Sydney Technical High School

Name: ....

Maths Class: ....

Year 12

MATHEMATICS EXTENSION 2

HSC COURSE

ASSESSMENT 4 – TRIAL HSC

AUGUST, 2017

Time allowed: 180 minutes

General Instructions:

- Write using black or blue pen
- In Questions 11–16, show relevant mathematical reasoning and/ or calculations
- Approved calculators may be used
- Full marks may not be awarded for careless work or illegible writing
- **Begin each question on a new page**
- All answers are to be in the writing booklet provided
- A reference sheet is provided at the back of this paper

Total Marks 100

Section 1 Multiple Choice
Questions 1-10
10 Marks

Section II Questions 11-16
90 Marks

Section 1

Multiple Choice (10 marks)
Use the multiple choice answer sheet for Question 1-10

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1. Write $\frac{40}{1-3i}$ in the form $a + ib$, where a and b are real.

   (A) $4 - 12i$

   (B) $4 + 12i$

   (C) $-5 - 15i$

   (D) $-5 + 15i$

2. Consider the hyperbola with equation $\frac{x^2}{4} - \frac{y^2}{3} = 1$.
   What are the co-ordinates of the vertices of the hyperbola?

   (A) $(\pm 2, 0)$

   (B) $(0, \pm 2)$

   (C) $(0, \pm 4)$

   (D) $(\pm 4, 0)$

3. Consider the equation $z^3 - 2z^2 + bz + c = 0$, where b and c are real numbers. If one of the roots of the equation is $2 - i$, what is the value of b?

   (A) -3

   (B) -19

   (C) 3

   (D) 19

4. What are the equations of the directrices of the ellipse $\frac{x^2}{4}+y^2=1$

(A) $x = \pm\frac{4}{\sqrt{3}}$

(B) $x = \pm\sqrt{3}$

(C) $x = \pm\frac{\sqrt{5}}{2}$

(D) $x = \pm\frac{2}{\sqrt{5}}$

5. A stone of mass m is dropped from rest and falls in a medium in which the resistance is directly proportional to the square of the velocity. Suppose mk is the constant of proportionality and that the displacement downwards from the initial position is x at time t. The acceleration due to gravity is g.

Which of the following is true?

(A) The terminal velocity is $\frac{g}{k}$.

(B) As $t \to \infty, x \to$ L where L is a positive constant.

(C) The equation of motion is given by $v\frac{dv}{dx} = g - kv^2$.

(D) The time for the stone to reach velocity V is given by $\int_0^V g - kv^2 \, dv$.

6. The polynomial P(x) with real coefficients has $x = 1$ as a root of multiplicity 2 and $x + i$ as a factor.

Which one of the following expressions could be a factorized form of P(x)?

(A) $(x^2 + 1)(x - 1)^2$

(B) $(x + i)^2(x - 1)^2$

(C) $(x - i)^2(x - 1)^2$

(D) $(x^2 + 1)(x - i)^2$

7. The horizontal base of a solid is the circle $x^2 + y^2 = 1$. Each cross section taken perpendicular to the x axis is a triangle with one side in the base of the solid. The length of this triangle side is equal to the altitude of the triangle through the opposite vertex. Which of the following is an expression for the volume of the solid?

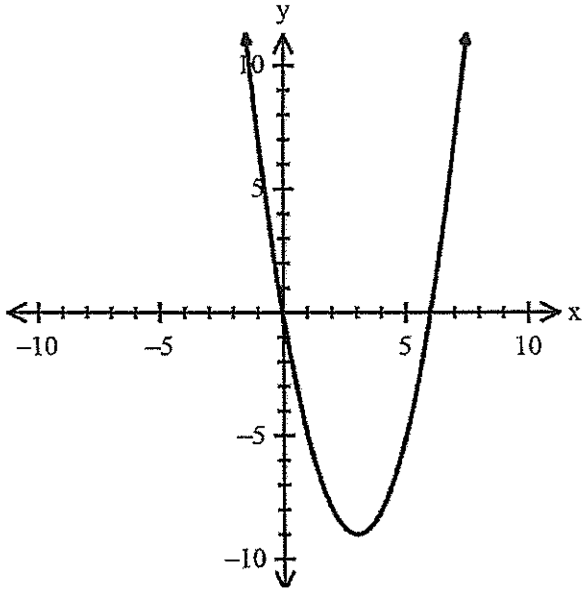
(A) $\frac{1}{2}\int_{-1}^{1} (1 - x^2)\ dx$

(B) $\int_{-1}^{1} (1 - x^2)\ dx$

(C) $\frac{3}{2}\int_{-1}^{1} (1 - x^2)\ dx$

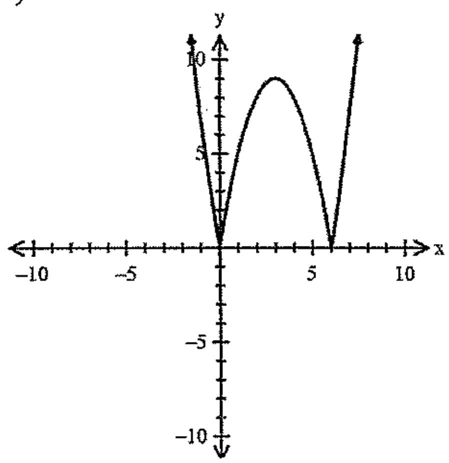
(D) $2\int_{-1}^{1} (1 - x^2)\ dx$

8. Consider the graph of $y = f(x)$ drawn below.

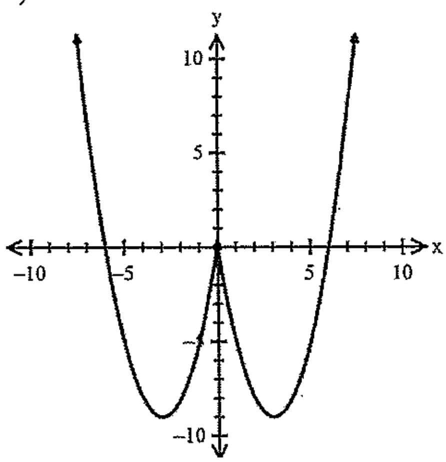


Which one of the following diagrams shows the graph of $y = f(|x|)$?

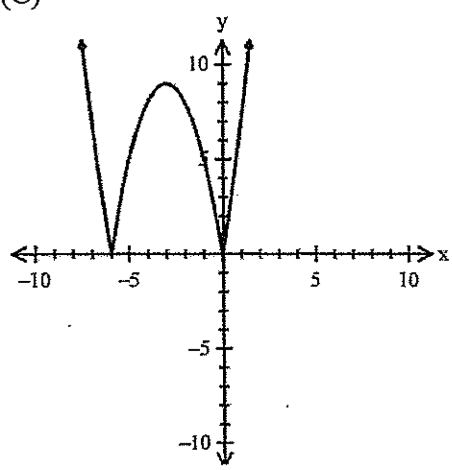
(A)



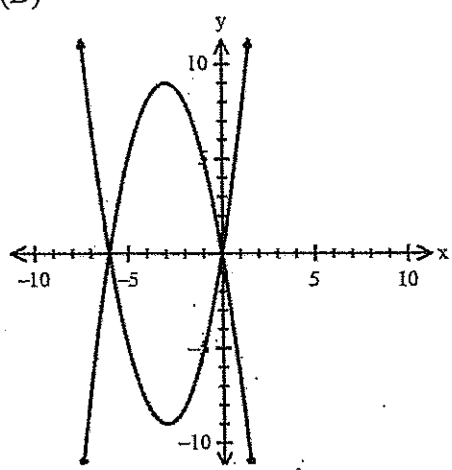
(B)



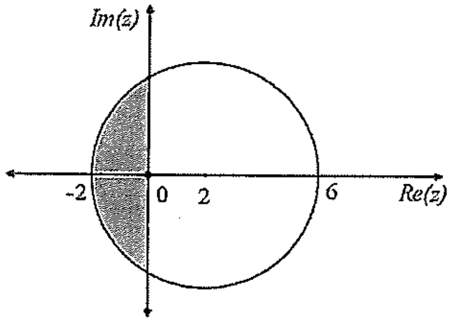
(C)



(D)



9. A circle with centre $(2,0)$ and radius 4 units is shown on an Argand diagram below.



Which of the following inequalities represents the shaded region?

(A) $Re(z) \leq 0$ and $|z-2| \leq 4$

(B) $Re(z) \leq 0$ and $|z-2| \leq 16$

(C) $Im(z) \leq 0$ and $|z-2| \leq 4$

(D) $Im(z) \leq 0$ and $|z-2| \leq 16$

10. Which of the following is the range of the function $f(x) = \sin^{-1} x + \tan^{-1} x$?

(A) $-\pi < y < \pi$

(B) $-\pi \leq y \leq \pi$

(C) $-\frac{3\pi}{4} \leq y \leq \frac{3\pi}{4}$

(D) $-\frac{\pi}{2} \leq y \leq \frac{\pi}{2}$

Section II

Total Marks (90)
Attempt Questions 11 – 16.

Answer each question in your writing booklet.
In Questions 11-16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 **(15 Marks)**

(a) Find $\displaystyle\int \frac{\cos 2x}{\cos^2 x}\,dx$. **2**

(b) Let $z = \dfrac{3+i}{1+2i}$.

i) Express z in the form $a + ib$ where a and b are real. **1**

ii) Hence express z^7 in modulus argument form. **2**

(c) (i) Find the square roots of $-24 - 10i$ **2**

(ii) Hence, or otherwise, solve $x^2 - (1-i)x + 6 + 2i = 0$ **2**

(d) On an Argand diagram shade the region where both $|z - 1| \geq 1$ and $-\dfrac{\pi}{4} \leq \arg z \leq \dfrac{\pi}{4}$. **3**

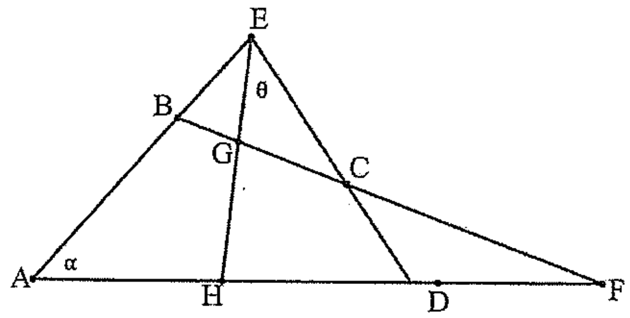
(e) (i) Show that $\cot\theta + \operatorname{cosec}\theta = \cot\left(\dfrac{\theta}{2}\right)$. **2**

(ii) Hence, or otherwise, find $\displaystyle\int (\cot\theta + \operatorname{cosec}\theta)\,d\theta$. **1**

End of Question 11

Question 12 (15 Marks) Use a Separate Sheet of paper

(a) Use the substitution $u = e^x + 1$ to find $\displaystyle\int \frac{e^{2x}}{(e^x+1)^2}\,dx$ **3**



(b) ABCD is a cyclic quadrilateral. AB produced and DC produced meet at E. AD produced and BC produced meet at F. EGH bisects $\angle AED$ where H lies on AD and G lies on BC. **4**

Copy the diagram and show that $FG = FH$.

(c) Find the equation of the tangent to the curve $x^2 - xy + y^3 = 1$ at the point $P(1,1)$ on the curve. **3**

(d) Let $I_n = \displaystyle\int_0^{\frac{\pi}{4}} \tan^n x\,dx$ for all integers $n \geq 0$.

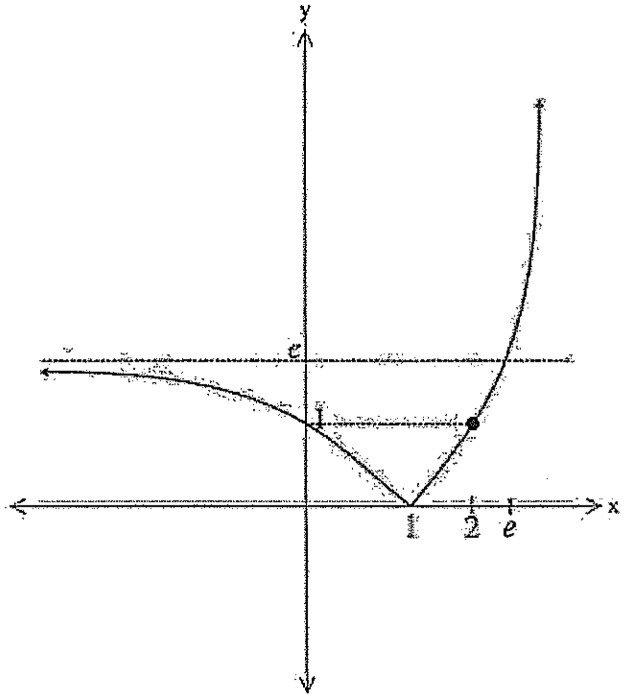
(i) Show that $I_n = \dfrac{1}{n-1} - I_{n-2}$ for integers $n \geq 2$. **3**

(ii) Hence find $\displaystyle\int_0^{\frac{\pi}{4}} \tan^5 x\,dx$. **2**

End of Question 12

Question 13 (15 Marks) Use a Separate Sheet of paper

(a) The diagram is a sketch of $y = f(x)$



Draw separate one third page sketches of the graphs of the following:

(i) $y = \dfrac{1}{f(x)}$ **2**

(ii) $y = f(x+1)$ **2**

(iii) $y = \sqrt{f(x)}$ **2**

(iv) $y = \ln(f(x))$ **2**

(b) For the hyperbola $\dfrac{y^2}{16} - \dfrac{x^2}{9} = 1$, find

(i) the eccentricity **1**

(ii) the coordinates of the foci S and S` and the equations of its directrices **2**

(iii) Sketch the hyperbola showing all the above features. **1**

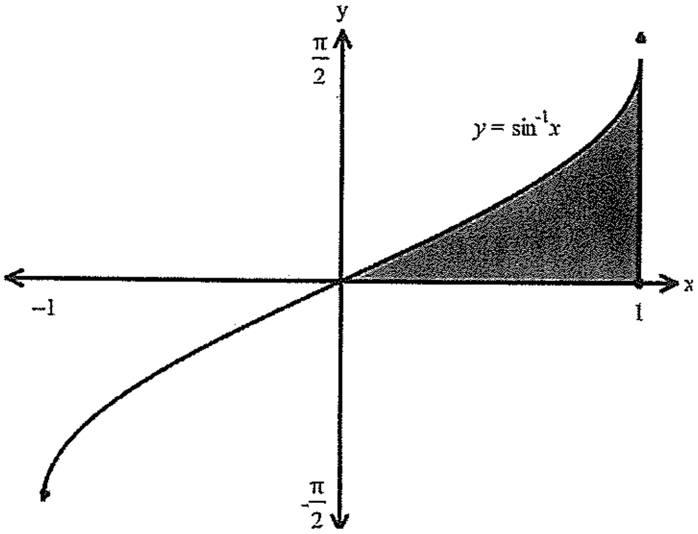
(c) Find $\displaystyle\int \frac{2x-3}{x^2-4x+5}\,dx$ **3**

End of Question 13

Question 14 (15 Marks) Use a Separate Sheet of paper

(a) (i) Evaluate $\displaystyle\int_0^1 x \sin^{-1}x \, dx$ **2**

(ii) The area bounded by $y = \sin^{-1}x$, the x-axis and the line $x = 1$ (as shown below) is rotated about the y-axis. Use the method of cylindrical shells to determine the volume of the solid generated. **2**



(b) A particle of mass 4kg is projected vertically upwards. It is subjected to a gravitational force of 40 Newtons and air resistance of $\frac{v^2}{10}$ Newtons.
The height of the particle at time t seconds is x metres and its velocity is v ms^{-1}.

(i) Given that $v^2 = 400\left(10e^{-\frac{x}{20}} - 1\right)$ until the particle reaches its maximum height, find the maximum height in exact form. **2**

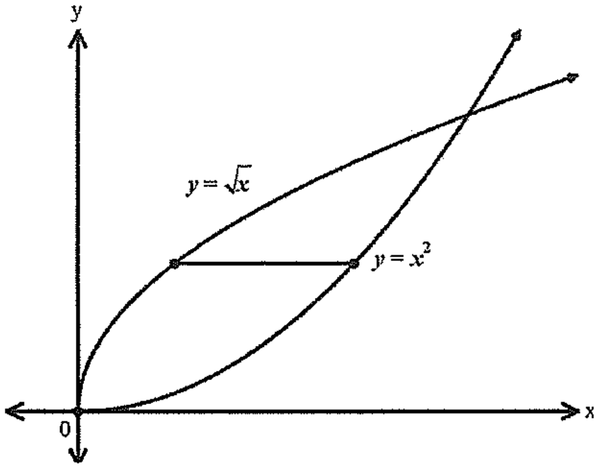
(ii) After reaching maximum height the particle begins to fall. **1**
Show that the equation of motion as it falls is $\ddot{x} = \dfrac{400 - v^2}{40}$.

(iii) How far has the particle fallen from its maximum height when the speed is 50% of its terminal velocity? (Leave your answer in exact form) **3**

(iv) Find the speed of the particle when it returns to its point of projection. (Leave your answer in exact form) **2**

(c) The area between curve $y = x^2$ and $y = \sqrt{x}$ is the base of the solid S. **3**

Cross sections perpendicular to the y-axis are squares.



Find the volume of S.

End of Question 14

Question 15 (15 Marks) Use a Separate Sheet of paper

(a) (i) Show that the normal to the hyperbola $xy = c^2, c \neq 0$, at $P\left(cp, \frac{c}{p}\right)$ is given by $px - \frac{y}{p} = c\left(p^2 - \frac{1}{p^2}\right)$. **2**

(ii) The normal at P meets the hyperbola again at $Q\left(cq, \frac{c}{q}\right)$. Show that $q = -\frac{1}{p^3}$. **3**

(b) A projectile is fired with initial velocity V at an angle of projection α.
The x and y components of its displacement at any time t are given by $x = Vt\cos\alpha$ and $y = Vt\sin\alpha - \frac{gt^2}{2}$, where g is the acceleration due to gravity.

(i) Show that the cartesian equation $y = x\tan\alpha - \frac{gx^2}{2V^2}(1 + \tan^2\alpha)$ describes the motion. **2**

(ii) A projectile with initial velocity 50ms^{-1} hits a point 100m away at a height of 3m above the point of projection. Taking $g = 10\text{ms}^{-2}$, calculate the two angles of projection which allow this to happen. (Answers to the nearest degree.) **3**

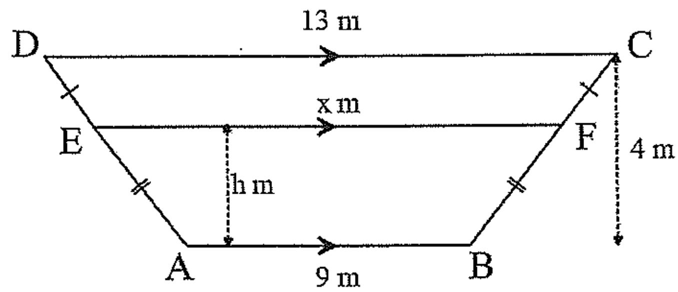
(c) (i) Show that if α is a zero of multiplicity 2 of a polynomial $f(x)$, then $f'(\alpha) = 0$ **2**

(ii) The polynomial $g(x) = px^3 - 3qx + r$ has a positive zero of multiplicity 2.
Show that $4q^3 = pr^2$. **3**

End of Question 15

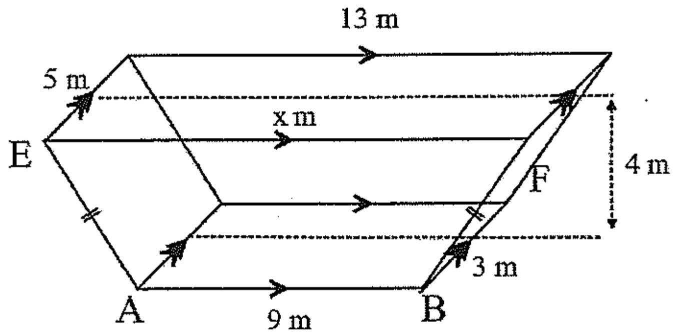
Question 16 **(15 Marks)** Use a Separate Sheet of paper

(a) (i) The diagram below shows a trapezium ABCD whose parallel sides AB and DC are 9cm and 13m respectively. The distance between the sides is 4cm and AD=BC. EF is parallel to AB at a distance of h m. **2**



Show that EF= (9+h) m.

(ii) The trench in the diagram below has a rectangular base with sides 9 m and 3 m. Its Top is also rectangular with dimensions of 13 m and 5 m. The trench has a depth of 4m and each of its four sides faces is a symmetrical trapezium. **3**



Find the volume of the trench.

(b) Find $\int \frac{x^3\,dx}{x^2+x+1}$ **3**

(c) Consider the polynomial equation $x^4 + Ax^2 + Bx + C = 0$ where A, B and C are real. Let the roots of this equation be α, β, γ and δ.

Show that:

(i) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = -2A$ **1**

Given that $(\alpha+\beta+\gamma+\delta)^2 = \alpha^2+\beta^2+\gamma^2+\delta^2+2(\alpha\beta+\alpha\gamma+\alpha\delta+\beta\gamma+\beta\delta+\gamma\delta)$

(ii) $\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = 2A^2 - 4C$ **2**

(d) (i) Show that for $k > 0$, **1**

$$\frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} < 0$$

(ii) Use mathematical induction to prove that for all integers $n \geq 2$, **3**

$$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \ldots + \frac{1}{n^2} < 2 - \frac{1}{n}$$

End of Examination

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Sydney Technical High School

Trial 2017

Extension 2.

Multiple Choice

1. $\dfrac{40}{1-3i} \times \dfrac{1+3i}{1+3i} = \dfrac{40(1+3i)}{1+9}$

$= 4(1+3i)$

$= 4+12i$ (B)

2. $\dfrac{x^2}{4} - \dfrac{y^2}{3} = 1$ hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$

$a=2 \quad b=\sqrt{3}$.

Vertex $(\pm a, 0)$

$= (\pm 2, 0)$ (A).

3. $2+i+2-i+a = 2$ (sum of roots)

$a=-2$

$b = -2(2+i) - 2(2-i) + 2(2+i)(2-i)$ sum of pairs of roots.

$b = -8+5$

$b = -3$ (A)

4. $\dfrac{x^2}{4} + y^2 = 1$ $\quad 1 = 4(1-e^2)$

$e^2 = \dfrac{3}{4}$

$a=2 \quad b=1$ $\quad e = \dfrac{\sqrt{3}}{2}$

$b^2 = a^2(1-e^2)$

$x = \pm \dfrac{a}{e}$ (A)

$x = \pm 2 \div \dfrac{\sqrt{3}}{2}$ $\quad x = \pm \dfrac{4}{\sqrt{3}}$

5. Equation of motion is

$m\ddot{x} = mg - mkv^2$

$\ddot{x} = g - kv^2$

Since $\ddot{x} = v\dfrac{dv}{dx}$

$v\dfrac{dv}{dx} = g - kv^2$ (C)

6. $x=1$ is a root

$(x-1)^2$ is a factor of $P(x)$

$(x+i)$ is a factor so $(x-i)$ is a factor.

$P(x) = (x+i)(x-i)(x-1)^2$

$= (x^2-i^2)(x-1)^2$

$= (x^2+1)(x-1)$ (A)

7. $A = \frac{1}{2}(2y)^2 = 2y^2$

$V = \displaystyle\int_{-1}^{1} 2y^2\,dx$

$V = 2\displaystyle\int_{-1}^{1} y^2\,dx$

$V = 2\displaystyle\int_{-1}^{1} (1-x^2)\,dx$

(D)

8. (B)

9. $x \le 0 \qquad Re(z) \le 0$.

Inside region of a circle with centre (2,0) radius 4 $\quad |z-2| \le 4$

$\therefore Re(z) \le 0$ and $|z-2| \le 4$ (A)

10. $-\sin^{-1}(-1) + \tan^{-1}(-1) \le y \le \sin^{-1}(1) + \tan^{-1}(1)$

$-\frac{\pi}{2} - -\frac{\pi}{4} \le y \le \frac{\pi}{2} + \frac{\pi}{4}$

$-\frac{3\pi}{4} \le y \le \frac{3\pi}{4}$ (c)

Question 11.

a) $\int \frac{\cos 2x}{\cos^2 x}\, dx = \int \frac{2\cos^2 x - 1}{\cos^2 x}\, dx$

$= \int 2 - \frac{1}{\cos^2 x}\, dx$

$= \int 2 - \sec^2 x\, dx$

$= 2x - \tan x + c$

b) i) $z = \frac{3+i}{1+2i} \times \frac{1-2i}{1-2i}$

$= \frac{3-6i+i-2i^2}{5}$

$= \frac{5-5i}{5}$

$= 1-i$

ii) $|z| = \sqrt{2} \qquad \arg z = -\frac{\pi}{4}$

$z = \sqrt{2}\left(\cos\left(-\frac{\pi}{4}\right) + i\sin\left(-\frac{\pi}{4}\right)\right)$

$= (\sqrt{2})^7\left[\cos\left(-\frac{7\pi}{4}\right) + i\sin\left(-\frac{7\pi}{4}\right)\right]$

$= 8\sqrt{2}\left[\cos\left(\frac{\pi}{4}\right) + i\sin\left(\frac{\pi}{4}\right)\right]$

11 c) i) $-24-10i = (a+ib)^2$

$a^2-b^2+2abi = -24-10i$

Real $= a^2-b^2 = -24$

Imaginary $= 2ab = -10$

$b = -\frac{5}{a}$

Sub $b = \frac{-5}{a}$ into $a^2-b^2=-24$

$a^4+24a^2-25=0$

$(a^2+25)(a^2-1) = 0$

$a = \pm 1$ since a is real

$b = \pm 5$

The roots are $1-5i$, $-1+5i$

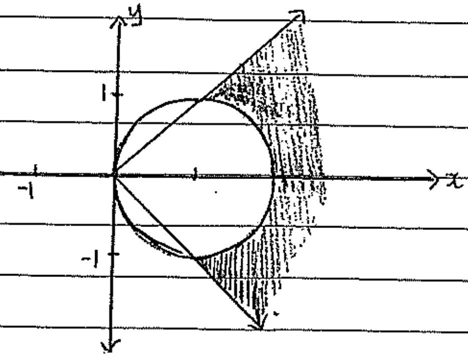
ii) $x = \frac{1-i \pm \sqrt{(1-i)^2-4(b+2i)}}{2}$

$x = \frac{1-i \pm \sqrt{-24-10i}}{2}$

$x = \frac{1-i \pm (1-5i)}{2}$

$x = 1-3i \qquad x = 2i$

d)



e i) Method 1

$\cot\theta + \operatorname{cosec}\theta$

$= \frac{\cos\theta}{\sin\theta} + \frac{1}{\sin\theta}$

$= \frac{\cos\theta + 1}{\sin\theta}$

$= \frac{2\cos^2\frac{\theta}{2}}{2\sin\frac{\theta}{2}\cos\frac{\theta}{2}}$

$= \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}$

$= \cot\frac{\theta}{2}$

Method 2

let $t = \tan\frac{\theta}{2}$

$\cot\theta + \operatorname{cosec}\theta$

$= \frac{1}{\tan\theta} + \frac{1}{\sin\theta}$

$= \frac{1-t^2}{2t} + \frac{1+t^2}{2t}$

$= \frac{2t}{2}$

$= \frac{1}{t}$

$= \cot\frac{\theta}{2}$

ii) $\int(\cot\theta + \operatorname{cosec}\theta)\,d\theta = \int \cot\frac{\theta}{2}\,d\theta$

$= \int \frac{\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,d\theta$

$= 2\int \frac{\frac{1}{2}\cos\frac{\theta}{2}}{\sin\frac{\theta}{2}}\,d\theta$

$= 2\ln\left(\sin\frac{\theta}{2}\right) + c$

Question 12

a) $u = e^x + 1$.

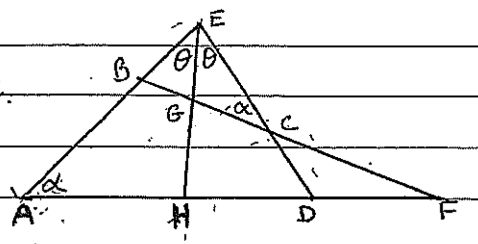
$du = e^x\,dx$ $\quad \int \frac{e^{2x}}{(e^x+1)^2}\,dx = \int \frac{e^x}{(e^x+1)^2}\,dx$

$= \int \frac{u-1}{u^2}\,du$

$= \ln u + \frac{1}{u} + c$

$= \ln(e^x+1) + \frac{1}{e^x+1} + c$

b).



$\angle FGH = \overset{\alpha}{\angle ECG} + \overset{\theta}{\angle CEG}$ (exterior angle of ΔGEC is equal to the sum of interior opposite angles)

$= \overset{\alpha}{\angle EAH} + \overset{\theta}{\angle CEG}$ (exterior angle of cyclic quadrilateral ABCD is equal to interior opposite angle)

$= \overset{\alpha}{\angle EAH} + \overset{\theta}{\angle AEH}$ (given EGH bisects $\angle AED$)

$= \angle FHG$ (exterior angle is equal to the sum of the two opposite interior angles)

$\therefore \Delta FGH$, $FG = FH$ (sides opposite equal angles are equal)

7.

c) $x^2 - xy + y^3 = 1$

$2x - \left(y + x\frac{dy}{dx}\right) + 3y^2\frac{dy}{dx} = 0$

$(3y^2 - x)\frac{dy}{dx} = y - 2x$

At $P(1,1)$ $\quad \frac{dy}{dx} = \frac{y-2x}{3y^2-x} = \frac{1-2}{3-1} = -\frac{1}{2}$

$\therefore$ Tangent at P has gradient $-\frac{1}{2}$ and equation

$y - 1 = -\frac{1}{2}(x-1)$

$x + 2y - 3 = 0$

d) $I_n = \int_0^{\frac{\pi}{4}} \tan^n x\,dx$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2} \tan^2 x\,dx$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,(\sec^2 x - 1)\,dx$

$= \int_0^{\frac{\pi}{4}} \tan^{n-2} x \sec^2 x\,dx - \int_0^{\frac{\pi}{4}} \tan^{n-2} x\,dx$

let $u = \tan x$

$du = \sec^2 x$

when $x = \frac{\pi}{4}$ $\quad u = 1$

$x = 0$ $\quad u = 0$

$\therefore I_n = \int_0^1 u^{n-2}\,du - I_{n-2}$

$= \left[\frac{u^{n-1}}{n-1}\right]_0^1 - I_{n-2}$

8.

$$= \left[\frac{1^{n-1}}{n-1} - 0\right] - I_{n-2}$$

$$= \frac{1}{n-1} - I_{n-2} \quad \text{as required}$$

ii) $\int_0^{\frac{\pi}{4}} \tan^5 x \, dx = I_5$

$I_5 = \frac{1}{4} - I_3$

$I_3 = \frac{1}{2} - I_1$

$I_1 = \int_0^{\pi/4} \tan x \, dx$

$= -\left[\log_e(\cos x)\right]_0^{\frac{\pi}{4}}$

$= -\left(\log_e\left(\cos\frac{\pi}{4}\right) - \log_e(\cos 0)\right)$

$= -\log_e \frac{1}{\sqrt{2}}$

$= -\log_e 2^{-\frac{1}{2}}$

$= \frac{1}{2}\log_e 2$

$I_3 = \frac{1}{2} - \frac{1}{2}\log_e 2$

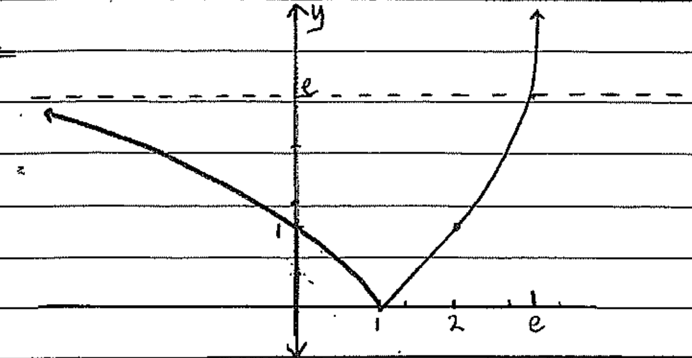
$I_5 = \frac{1}{4} - \left(\frac{1}{2} - \frac{1}{2}\log_e 2\right)$

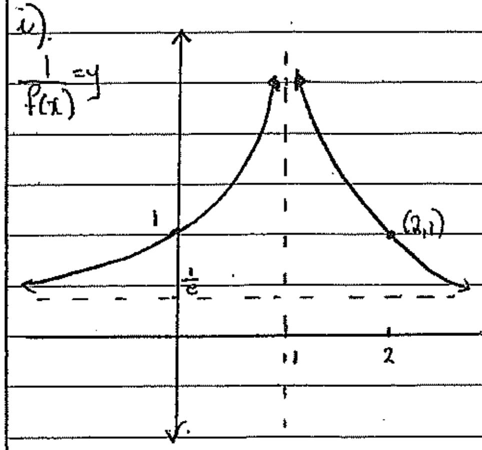
$\therefore \int_0^{\frac{\pi}{4}} \tan^5 x \, dx = \frac{1}{2}\log_2 2 - \frac{1}{4}$

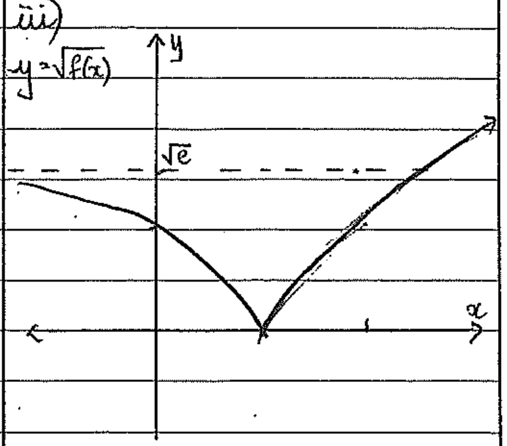
Student Name: Teacher Name:

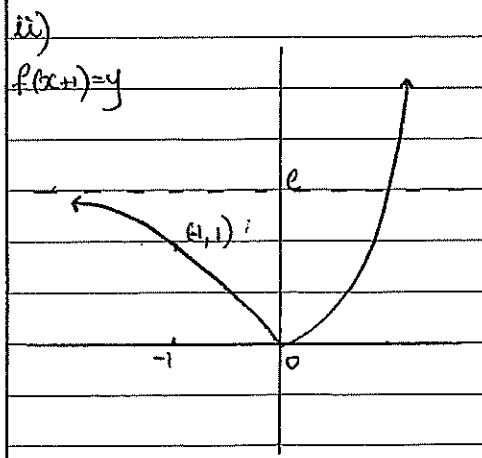
Question 13.

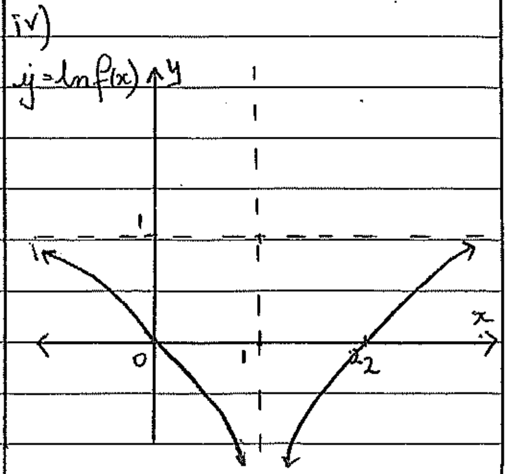
a).











b i) $\frac{y^2}{16} - \frac{x^2}{9} = 1$

$a^2 = b^2(e^2-1)$

$9 = 16(e^2-1)$

$\frac{9}{16} = e^2 - 1$

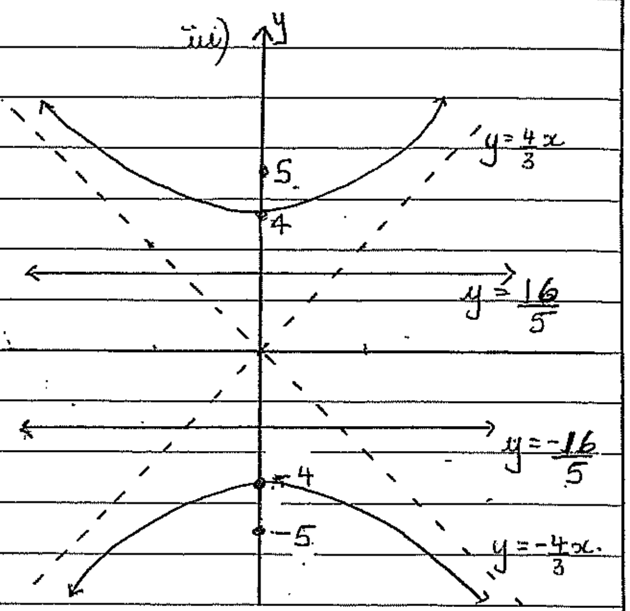
$e^2 = \frac{25}{16}$

$e = \frac{5}{4}$

ii) Foci $(0, \pm 5)$

Directrices $y = \pm \frac{16}{5}$

iii)



c) $\int \frac{2x-3}{x^2-4x+5}\,dx = \int \frac{2x-3-1+1}{x^2-4x+5}\,dx$

$= \int \frac{2x-4}{x^2-4x+5} + \frac{1}{x^2-4x+5}\,dx$

$= \ln(x^2-4x+5) + \int \frac{1}{(x-2)^2+1}\,dx$

$= \ln(x^2-4x+5) + \tan^{-1}(x+2) + C$

Student Name: ______ Teacher Name: ______

Question 14

a i) By parts $u = \sin^{-1}x$ $v = \frac{x^2}{2}$

$u' = \frac{1}{\sqrt{1-x^2}}$ $v' = x$

$= \left[\frac{x^2}{2}\sin^{-1}x\right]_0^1 - \frac{1}{2}\int_0^1 \frac{x^2}{\sqrt{1-x^2}}\,dx$

$= \left[\frac{1}{2}\sin^{-1}1 - 0 - \frac{1}{2}\times\frac{\pi}{4}\right]$

$= \frac{\pi}{4} - \frac{\pi}{8}$

$= \frac{\pi}{8}$

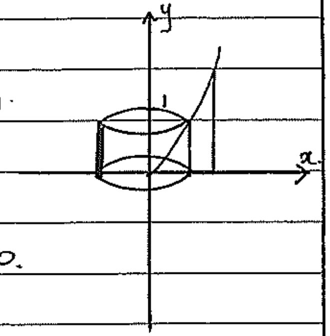
b ii) $V_{shell} = \pi R^2 h - \pi r^2 h$

$= \pi y(x+\delta x+x)(x+\delta x-x)$

$= \pi y(2x+\delta x)(\delta x)$

$= 2\pi xy\,\delta x$

as δx^2 is negligible $\delta x \to 0$.



Volume of Solid $= 2\pi\int_0^1 xy\,dx$

$= 2\pi\int_0^1 x\sin^{-1}x\,dx$

$= 2\pi \times \frac{\pi}{8}$

$= \frac{\pi^2}{4}$ units^3

Student Name: Teacher Name:

b i) $v^2 = 400\left(10e^{-x/20} - 1\right)$ At maximum height $v = 0$

ie $10e^{-x/20} - 1 = 0$

$e^{-\frac{x}{20}} = \frac{1}{10}$

$e^{\frac{x}{20}} = 10$

$\frac{x}{20} = \log_e 10$

$x = 20 \log_e 10$ max height

b ii) $4\ddot{x} = 40 - \frac{v^2}{10}$

$4\ddot{x} = \frac{400 - v^2}{10}$

$\ddot{x} = \frac{400 - v^2}{40}$

b iii) $\ddot{x} = \frac{400 - v^2}{40}$

$-v\frac{dv}{dx} = \frac{400 - v^2}{40v}$

$\frac{dx}{dv} = \frac{40v}{400 - v^2}$

$x = -20 \log_e(400 - v^2) + c$ when $x = 0$ $v = 0$

$c = 20 \log_e 400$

$\therefore$ $x = 20 \log_e \frac{400}{400 - v^2}$

Student Name: Teacher Name:

From ii) $\ddot{x} \to 0$ as $v^2 \to 400$.

$\therefore$ Terminal velocity is $20 ms^{-1}$

When $v = 10$ $x = 20 \log_e \frac{400}{400 - 100}$

$x = 20 \log_e \frac{4}{3}$

Body has fallen $20 \log_e \frac{4}{3}$ metres when its speed is 50% of terminal velocity

iv) $x = 20 \log_e \frac{400}{400 - v^2}$

When $x = 20 \log_e 10$ when body returns to point of projection $\therefore$ $20 \log_e 10 = 20 \log_e \frac{400}{400 - v^2}$

$\frac{400}{400 - v^2} = 10$

$400 - v^2 = 40$

$v^2 = 360$

$v = 6\sqrt{10}\ ms^{-1}$

c) Curves intersect at $(0,0)$ and $(1,1)$

length of slice $\sqrt{y} - y^2$

thickness of slice $= \delta y$

Area of cross section $= (\sqrt{y} - y^2)^2$

Volume of slice $= (\sqrt{y} - y^2)^2 \, \delta y$

Volume of solid $= \int_0^1 (\sqrt{y} - y^2)^2 \, dy$

$$= \int_0^1 y - 2y^{\frac{5}{2}} + y^4 \, dy$$

$$= \left[\frac{y^2}{2} - \frac{4}{7} y^{\frac{7}{2}} + \frac{1}{5} y^5\right]_0^1$$

$$= \left[\frac{1}{2} - \frac{4}{7} + \frac{1}{5}\right]$$

$$= \frac{9}{70} \text{ units}^3$$

15.

Student Name: ______ Teacher Name: ______

Question 15

a i) $xy = c^2 \qquad y = c^2 x^{-1}$

$$\therefore \frac{dy}{dx} = -\frac{c^2}{x^2}$$

At P $\frac{dy}{dx} = \frac{-c^2}{c^2 p^2} = -\frac{1}{p^2}$

$\therefore$ gradient of the normal at P is p^2

Equation of normal $y - \frac{c}{p} = p^2(x - cp)$

$$\frac{y}{p} - \frac{c}{p^2} = px - cp^2$$

$$px - \frac{y}{p} = cp^2 - \frac{c}{p^2}$$

$$px - \frac{y}{p} = c\left(p^2 - \frac{1}{p^2}\right)$$

as required

Method 1

$$M_{PQ} = \frac{\frac{c}{p} - \frac{c}{q}}{cp - cq}$$

$$= \frac{c(q-p)}{cpq(p-q)}$$

$\therefore$ The gradient of the normal is p^2

$$\therefore p^2 = -\frac{1}{pq}$$

$$q = -\frac{1}{p^3}$$

16.

a ii) Method 2

Solving $y=\frac{c^2}{x}$ and $px+\frac{y}{p}=c(p^2-\frac{1}{p^2})$ simultaneously

$px+\frac{c^2}{px}=c\left(p^2-\frac{1}{p^2}\right)$

$p^2x^2-cpx\left(p^2-\frac{1}{p^2}\right)+c^2=0$

The roots of this equation are the x-values of the points P and Q : cp and cq

Using the product of the roots.

$cp\,cq=-\frac{c^2}{p^2}$

$q=\frac{-1}{p^3}$ OR

Using the sum of the roots.

$cp+cq=\frac{pc\left(p^2-\frac{1}{p^2}\right)}{p^2}$

$p+q=p-\frac{1}{p^3}$

$q=-\frac{1}{p^3}$

Method 3

The normal meets the hyperbola $y=\frac{c^2}{x}$, again at $Q(cq,\frac{c}{q})$ the co-ordinates of Q must satisfy the equation of the normal.

$\therefore p(cq)-\frac{\left(\frac{c}{q}\right)}{p}=c\left(p^2-\frac{1}{p^2}\right)$

$cpq-\frac{c}{pq}=c\left(p^2-\frac{1}{p^2}\right)$

$pq-\frac{1}{pq}=p^2-\frac{1}{p^2}$

$pq-p^2=\frac{1}{pq}-\frac{1}{p^2}$

$pq-p^2=\frac{p^2-pq}{p^3q}$

$p(q-p)=\frac{p(p-q)}{p^3q}$

$-1=\frac{1}{p^3q}$

$\therefore q=-\frac{1}{p^3}$

b i) $t=\frac{x}{V\cos\alpha}$ from the 1st equation sub into

$y=Vt\sin\alpha-\frac{gt^2}{2}$

$y=V\sin\alpha\left(\frac{x}{V\cos\alpha}\right)-\frac{g}{2}\left(\frac{x}{V\cos\alpha}\right)^2$

$y=x\tan\alpha-\frac{gx^2}{2V^2}\sec^2\alpha$ but $\sec^2\alpha=1+\tan^2\alpha$

$y=x\tan\alpha-\frac{gx^2}{2V^2}\left(1+\tan^2\alpha\right)$

b ii) $y=3$
$x=100.$
$g=10.$
$V=50$

$3=100\tan\alpha-\frac{10\times100^2}{2\times50^2}-\frac{10\times100^2\tan^2\alpha}{2\times50^2}$

$3=100\tan\alpha-\frac{100\,000}{5000}-\frac{100\,000\tan^2\alpha}{5000}$

$20\tan^2\alpha-100\tan\alpha+23=0$

$\tan\alpha=\frac{100\pm\sqrt{100^2-4\times20\times23}}{40}$

$\tan\alpha=3.5723\ ,\ 1.4276$

$\alpha=78^\circ,14^\circ$

c.i) Given α is a zero of multiplicity 2.

$f(x) = (x-\alpha)^2 q(x)$ for some polynomial $q(x)$.

$$f'(x) = (x-\alpha)^2 q'(x) + 2(x-\alpha) q(x)$$
$$= (x-\alpha)\left[(x-\alpha) q'(x) + 2q(x)\right]$$

$$\therefore f(\alpha) = (\alpha-\alpha)^2 q(\alpha)$$
$$= 0 \times q(\alpha)$$
$$= 0$$

$$f'(\alpha) = (\alpha-\alpha)\left[(\alpha-\alpha) q'(\alpha) + 2q(\alpha)\right]$$
$$= 0\left(0 + 2q(\alpha)\right)$$
$$= 0$$

Hence if α is a zero multiplicity 2 of a polynomial $f(x)$, then $f(\alpha) = f'(\alpha) = 0$

c ii) $g(x) = px^3 - 3qx + r$

$g'(x) = 3px^2 - 3q$

Let α be the root of multiplicity 2.

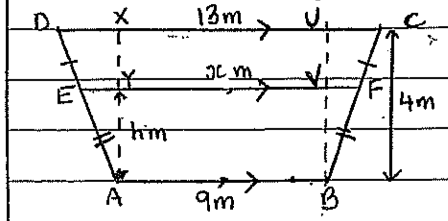
Then $g(\alpha) = p\alpha^3 - 3q\alpha + r = 0$ and

$g'(\alpha) = 3p\alpha^2 - 3q = 0$.

$$3p\alpha^2 - 3q = 0.$$
$$\alpha^2 = \frac{q}{p}.$$

$$p\alpha^3 - 3q\alpha + r = 0$$
$$r = \alpha(3q - p\alpha^2)$$
$$r^2 = \alpha^2(3q - p\alpha^2)^2$$
$$= \frac{q}{p}\left(3q - p \times \frac{q}{p}\right)^2$$
$$= \frac{q}{p}(2q)^2$$
$$r^2 = \frac{4q^3}{p}$$
$$\therefore 4q^3 = pr^2$$

Question 16.



ΔAXD and ΔBUC are congruent (RHS)

$DX = CU = 2$

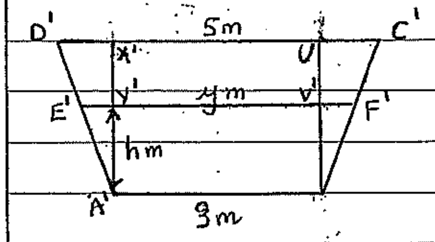
By similar triangles

$$\frac{EY}{AY} = \frac{DX}{AX}$$

$$\frac{EY}{h} = \frac{2}{4} = \frac{1}{2}$$

$$EY = \frac{h}{2}$$

Vertical cross-sections through centre of the base and parallel to the sides of the base will be symmetric trapeziums



Similarly $VF = \frac{h}{2}$

$EF = EY + 9 + VF$

$EF = 9 + h$

ii) Cross sections parallel to the base of the trench will be rectangles with sides x and y as shown in diagram above.

$x = 9 + h$ from b i

From second diagram $D'X' = C'U' = 1$

Using similar triangles $\frac{E'Y'}{h} = \frac{1}{4}$

$y = 3 + 2 \times \frac{h}{4}$

$y = 3 + \frac{h}{2}$

$\therefore$ Area of cross-section $= (9+h)(3+\frac{h}{2})$

$= \frac{h^2}{2} + \frac{15h}{2} + 27$

$\therefore$ Volume $= \int_0^4 \left(\frac{h^2}{2} + \frac{15h}{2} + 27\right) dh$

$= \left[\frac{h^3}{6} + \frac{15h^2}{4} + 27h\right]_0^4$

$= 178\frac{2}{3}\ m^3$

b) $\int \frac{x^3}{x^2+x+1}\, dx = \int \frac{x^3 - 1 + 1}{x^2+x+1}\, dx$

$= \int x - 1 + \frac{1}{(x+\frac{1}{2})^2 + \frac{3}{4}}\, dx$

$= \frac{x^2}{2} - x + \frac{2}{\sqrt{3}} \tan^{-1}\left(\frac{x+\frac{1}{2}}{\frac{\sqrt{3}}{2}}\right) + c$

$= \frac{x^2}{2} - x + \frac{2}{\sqrt{3}} \tan^{-1} \frac{(2x+1)}{\sqrt{3}} + c$

c) $\alpha^2 + \beta^2 + \gamma^2 + \delta^2 = (\alpha+\beta+\gamma+\delta)^2 - 2(\alpha\beta + \alpha\gamma + \alpha\delta + \beta\gamma + \beta\delta + \gamma\delta)$

$= \left(-\frac{b}{a}\right)^2 + -2 \times \frac{A}{1}$

$= 0 - 2A$

$= -2A$ as required

cii) Since α, β, γ and δ are roots then

$\alpha^4 + A\alpha^2 + B\alpha + C = 0$

$\beta^4 + A\beta^2 + B\beta + C = 0$

$\gamma^4 + A\gamma^2 + B\gamma + C = 0$

$\delta^4 + A\delta^2 + B\delta + C = 0$

Adding $\alpha^4 + \beta^4 + \gamma^4 + \delta^4 = -A(\alpha^2 + \beta^2 + \gamma^2 + \delta^2) - B(\alpha+\beta+\gamma+\delta) - 4C$

$= -A(-2A) - B(0) - 4C$

$= 2A^2 - 4C$ as required

d i) $\frac{1}{(k+1)^2} - \frac{1}{k} + \frac{1}{k+1} = \frac{k-(k+1)^2+k(k+1)}{k(k+1)^2}$

$= \frac{k(k^2+2k+1)+k^2+k}{k(k+1)^2}$

$= \frac{-1}{k(k+1)^2} < 0$ since $k > 0$

ii) For $n = 2$

$L.H.S = \frac{1}{1^2} + \frac{1}{2^2} = 1\frac{1}{4}$

$R.H.S = 2 - \frac{1}{2} = 1\frac{1}{2}$

$\therefore$ True for $n = 2$ Since $1\frac{1}{4} < 1\frac{1}{2}$

Let $\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{k^2} < 2 - \frac{1}{k}$

be true for some integer k

Then for $n = k+1$ we need to prove that:

$\frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$

$L.H.S = \frac{1}{1^2} + \frac{1}{2^2} + \dots + \frac{1}{k^2} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k} + \frac{1}{(k+1)^2}$ from (i)

but $\frac{1}{(k+1)^2} - \frac{1}{k} < \frac{-1}{k+1}$

So $2 - \frac{1}{k} + \frac{1}{(k+1)^2} < 2 - \frac{1}{k+1}$

$< R.H.S$

$\therefore$ By Mathematical Induction

$\frac{1}{1^2} + \frac{1}{2^2} + \frac{1}{3^2} + \dots + \frac{1}{n^2} < 2 - \frac{1}{n}$

is true for all integers $n \geq 2$

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