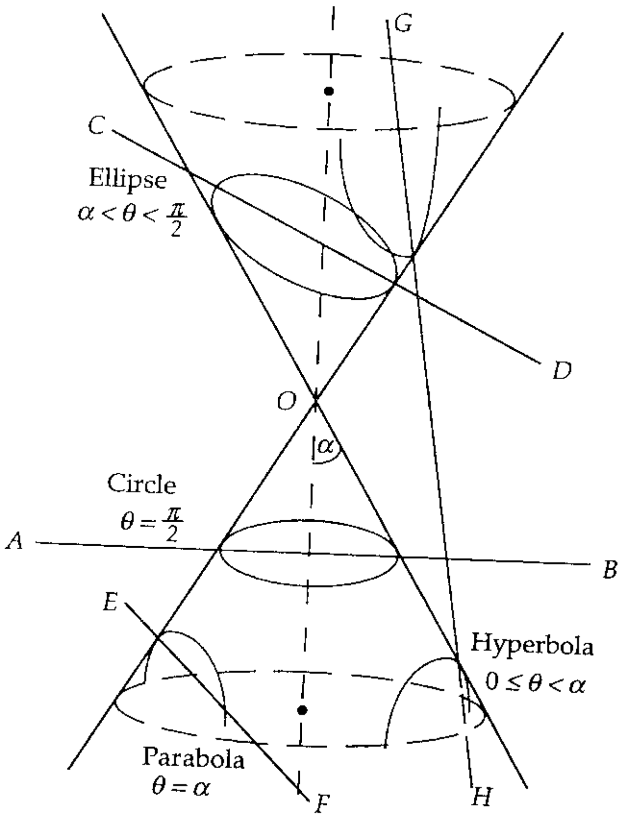
Conics

The conic sections were first discovered by the Greeks about 350 – 400BC

They again came into prominence at the time of Galileo and Kepler, but it was not until the work of Descartes and de Fermat in the 1600's that the curves were described algebraically.

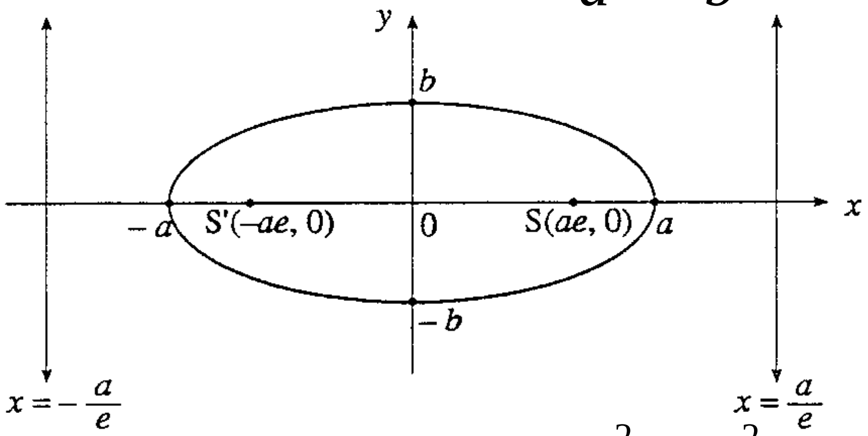


Generally there are two Conics questions in the HSC.

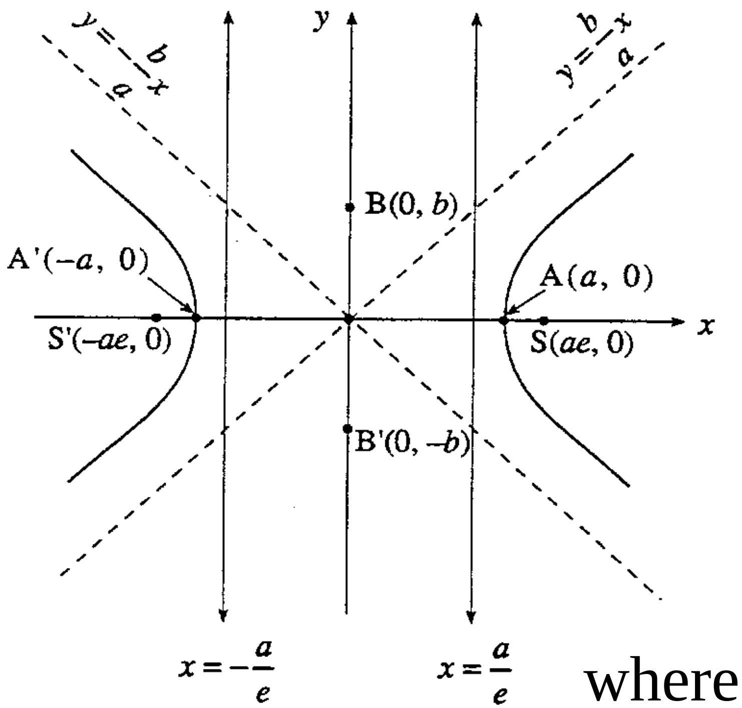
You will need to know how to find the **eccentricity** and how to find **focal points**, the equations of **directrices** and **asymptotes**, etc

You will need to be able to derive the equations of **chords**, **tangents** and **normals.**

Ellipse (0 < e < 1) $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ where; $b^2=a^2(1-e^2)$



Hyperbola (e > 1) $\dfrac{x^2}{a^2}-\dfrac{y^2}{b^2}=1$



where; $b^2=a^2(e^2-1)$

focus: $(\pm ae, 0)$

directrices: $x=\pm\dfrac{a}{e}$

asymptotes: $y=\pm\dfrac{b}{a}x$

(hyperbola only)

Note: If $b > a$ *(ellipse)*

OR

If $\dfrac{y^2}{b^2}-\dfrac{x^2}{a^2}=1$

foci on the y axis

focus: $(0, \pm be)$

directrices: $y=\pm\dfrac{b}{e}$

Examples

1. A conic has the equation $\dfrac{x^2}{k}+\dfrac{y^2}{2k-3}=1$

   Find the value of *k* if the equation represents:

(i) a hyperbola with focii on the *x* axis.

$$k>0 \qquad 2k-3<0$$

$$2k<3$$

$$k<\frac{3}{2}$$

$$0<k<\frac{3}{2}$$

(ii) an ellipse with its major axis along the *x* axis.

$$k>2k-3 \qquad 2k-3>0$$

$$-k>-3 \qquad 2k>3$$

$$k<3 \qquad k>\frac{3}{2}$$

$$\frac{3}{2}<k<3$$

2. Find the eccentricity, foci and directrices of $\dfrac{x^2}{16}+\dfrac{y^2}{9}=1$

$$e^2=\frac{a^2-b^2}{a^2}$$

$$e^2=\frac{16-9}{16}$$

$$e^2=\frac{7}{16}$$

$$e=\frac{\sqrt{7}}{4}$$

$\therefore \text{eccentricity}=\dfrac{\sqrt{7}}{4}$

$\text{foci}:\left(\pm\sqrt{7},0\right)$

$\text{directrices}: x=\pm 4\times\dfrac{4}{\sqrt{7}}$

$$x=\pm\frac{16}{\sqrt{7}}$$

3. A hyperbola in standard form has a vertex at (5,0) and a focus at (-8,0). Find:

a) its equation

$a = 5 \qquad ae = 8$

$5e = 8$

$e = \frac{8}{5}$

$b^2 = a^2(e^2 - 1)$

$b^2 = 25\left(\frac{64}{25} - 1\right)$

$b^2 = 25\left(\frac{39}{25}\right)$

$= 39$

$\therefore \frac{x^2}{25} - \frac{y^2}{39} = 1$

b) the eccentricity

$\text{eccentricity} = \frac{8}{5}$

c) the equations of the asymptotes

$y = \pm\frac{\sqrt{39}}{5}x$

Parametric Coordinates

1. Circle

$$x = a\cos\theta \qquad y = a\sin\theta$$

2. Ellipse

$$x = a\cos\theta \qquad y = b\sin\theta$$

3. Hyperbola

$$x = a\sec\theta \qquad y = b\tan\theta$$

For ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$

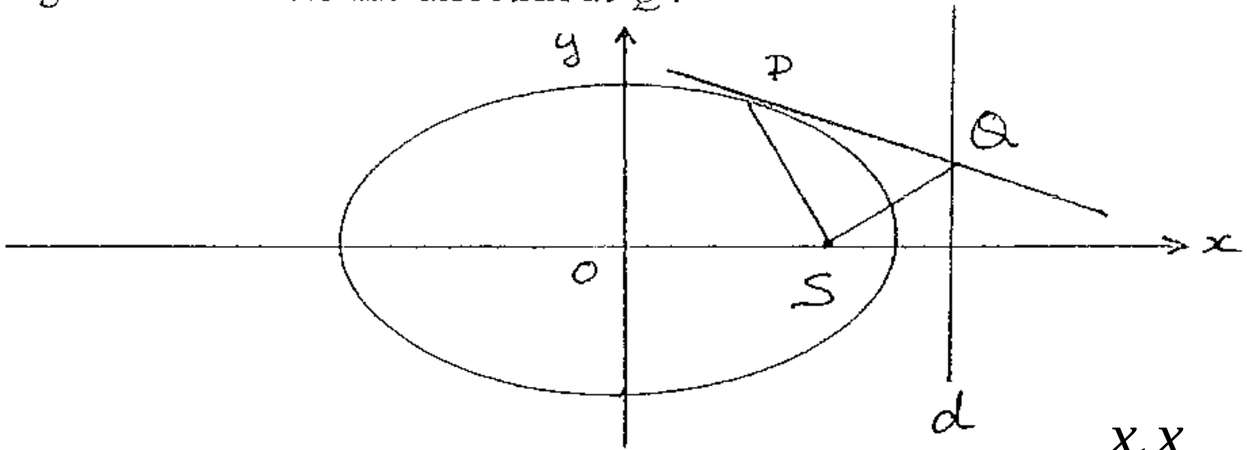
tangent at (x_1, y_1)	normal at (x_1, y_1)
$\frac{x_1 x}{a^2}+\frac{y_1 y}{b^2}=1$	$\frac{a^2 x}{x_1}-\frac{b^2 y}{y_1}=a^2-b^2$
tangent at $(a\cos\theta, b\sin\theta)$	normal at $(a\cos\theta, b\sin\theta)$
$\frac{x\cos\theta}{a}+\frac{y\sin\theta}{b}=1$	$\frac{ax}{\cos\theta}-\frac{by}{\sin\theta}=a^2-b^2$

For hyperbola $\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$

tangent at (x_1, y_1) $\frac{x_1 x}{a^2} - \frac{y_1 y}{b^2} = 1$	normal at (x_1, y_1) $\frac{a^2 x}{x_1} + \frac{b^2 y}{y_1} = a^2 + b^2$
tangent at $(a\sec\theta, b\tan\theta)$ $\frac{x\sec\theta}{a} - \frac{y\tan\theta}{b} = 1$	normal at $(a\sec\theta, b\tan\theta)$ $\frac{ax}{\sec\theta} + \frac{by}{\tan\theta} = a^2 + b^2$

Examples

1. $P(x_1, y_1)$ is a point on the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ with focus S and directrix d. The tangent at P meets the directrix at Q.



(i) Show that the equation of the tangent at P is $\frac{x_1 x}{a^2}+\frac{y_1 y}{b^2}=1$

$$\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$$

$$\frac{2x}{a^2}+\frac{2y}{b^2}\cdot\frac{dy}{dx}=0$$

$$\frac{2y}{b^2}\cdot\frac{dy}{dx}=-\frac{2x}{a^2}$$

$$\frac{dy}{dx}=-\frac{b^2x}{a^2y}$$

at $P(x_1, y_1)$

$$\frac{dy}{dx}=-\frac{b^2x_1}{a^2y_1}$$

$$y-y_1=-\frac{b^2x_1}{a^2y_1}(x-x_1)$$

$$a^2y_1y-a^2y_1^2=-b^2x_1x+b^2x_1^2$$

$$b^2x_1x+a^2y_1y=b^2x_1^2+a^2y_1^2$$

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=\frac{x_1^2}{a^2}+\frac{y_1^2}{b^2}$$

$$\frac{x_1x}{a^2}+\frac{y_1y}{b^2}=1$$

(ii) Find the coordinates of Q

$$\text{when } x = \frac{a}{e}, \frac{\frac{a}{e}x_1}{a^2} + \frac{yy_1}{b^2} = 1$$

$$\frac{yy_1}{b^2} = 1 - \frac{x_1}{ae}$$

$$y = \frac{b^2(ae - x_1)}{aey_1}$$

$$\therefore Q\left(\frac{a}{e}, \frac{b^2(ae - x_1)}{aey_1}\right)$$

(iii) Prove that $\angle PSQ = 90^\circ$

$$m_{PS} = \frac{y_1 - 0}{x_1 - ae}$$

$$= \frac{y_1}{x_1 - ae}$$

$$m_{QS} = \frac{\frac{b^2(ae - x_1)}{aey_1} - 0}{\frac{a}{e} - ae}$$

$$= \frac{b^2(ae - x_1)}{aey_1} \times \frac{e}{a - ae^2}$$

$$= \frac{b^2(ae - x_1)}{a^2(1 - e^2)y_1}$$

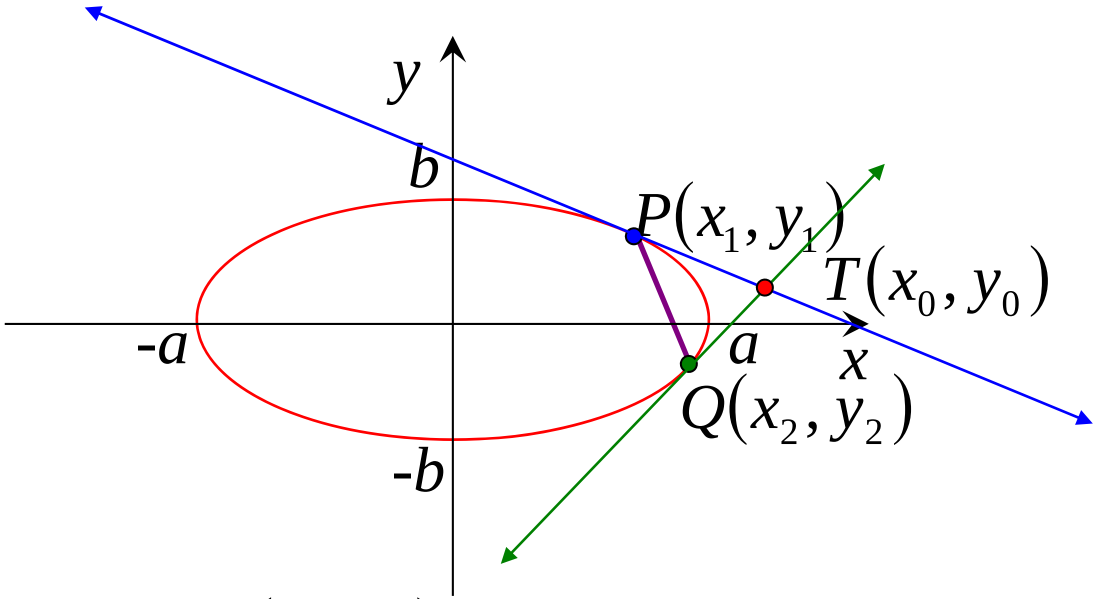
$$m_{PS} \times m_{QS} = \frac{y_1}{(x_1 - ae)} \times \frac{b^2(ae - x_1)}{a^2(1-e^2)y_1}$$

$$= \frac{-b^2}{a^2(1-e^2)} \qquad \text{But } b^2 = a^2(1-e^2)$$

$$= -1$$

$$\therefore \angle PSQ = 90^\circ$$

Chord of Contact



Take external point as $T(x_0, y_0)$ and the tangents from T touch the ellipse at $P(x_1, y_1)$ and $Q(x_2, y_2)$

Equation of tangent at P is $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$

T lies on the tangent, coordinates of T must satisfy its equation.

$$\therefore \frac{x_1 x_0}{a^2} + \frac{y_1 y_0}{b^2} = 1$$

Not only is this the condition for (x_0, y_0) to lie on $\frac{x_1 x}{a^2} + \frac{y_1 y}{b^2} = 1$, but it is the condition for (x_1, y_1) to lie on $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$

i.e. *P* lies on this line.

Similarly, *T* lies on tangent *TR*, leading to a similar condition for *Q* lying on $\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1$

Thus $\boxed{\frac{x_0 x}{a^2} + \frac{y_0 y}{b^2} = 1}$ must be the equation of line *PQ*.

i.e. Equation of the chord of contact from (x_0, y_0)

Example

Prove that the chord of contact from a point on the directrix is a focal chord.

As *T* is on the directrix it has coordinates $\left(\frac{a}{e}, y_0\right)$ i.e. $x_0 = \frac{a}{e}$

$\therefore$ chord of contact will have the equation;

$$\frac{x\left(\frac{a}{e}\right)}{a^2} + \frac{yy_0}{b^2} = 1$$

$$\frac{x}{ae} + \frac{yy_0}{b^2} = 1$$

Substitute in focus $(ae, 0)$

$$\frac{ae}{ae} + 0 = 1 + 0$$

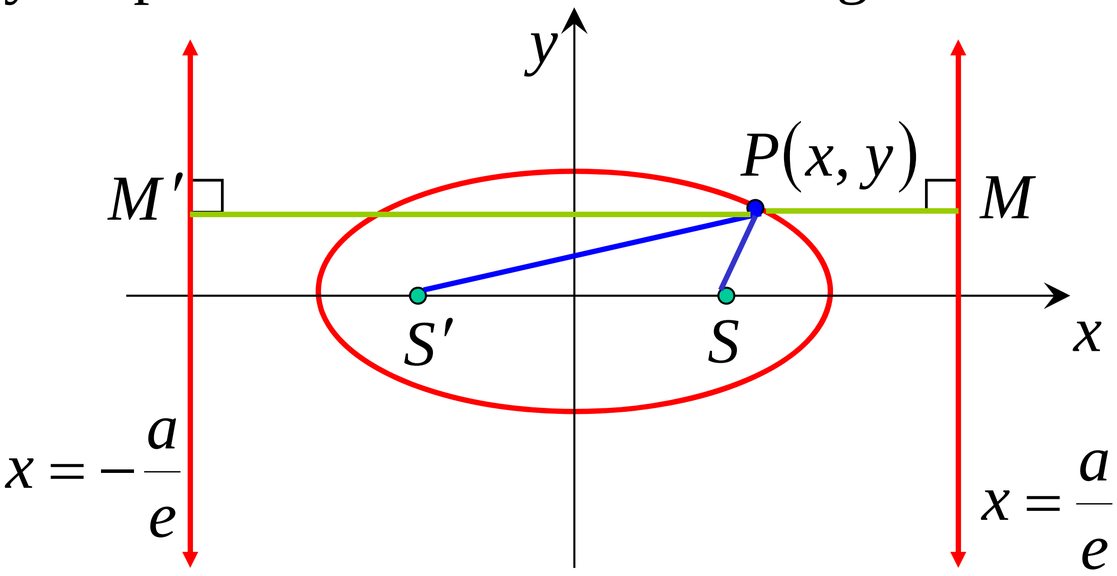
$$= 1$$

$\therefore$ focus lies on chord of contact

i.e. it is a focal chord

Some Geometrical Properties

For any ellipse the sum of the focal lengths is a constant.



By definition of an ellipse;

$$PS + PS' = ePM + ePM'$$

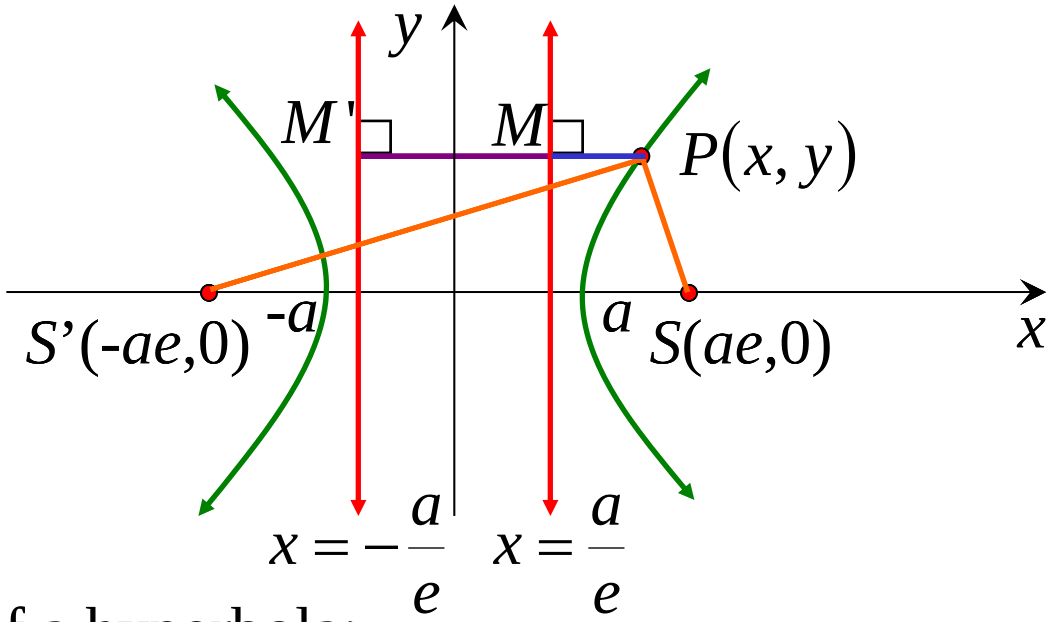
$$= e(PM + PM')$$

$$= e\left(\frac{2a}{e}\right)$$

$$= 2a$$

Example

For any hyperbola the difference of the focal lengths is a constant.



By definition of a hyperbola;

$$PS' - PS = ePM' - ePM$$

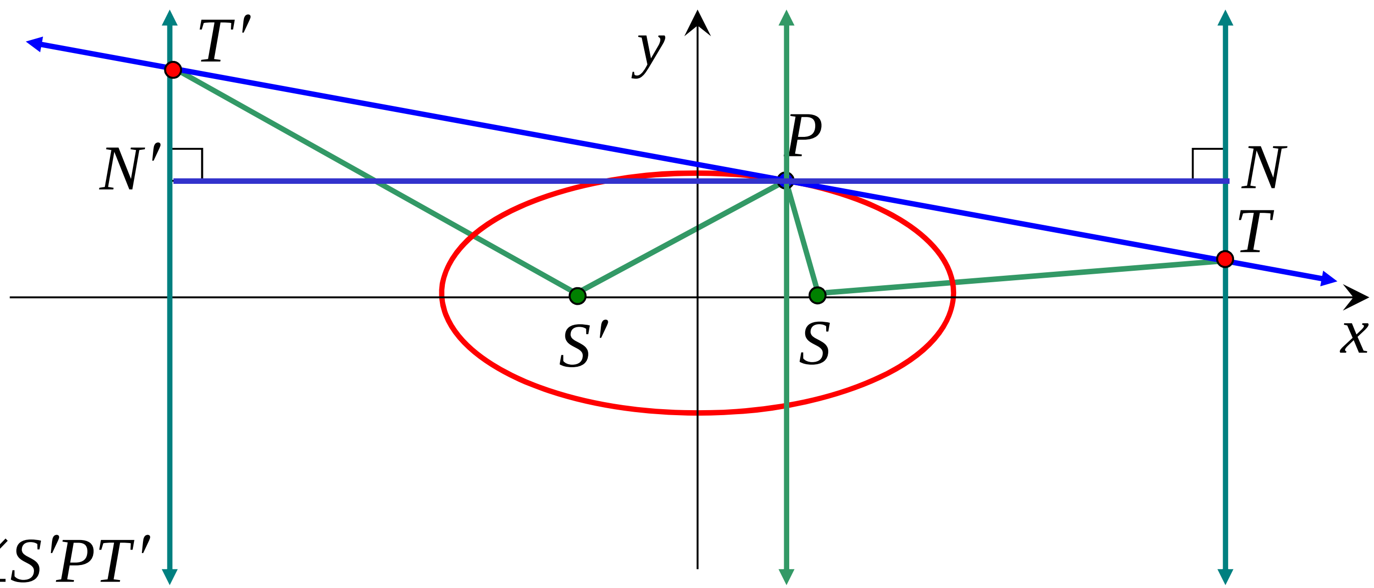
$$= e(PM' - PM)$$

$$= e\left(\frac{2a}{e}\right)$$

$$= 2a$$

Reflection Property

Tangent to an ellipse at a point P on it is equally inclined to the focal chords through P.



Prove: $\angle SPT = \angle S'PT'$

Construct a line || y axis passing through P

$$\frac{PT}{PT'} = \frac{PN}{PN'}$$ (ratio of intercepts of || lines)

$$\therefore \frac{PT}{PN} = \frac{PT'}{PN'}$$

$$ePN = PS \quad \text{and} \quad ePN' = PS'$$

$$\therefore \frac{PT}{\frac{PS}{e}} = \frac{PT'}{\frac{PS'}{e}}$$

$$\frac{PT}{PS} = \frac{PT'}{PS'}$$

$$\angle PST = \angle PS'T' = 90^\circ \quad \text{(proven earlier)}$$

$$\therefore \sec\angle SPT = \sec\angle S'PT'$$

$$\underline{\angle SPT = \angle S'PT'}$$

Rectangular Hyperbola

A hyperbola whose asymptotes are perpendicular to each other

$$x^2 - y^2 = a^2$$

The rectangular hyperbola with *x* and *y* axes as aymptotes, has the equation;

$$xy = \frac{1}{2}a^2$$

where;

$\text{foci}: (\pm a, \pm a)$

$\text{directrices}: x + y = \pm a$

$\text{eccentricity} = \sqrt{2}$

Parametric Coordinates

$$x = ct \qquad y = \frac{c}{t}$$

Tangent: $x + t^2 y = 2ct$

Normal: $t^3 x - ty = c(t^4 - 1)$

Locus questions in the HSC will be restricted to the rectangular hyperbola

Example

1. $P\left(ct, \frac{c}{t}\right)$ lies on the rectangular hyperbola $xy = c^2$

a) Show that the normal at P cuts the hyperbola again at the point Q with coordinates $\left(-\frac{c}{t^3}, -ct^3\right)$

$$y = \frac{c^2}{x}$$

$$\frac{dy}{dx} = \frac{-c^2}{x^2}$$

when $x = ct, \frac{dy}{dx} = \frac{-c^2}{c^2t^2}$

$$= -\frac{1}{t^2}$$

$$y - \frac{c}{t} = t^2(x - ct)$$

$$ty - c = t^3x - ct^4$$

$$t^3x - ty = ct^4 - c$$

when $y = -ct^3$

$$t^3x + ct^4 = ct^4 - c$$

$$t^3x = -c$$

$$x = -\frac{c}{t^3}$$

$\therefore Q$ is on the normal

$$xy = -\frac{c}{t^3} \times -ct^3$$

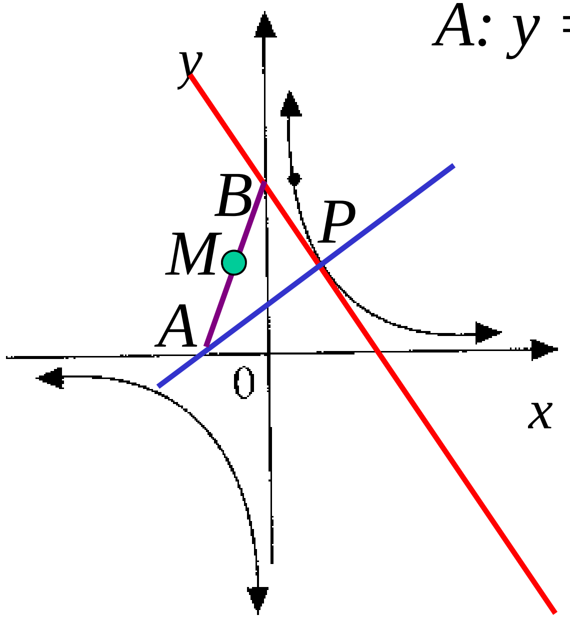
$$= c^2$$

Q also lies on the hyperbola, so the normal intersects the hyperbola at Q

b) Hence find the coordinates of the point R where the normal at Q cuts the hyperbola again.

$$R\left(-\frac{c}{\left(-\frac{1}{t^3}\right)^3}, -c\left(-\frac{1}{t^3}\right)^3\right) = \left(ct^9, \frac{c}{t^9}\right)$$

c) The normal at P meets the x axis at A and the tangent at P meets the y axis at B. M is the midpoint of AB. Find the locus of M.



A: $y = 0$ $\quad t^3x = c\left(t^4 - 1\right)$

$$x = \frac{c\left(t^4 - 1\right)}{t^3}$$

B: $x = 0$ $\quad t^2y = 2ct$

$$y = \frac{2c}{t}$$

$$M\left\{\frac{c\left(t^4 - 1\right)}{2t^3}, \frac{c}{t}\right\}$$

$$t = \frac{c}{y} \qquad x = \frac{c\left(\left(\frac{c}{y}\right)^4 - 1\right)}{2\left(\frac{c}{y}\right)^3}$$

$$x = \frac{c\left(\left(\frac{c}{y}\right)^4 - 1\right)}{2\left(\frac{c}{y}\right)^3}$$

$$x = \frac{c\left(c^4 - y^4\right)}{y^4} \times \frac{2y^3}{c^3}$$

$$x = \frac{2\left(c^4 - y^4\right)}{c^2 y}$$

2006 Extension 2 HSC Q4c)

Let $P\left(p, \frac{1}{p}\right)$, $Q\left(q, \frac{1}{q}\right)$ and $R\left(r, \frac{1}{r}\right)$ be three distinct points on the hyperbola $xy = 1$

(i) Show that the equation of the line, *l*, through *R*, perpendicular to *PQ* is

$$y = pqx - pqr + \frac{1}{r}$$

$$m_{PQ} = \frac{\frac{1}{q} - \frac{1}{p}}{q - p}$$

$$= \frac{\frac{p-q}{pq}}{q-p}$$

$$= \frac{-1}{pq}$$

$$y - \frac{1}{r} = pq(x - r)$$

$$y - \frac{1}{r} = pqx - pqr$$

$$y = pqx - pqr + \frac{1}{r}$$

(ii) Write down the equation of *m*, through *P*, perpendicular to *QR*.

$$y = qrx - pqr + \frac{1}{p}$$

(iii) The lines *l* and *m* intersect at *T*. Show that *T* lies on the hyperbola.

$$y = pqx - pqr + \frac{1}{r}$$

$$y = qrx - pqr + \frac{1}{p}$$

$$0 = (pq - qr)x + \frac{1}{r} - \frac{1}{p}$$

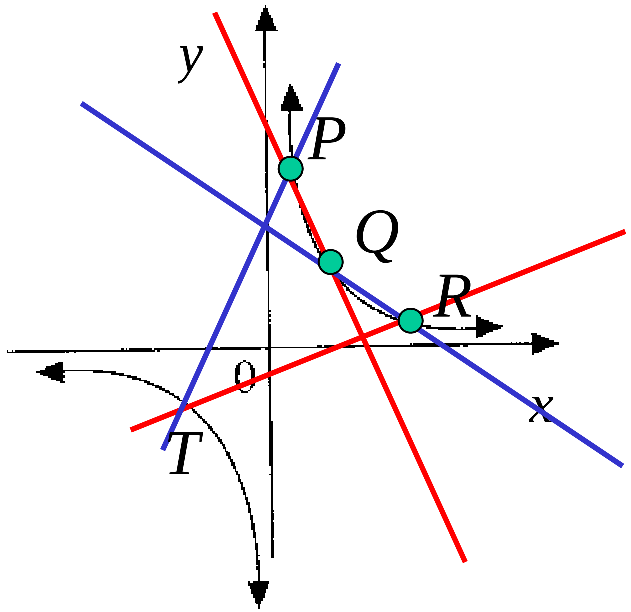
$$q(p - r)x = \frac{r - p}{pr}$$

$$x = \frac{-1}{pqr} \qquad y = \frac{-qr}{pqr} - pqr + \frac{1}{p}$$

$$= \frac{-1}{p} - pqr + \frac{1}{p}$$

$$= -pqr$$

$$\therefore T\left(\frac{-1}{pqr}, -pqr\right)$$



$$xy = \frac{-1}{pqr} \times -pqr$$

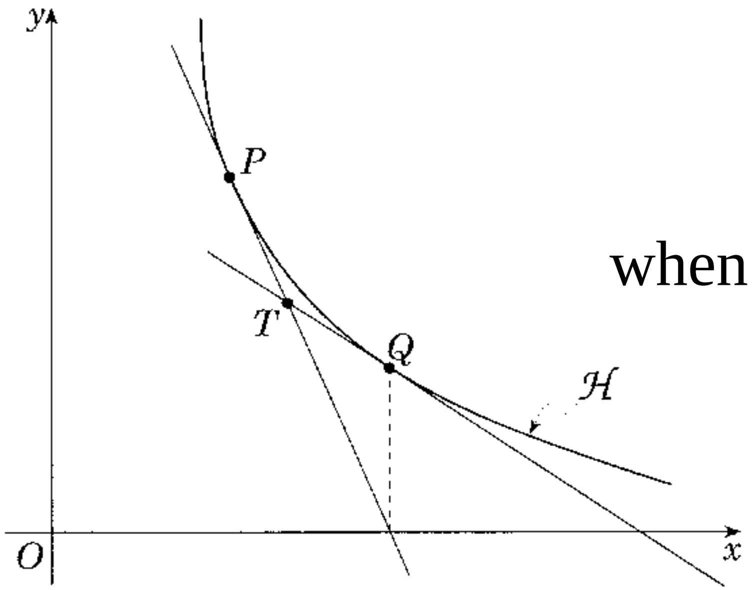
$$= 1$$

$\therefore$ *T* lies on the hyperbola

2002 Extension 2 HSC Q3b)

The distinct points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$ are on the same branch of the hyperbola *H* with equation $xy = c^2$. The tangents to *H* at *P* and *Q* meet at the point *T*.

(i) Show that the equation of the tangent is $x + p^2 y = 2cp$



$$y = \frac{c^2}{x}$$

$$\frac{dy}{dx} = \frac{-c^2}{x^2}$$

$$\text{when } x = cp, \frac{dy}{dx} = \frac{-c^2}{c^2 p^2}$$

$$= -\frac{1}{p^2}$$

$$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$$

$$p^2 y - cp = -x + cp$$

$$x + p^2 y = 2cp$$

(ii) Show that T is the point $\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$

$$x + p^2 y = 2cp$$
$$x + q^2 y = 2cq$$
$$\left(p^2 - q^2\right)y = 2c(p-q)$$
$$y = \frac{2c}{p+q}$$

$$x + \frac{2cp^2}{p+q} = 2cp$$
$$x = \frac{2cp(p+q) - 2cp^2}{p+q}$$
$$= \frac{2cpq}{p+q}$$

$$\therefore T\left(\frac{2cpq}{p+q}, \frac{2c}{p+q}\right)$$

(iii) Suppose P and Q move so that the tangent at P intersects the x axis at $(cq,0)$. Show that the locus of T is a hyperbola, and state its eccentricity.

$$(cq,0): cq = 2cp$$
$$q = 2p$$

$$x = \frac{2cp \times 2p}{p+2p} = \frac{4cp}{3}$$

$$y = \frac{2c}{p+2p} = \frac{2c}{3p}$$

$$xy = \frac{8c^2}{9}$$

$\therefore$ locus of T is the rectangular hyperbola $xy = \frac{8c^2}{9}$, with eccentricity $= \sqrt{2}$

1998 Extension 2 HSC Q5a)

$P\left(4p, \frac{4}{p}\right)$ and $Q\left(4q, \frac{4}{q}\right)$ where $p > 0$ and $q > 0$, are two points on the hyperbola $xy = 16$.

(i) Find the equation of the chord PQ.

$$m_{PQ} = \frac{\frac{4}{q} - \frac{4}{p}}{4q - 4p}$$

$$= \frac{\frac{4p - 4q}{pq}}{4q - 4p}$$

$$= \frac{-1}{pq}$$

$$y - \frac{4}{p} = \frac{-1}{pq}(x - 4p)$$

$$pqy - 4q = -x + 4p$$

$$x + pqy = 4(p + q)$$

(ii) Prove the equation of the tangent at P is $x + p^2 y = 8p$

$$y = \frac{16}{x}$$

$$\frac{dy}{dx} = \frac{-16}{x^2}$$

$$\text{when } x = 4p, \frac{dy}{dx} = \frac{-16}{16p^2}$$

$$= -\frac{1}{p^2}$$

$$y - \frac{4}{p} = -\frac{1}{p^2}(x - 4p)$$

$$p^2 y - 4p = -x + 4p$$

$$x + p^2 y = 8p$$

(iii) The tangents at P and Q intersect at T. Find the coordinates of T.

$$x + p^2 y = 8p$$

$$x + q^2 y = 8q$$

$$(p^2 - q^2)y = 8(p - q)$$

$$y = \frac{8}{p+q}$$

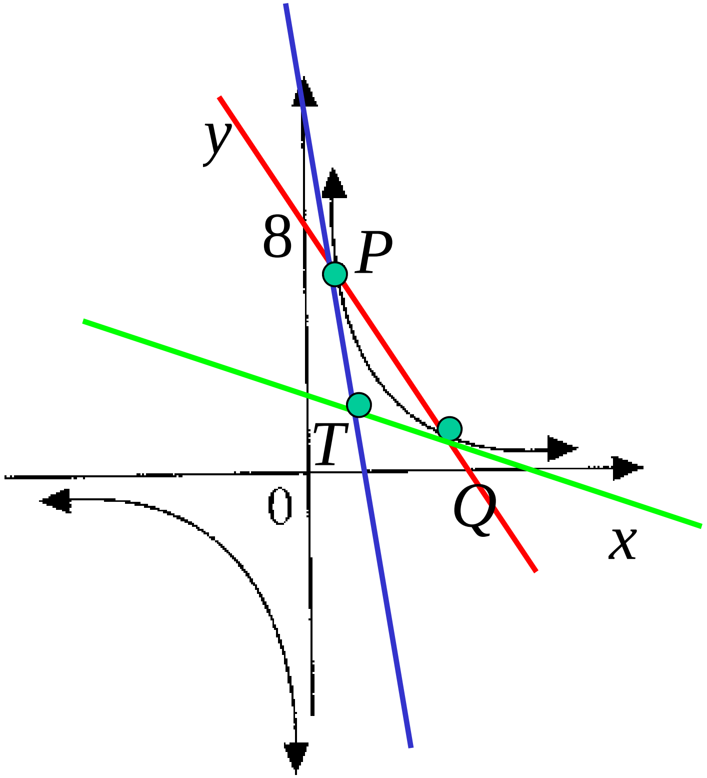
$$x + \frac{8p^2}{p+q} = 8p$$

$$x = \frac{8p(p+q) - 8p^2}{p+q}$$

$$= \frac{8pq}{p+q}$$

$$\therefore T\left(\frac{8pq}{p+q}, \frac{8}{p+q}\right)$$

(iv) The chord PQ produced passes through the point $N(0,8)$. Find the locus of T.



$$(0,8): 8pq = 4(p+q)$$

$$x = \frac{4(p+q)}{p+q}$$

$$x = 4$$

However tangents could only meet in the area between the x axis and the hyperbola

$\therefore$ the locus is $x = 4$, with a range of $0 < y < 4$