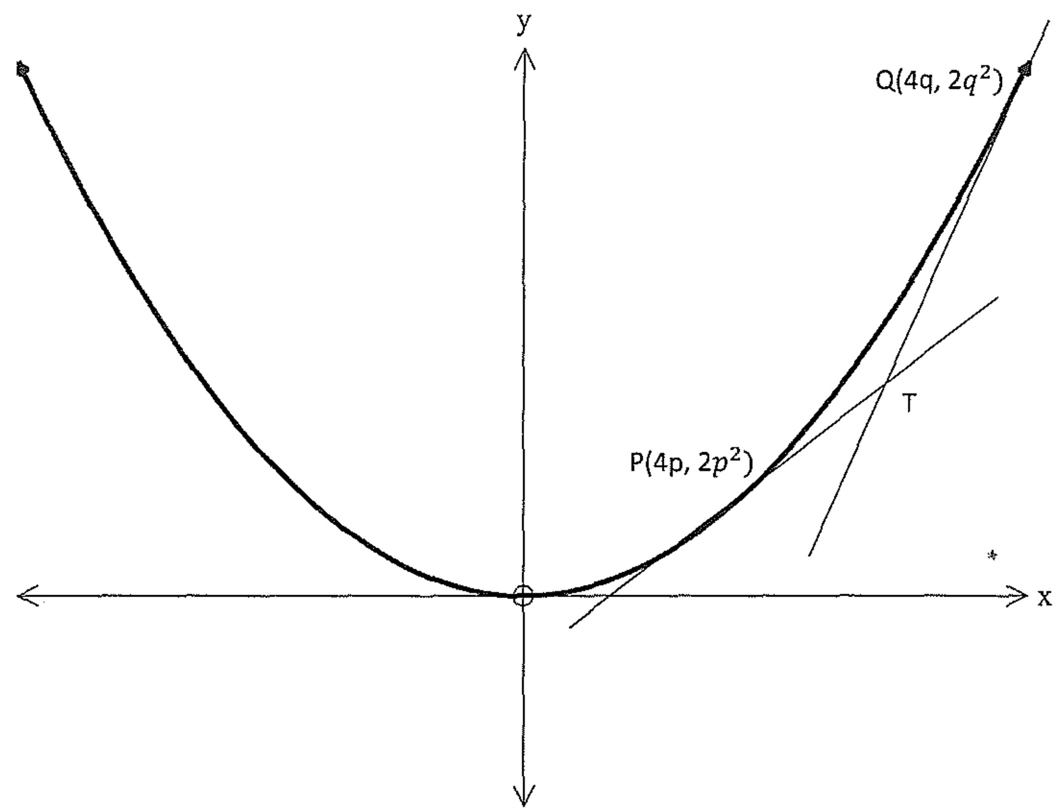
(iii) Find the temperature of the body after another 30 minutes. **1**

End of Question 11

QUESTION 12: (15 Marks) ***Start a new page***

Marks

(a)



(i) Given the point P $(4p, 2p^2)$ lies on the parabola $8y = x^2$, prove that the equation of the tangent at the point P is given by **1**

$$y = px - 2p^2$$

(ii) If Q $(4q, 2q^2)$ also lies on the same arc of the parabola as P, find the equation of the chord PQ. **1**

(iii) PQ passes through the point (0, -4). Prove that $pq = 2$. **1**

(iv) T is the point of intersection of the tangents at P and Q. Find the equation of the locus of T. **2**

(v) Find any restrictions on x in the locus of T. **1**

QUESTION 12 continued overleaf.......

QUESTION 12 continued.......

(b) A particle is moving with simple harmonic motion in a straight line. Its speed, v, when it is a distance x from the centre of the oscillation is given by

$$v^2 = \pi^2(9 - x^2)$$

(i) What is the period of the motion? **1**

(ii) What is the maximum acceleration? **1**

(c) (i) Find $\frac{d}{dx}(\sin^{-1}x + \sin^{-1}\sqrt{1-x^2})$ **2**

(ii) Explain the meaning of the answer to part (i) above. **1**

(d) The Volume and the Surface Area of a sphere are given by the formulae:

$$V = \frac{4}{3}\pi r^3 \qquad \text{and } S = 4\pi r^2$$

(i) Show that $\frac{dV}{dt} = S\frac{dr}{dt}$ **1**

(ii) A spherical ball of radius 48 mm has its Volume changing at a rate equal to 6 times its Surface Area, while remaining spherical. (The rate is in mm^3/sec). Show that the rate of change of the radius is a constant. **1**

(iii) How many seconds does it take for the Volume to reduce to $\frac{1}{8}$ of its original? **2**

End of Question 12

QUESTION 13: (15 Marks) *Start a new page*

Marks

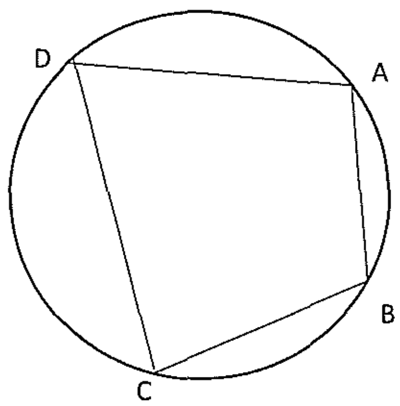
(a) A man walks at a speed of $\frac{16}{4+t}$ km per hour, after walking for t hours.

(i) How long does he walk for his speed to reduce to 3 kph? **1**

(ii) How far has he walked in that time? (give to 2 decimal places) **2**

(b) Prove that $\frac{d}{dx}\ln(secx) = tanx$ **2**

(c) The points A, B, C and D lie on the circumference of the circle below. **3**



Prove (giving all reasons) that tan A + tan B + tan C + tan D = 0

(d) Find $\frac{d}{dx}\ln(x+\sqrt{x^2+1})$ and hence find the value, in exact form, of **3**

$$\int_0^1 \frac{dx}{\sqrt{x^2+1}}$$

(e) Prove, by the process of Mathematical Induction, that $3^{2n}+7$ is divisible by 8 for all positive integral *n*. **4**

End of Question 13

QUESTION 14: (15 Marks) *Start a new page*

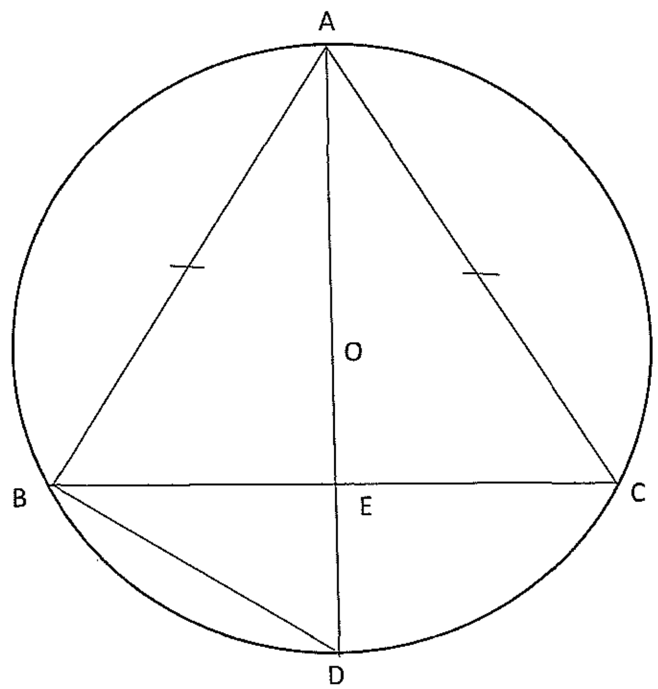
Marks

(a) If $\frac{dy}{dx} = 1 + y^2$ and at $x = \frac{\pi}{2}$, $y = 1$, show that $y = \tan(x - \frac{\pi}{4})$ **2**

(b) (i) If $f(x) = cosec\, x$ show that $f^{-1}(x) = sin^{-1}\left(\frac{1}{x}\right)$ **2**

(ii) Find the Domain of y $= f^{-1}(x)$ **1**

(c)



AD is a diameter of the circle, centre O
ΔABC is isosceles, with AC = AB

(i) Redraw the diagram onto your answer booklet (at least one third of a page)

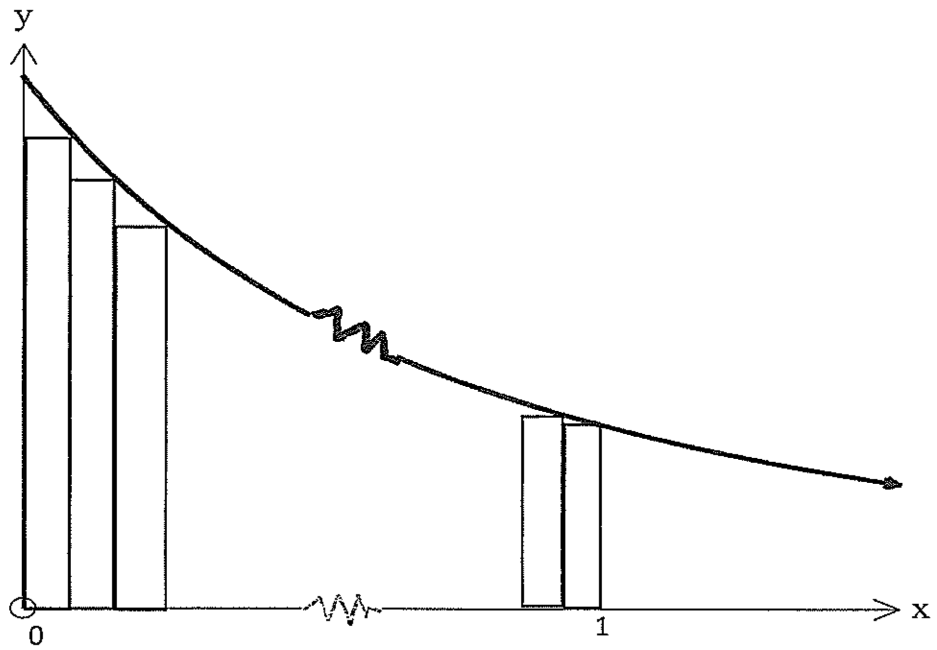
(ii) Let ∠ABC = x. Prove that ∠BDE = x (give all reasons) **1**

(iii) Prove that BC is perpendicular to AD (give all reasons)
***(DO NOT use congruency or the properties of an isosceles triangle)*)** **2**

QUESTION 14 continued overleaf.......

QUESTION 14 continued…….

(d) The curve $y = \frac{1}{(x+1)^2}$ shown below, has n rectangles of equal width inscribed on it, between the values x = 0 and x = 1.
The height of the rectangles is determined by the x – value on the right of the rectangle, forming what are called *Lower Rectangles*.



(i) Give the area of the first rectangle on the left. **2**

(ii) Find a simplified expression for the area of the *kth* rectangle from the left **2**

(iii) Deduce that **3**

$$\lim_{n\to\infty} n\left\{\frac{1}{(n+1)^2} + \frac{1}{(n+2)^2} + \;\cdots\cdots + \frac{1}{(2n)^2}\right\} = \frac{1}{2}$$

End of Examination

SOLUTIONS [EXT 1 2017]

Multiple Choice

(1) $EB \times BD = CB^2$

$5x \times 4x = CB^2$

$CB = 2x\sqrt{5}$

and $AB = CB = 2\sqrt{5}x$

$\therefore$ **A**

(2) $(2,3)$ $(5,-1)$

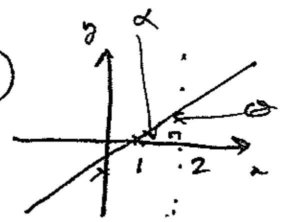
$-2 : 3$

$\therefore M$ is $\left(\dfrac{-2\times5+3\times2}{1}, \dfrac{-2\times-1+9}{1}\right)$

$= (-4, 11)$ **A**

(3) $\dfrac{2}{2x+1} - \dfrac{3}{3x+2}$

$= \dfrac{1}{(2x+1)(3x+2)}$ **D**

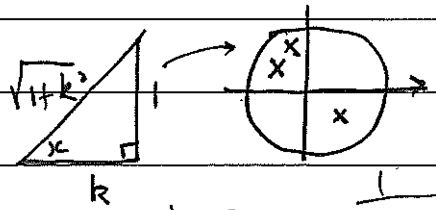
(4) 

$\tan\alpha = \frac{1}{\sqrt{3}}$

$\therefore \alpha = \pi/6$

$\therefore \theta = \pi/3$

B

(5) 

$\therefore \sec x = \dfrac{1}{\cos x}$

$= -\dfrac{\sqrt{1+k^2}}{k}$

B

(6) **C**

(7) $\displaystyle\int \frac{1/5}{\sqrt{9/25 - x^2}}\,dx = \frac{1}{5}\sin^{-1}\frac{x}{3/5}$

$= \frac{1}{5}\sin^{-1}\frac{5x}{3}$

C

(8) $\displaystyle\int_{\pi/3}^{\pi/2} \sin x \cos^2 x\,dx$

$= -\frac{1}{3}\cos^3 x\Big]_{\pi/3}^{\pi/2}$

$= \frac{1}{3}\left(\frac{1}{8}\right)$

$= \frac{1}{24}$. **C**

(9) $\dfrac{1-(1-2\sin^2 x)}{1+(2\cos^2 x-1)} = \dfrac{2\sin^2 x}{2\cos^2 x}$

$= \tan^2 x$

A

(10) For max ratio (r) $\frac{dr}{dx} = 0$

$\therefore \frac{d}{dx}\left(\frac{\ln x}{x}\right) = 0$

$\therefore \dfrac{x \cdot 1/x - \ln x}{x^2} = 0$

$\therefore \ln x = 1$

$\therefore x = e$ **C**

SECTION II

QUESTION 11:

(a) (i) $\int (2x+3)^{-3/2}\,dx$

$= -(2x+3)^{-1/2} + k$

$= \dfrac{-1}{\sqrt{2x+3}} + k$

(ii) $\cos 2A = 1 - 2\sin^2 A$

$\therefore \sin^2 A = \dfrac{1-\cos 2A}{2}$

$\therefore \int \sin^2 4x\,dx = \int \dfrac{1-\cos 8x}{2}\,dx$

$= \dfrac{x}{2} - \dfrac{1}{16}\sin 8x + k$

(b) Let the roots be α and 2α

Sum $3\alpha = 3k \Rightarrow \alpha = k$

Product $2\alpha^2 = k+3$

$\therefore 2k^2 - k - 3 = 0$

$(2k-3)(k+1) = 0 \Rightarrow k = -1$ or $k = \dfrac{3}{2}$

(c) $\dfrac{d}{dx} e^{x^3} = 3x^2 e^{x^3}$

$\dfrac{d^2y}{dx^2} = e^{x^3} 6x + 3x^2 . 3x^2 e^{x^3}$

$= 3xe^{x^3}(2 + 3x^3 e^{x^3})$

(d) $u = 1 - x^3 \Rightarrow \dfrac{du}{dx} = -3x^2$

$\therefore dx = -du/3x^2$

$\therefore$ Integral $= \int x^2 . u^4 (-du/3x^2)$

$= -\frac{1}{3}\int u^4\,du$

$= -\frac{1}{15}u^5 + k$

$= -\frac{1}{15}(1-x^3)^5 + k$

(e) Sum $= \alpha + \beta = \frac{1}{3}$

Product $= \alpha\beta = \frac{2}{3}$

Sum new $= 2 - (\alpha+\beta) = \frac{5}{3}$

Product new $= 1 - (\alpha+\beta) + \alpha\beta$

$= 1 - \frac{1}{3} + \frac{2}{3}$

$= \frac{4}{3}$

$\therefore$ Q.E is

$x^2 - \frac{5}{3}x + \frac{4}{3} = 0$

$\therefore 3x^2 - 5x + 4 = 0$

11.(f) $T = 25 + 125e^{-kt}$ (i). $\frac{dT}{dt} = -125ke^{-kt}$

$= -k(25 + 125e^{-kt} - 25)$

$= -k(T - 25)$

(ii) At $t = 30$, $T = 100$

$\therefore 75 = 125e^{-30k}$

$\therefore e^{-30k} = 3/5$

$\therefore -30k = \ln(3/5)$

$\therefore k = \frac{1}{30}\ln(5/3)$

(iii) At $t = 60$

$T = 25 + 125e^{-2\ln(5/3)}$

$= 25 + 125e^{\ln(9/25)}$

$= 25 + 125 \times 9/25$

$= 70°$

QUESTION 12:

(i) $\frac{dy}{dx} = 2x/8 = x/4$

At P, $m_T = p$.

$\therefore$ Equation is $y - 2p^2 = p(x - 4p)$

$\therefore y - 2p^2 = px - 4p^2$

$y = px - 2p^2$

(ii) $m_{PQ} = \frac{2q^2 - 2p^2}{4q - 4p}$

$= \frac{p+q}{2}$

Equation PQ: $y - 2q^2 = \frac{p+q}{2}(x - 4q)$

$\therefore 2y - 4q^2 = (p+q)x - 4pq - 4q^2$

$\therefore 2y = (p+q)x - 4pq$

(iii) If PQ passes through $(0, -4)$

then $-8 = -4pq$

$\therefore pq = 2$.

Q 13 (a) (iv) $y = px - 2p^2$ (1)

$y = qx - 2q^2$ (2)

(1)−(2) $x(p-q) = 2(p^2-q^2)$

$\therefore x = 2(p+q)$

$\therefore y = 2p^2 + 2pq - 2p^2$

$\therefore y = 2pq$

$\therefore y = 4$ since $pq = 2$ from part (iii)

(v) $y = 4$ intersects the parabola at $(\sqrt{32}, 0)$ and $(-\sqrt{32}, 0)$

Since P must lie OUTSIDE the parabola,

then $x > \sqrt{32}$ or $x < -\sqrt{32}$.

Question 12(b) $n = \pi$

(i) $\therefore T = 2\pi/\pi$

$= 2$

(ii) At max acceleration, $v = 0$

$\therefore x = \pm 3$ (positive for min, negative for max)

$a = \frac{d}{dx} \frac{1}{2}v^2$

$= -\pi^2 x$

$\therefore$ max acceleration is $3\pi^2$

(c) (i) $\frac{1}{\sqrt{1-x^2}} + \frac{\frac{1}{2}(1-x^2)^{-\frac{1}{2}} \cdot (-2x)}{\sqrt{1-(1-x^2)}}$

$= \frac{1}{\sqrt{1-x^2}} - \frac{x}{x\sqrt{1-x^2}}$

$= 0$

(ii) this means that $\sin^{-1}x + \sin^{-1}\sqrt{1-x^2}$ is a constant.

(d) (i) $\frac{dV}{dt} = \frac{dV}{dr} \cdot \frac{dr}{dt}$

$= 4\pi r^2 \frac{dr}{dt}$

$= 8\, dr/dt$

(ii) $\frac{dv}{dt} = 6S = S\,\frac{dr}{dt}$

$\therefore \frac{dr}{dt} = 6$

(iii) If the volume is $\frac{1}{8}$ of its original, the radius is $\frac{1}{2}$ of its original. (Since $\left(\frac{1}{2}\right)^3 = \frac{1}{8}$)

Which means 48 mm becomes 24 mm, and since the rate of change of the radius is 6 mm/sec (see part (ii)) then 24 mm takes 4 seconds.

QUESTION 13:

(a) (i) $\frac{16}{4+t} = 3$

$\therefore 12 + 3t = 16$

$t = \frac{4}{3}$ hrs.

(ii) $\frac{dx}{dt} = \frac{16}{4+t}$

$\therefore x = 16\ln(4+t) + k$

At $x = 0,\ t = 0$

$\therefore -16\ln(4) = k$

$\therefore x = 16\ln(4+t) - 16\ln(4)$

At $t = \frac{4}{3}$

$x = 16\ln\left(\frac{\frac{16}{3}}{4}\right)$

$= 16\ln\left(\frac{16}{12}\right)$

$= 16\ln\left(\frac{4}{3}\right)$ km

≈ 4.60 km.

(b) $\sec x = \cos^{-1} x$

$\therefore \frac{d}{dx}\ln\sec x = \frac{d}{dx}(-\ln\cos x)$

$= -\frac{-\sin}{\cos x}$

$= \tan x$

OR

$\frac{d}{dx}\sec x = -(\cos x)^{-2}(-\sin x)$

$= \frac{\sin x}{\cos^2 x}$

$\therefore \frac{d}{dx}\ln\sec x = \frac{\sin x/\cos^2 x}{\sec x} = \tan x$

(c) Since ABCD is cyclic,

$\angle D + \angle B = 180^\circ$

$\therefore \tan(D + B) = \tan 180^\circ = \dfrac{\tan D + \tan B}{1 - \tan D \tan B}$

$\Rightarrow \tan D + \tan B = 0$

Similarly $\tan A + \tan C = 0$

$\therefore \tan A + \tan B + \tan C + \tan D = 0$

(d) $\dfrac{d}{dx}\left(x + \sqrt{x^2+1}\right) = 1 + \tfrac{1}{2} \cdot 2x \left(x^2+1\right)^{-1/2}$

$= 1 + x/\sqrt{x^2+1}$

$\therefore \dfrac{d}{dx} \ln\left(x + \sqrt{x^2+1}\right) = \dfrac{1 + x/\sqrt{x^2+1}}{x + \sqrt{x^2+1}}$

$= \dfrac{\sqrt{x^2+1} + x}{\sqrt{x^2+1}\left[x + \sqrt{x^2+1}\right]}$

$= 1/\sqrt{x^2+1}$

$\therefore \displaystyle\int_0^1 \frac{dx}{\sqrt{x^2+1}} = \ln\left(x + \sqrt{x^2+1}\right)\Big]_0^1$

$= \ln\left(1 + \sqrt{2}\right) - \ln\left(\sqrt{1}\right)$

$= \ln\left(1 + \sqrt{2}\right)$

(e) For $n = 1$, $3^2 + 7 = 16$ which is divisible by 8

$\therefore$ The formula is true for $n = 1$

Assume it is true for $n = k$

i.e., $3^{2k} + 7 = 8m$ where $m \in \mathbb{J}$

For $n = k+1$

$3^{2k+2} + 7 = 3^2\left(3^{2k} + 7\right) + 7 - 63$

$= 9 \cdot 8m - 56$

$= 8(9m - 7)$

$= 8M$ where $M \in \mathbb{J}$

∴ If the statement is true for $n = k$,
it is also true for $n = k+1$
AND it is true for $n = 1$
∴ it is true for $n = 2$ and so on
ie true $\forall\ n \in \mathbb{Z}$.

QUESTION 14:

(a) $dy/dx = 1 + y^2$

$dx/dy = \frac{1}{1+y^2}$

∴ $x = \tan^{-1} y + k$

At $x = \pi/2$, $y = 1$

∴ $\pi/2 = \tan^{-1}(1) + k \Rightarrow k = \pi/4$

∴ $x = \tan^{-1} y + \pi/4$

∴ $y = \tan(x - \pi/4)$

(b) (i) $y = \operatorname{cosec} x \longrightarrow$ for Inverse $x = \operatorname{cosec} y$

∴ $1/x = \sin y$

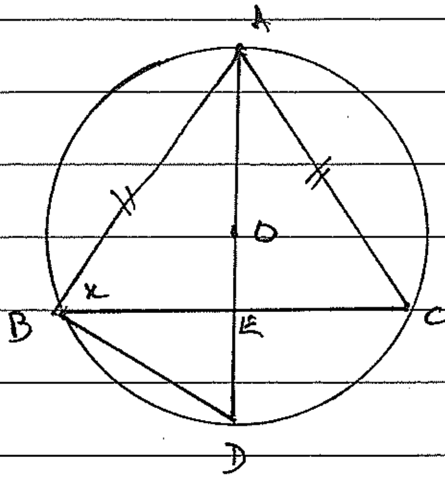
∴ $y = \sin^{-1}(1/x)$

(ii) for Inverse sine normally $\mathcal{D}: -1 \leq x \leq 1$

In this case $-1 \leq 1/x \leq 1$

∴ $x \leq -1$ or $x \geq 1$ for $\mathcal{D}_{f^{-1}}$

(c)



(ii) Since $\triangle ABC$ is isosceles $\angle ACB = x$

$\therefore \angle ADB = \angle ACB = x$ (angles at the circumference standing on arc AB)

$\angle BDE$

(iii) Since AD is a diameter, $\angle ABD = 90^\circ$ (angle in a semi-circle)

$\therefore \angle EBD = (90 - x)^\circ$

$\therefore \angle BED = 90^\circ$ (angle sum of $\triangle BED$)

(d) (i) width of each rectangle is $1/n$

Height of first rectangle is $f(1/n) = \dfrac{1}{(1/n+1)^2}$

$\therefore$ Area $= 1/n \left(\dfrac{1}{(1/n+1)^2}\right)$

$= \dfrac{1}{n} \dfrac{n^2}{(1+n)^2}$

$= n/(1+n)^2$

(ii) The height of the k^{th} rectangle is $\dfrac{1}{(k/n+1)^2}$

$\therefore$ Area $= 1/n \left[\dfrac{1}{(k/n+1)^2}\right] = \dfrac{n}{(k+n)^2}$

(iii) The sum of the area of all rectangle is

$\dfrac{n}{(n+1)^2} + \dfrac{n}{(n+2)^2} + \dots + \dfrac{n}{(n+k)^2} + \dots + \dfrac{1}{(2n)^2}$

As $n \to \infty$, this area becomes the exact area under $y = 1/(x+1)^2$

$\therefore \lim_{n \to \infty} n \left\{ \dfrac{1}{(n+1)^2} + \dots + \dfrac{1}{(2n)^2} \right\} = \int_0^1 \dfrac{1}{(x+1)^2}\,dx$

$= 1/2$