Factorial Notation

$$\begin{cases} 0! = 1 \\ n! = n \times (n-1)! \text{ , for } n \geq 1 \end{cases}$$

From this definition we can conclude;

$$1! = 1 \times 0! = 1 \times 1 = 1$$

$$2! = 2 \times 1! = 2 \times 1 = 2$$

$$3! = 3 \times 2! = 3 \times 2 = 6$$

$$4! = 4 \times 3! = 4 \times 6 = 24$$

$$\vdots$$

$$\vdots$$

$$n! = n \times (n-1)! = n(n-1) \times (n-2)!$$

$$n! = n(n-1)(n-2)(n-3)\cdots 3 \times 2 \times 1$$

e.g. $(i)\ \frac{12!}{9!} = \frac{12\times 11!}{9!}$

$$= \frac{12\times 11\times 10!}{9!}$$

$$= \frac{12\times 11\times 10\times 9!}{9!}$$

$$= 12\times 11\times 10$$

$$= 1320$$

$(ii)\ \frac{16!}{10!\times 7!} = \frac{16\times 15\times 14\times 13\times 12\times 11}{7\times 6\times 5\times 4\times 3\times 2\times 1}$

$$= \frac{16\times 15\times 13\times 12\times 11}{6\times 5\times 4\times 3}$$

$$= \frac{\overset{4}{16}\times 13\times \overset{2}{12}\times 11}{6\times 4}$$

$$= 4\times 13\times 2\times 11$$

$$= 4\times 26\times 11$$

$$= 104\times 11$$

$$= 1144$$

$(iii)\ \frac{1}{k!} + \frac{1}{(k-2)!} = \frac{k(k-1)+1}{k(k-1)(k-2)!}$

$$= \frac{k^2 - k + 1}{k!}$$

The Basic Counting Principle

If one event can happen in *m* different ways and after this another event can happen in *n* different ways, then the two events can occur in *mn* different ways.

e.g. 3 dice are rolled

(*i*) How many ways can the three dice fall?

the 1st die has 6 possibilities

the 2nd die has 6 possibilities

the 3rd die has 6 possibilities

$$\text{Ways} = 6 \times 6 \times 6$$
$$= \underline{216}$$

The number of ways of arranging *n* distinct objects, with replacement, in *k* different ways is n^k

(ii) How many ways can all three dice show the same number?

the 1st die has 6 possibilities

the 2nd die now has only 1 possibility

$$\text{Ways} = 6 \times 1 \times 1$$

$$= 6$$

the 3rd die now has only 1 possibility

(iii) What is the probability that all three dice show the same number?

$$P(\text{all 3 the same}) = \frac{6}{216}$$

$$= \frac{1}{36}$$

1996 Extension 1 HSC Q5c)

Mice are placed in the centre of a maze which has five exits.

Each mouse is equally likely to leave the maze through any of the five exits. Thus, the probability of any given mouse leaving by a particular exit is $\frac{1}{5}$

Four mice, *A*, *B*, *C* and *D* are put into the maze and behave independently.

(i) What is the probability that *A*, *B*, *C* and *D* all come out the same exit?

$$\text{Total possibilities} = 5^4$$
$$= 625$$

First mouse can go through any door

$$\text{Ways go through same door} = 5 \times 1 \times 1 \times 1$$
$$= 5$$

Other mice must go through same door

$$P(\text{all use same exit}) = \frac{5}{625}$$
$$= \frac{1}{125}$$

(ii) What is the probability that A, B and C come out the same exit and D comes out a different exit?

D can go through any door

Ways ABC use same exit, D uses different exit $= 5 \times 4 \times 1 \times 1$

$= 20$

Next mouse has 4 doors to choose

Other mice must go through the same door as D

$$P(ABC \text{ use same exit, } D \text{ uses different exit}) = \frac{20}{625}$$

$$= \frac{4}{125}$$

(iii) What is the probability that *any* three of the four mice come out the same exit and the other comes out a different exit?

$$P(D \text{ uses different exit}) = \frac{4}{125}$$

$$\therefore P(A \text{ uses different exit}) = \frac{4}{125}$$

$$P(B \text{ uses different exit}) = \frac{4}{125}$$

$$P(C \text{ uses different exit}) = \frac{4}{125}$$

$$\therefore P(\text{any mouse uses different exit}) = 4 \times \frac{4}{125}$$

$$= \frac{16}{125}$$

(iv) What is the probability that no more than two mice come out the same exit?

$$P(\text{no more than 2 use same exit}) = 1 - P(\text{all same}) - P(\text{3 use same})$$

$$= 1 - \frac{1}{125} - \frac{16}{125}$$

$$= \frac{108}{125}$$

Permutations

A permutation is an **ordered** set of objects
i.e. an **arrangement**

Case 1: Ordered Sets of *n* Different Objects, from a Set of *n* Such Objects

(i.e. use all of the objects)

If we arrange *n* different objects in a line, the number of ways we could arrange them are;

possibilities for object 1

possibilities for object 2

possibilities for object 3

possibilities for last object

$$\text{Number of Arrangements} = n \times (n-1) \times (n-2) \times \cdots \times 1$$
$$= n!$$

e.g. In how many ways can 5 boys and 4 girls be arranged in a line if;

(i) there are no restrictions?

$$\text{Arrangements} = 9!$$
$$= \underline{362880}$$

With no restrictions, arrange 9 people
gender does not matter

(ii) boys and girls alternate?

(ALWAYS look after any restrictions first)

first person MUST be a boy

number of ways of arranging the boys

$$\text{Arrangements} = 1 \times 5! \times 4!$$
$$= \underline{2880}$$

number of ways of arranging the girls

(iii) What is the probability of the boys and girls alternating?

$$P(\text{boys \& girls alternate}) = \frac{2880}{362880}$$

$$= \underline{\frac{1}{126}}$$

(iv) Two girls wish to be together?

the number of ways the girls can be arranged

number of ways of arranging 8 objects (2 girls) + 7 others

$$\text{Arrangements} = 2! \times 8!$$

$$= \underline{80640}$$

Case 2: Ordered Sets of *k* Different Objects, from a Set of *n* Such Objects (*k* < *n*)

(i.e. use some of the objects)

If we have *n* different objects in a line, but only want to arrange *k* of them, the number of ways we could arrange them are;

possibilities for object 1; possibilities for object 2; possibilities for object 3; possibilities for object k

$$\text{Number of Arrangements} = n \times (n-1) \times (n-2) \times \cdots \times (n-k+1)$$

$$= n(n-1)(n-2)\cdots(n-k+1) \times \frac{(n-k)(n-k-1)\cdots(3)(2)(1)}{(n-k)(n-k-1)\cdots(3)(2)(1)}$$

$$= \frac{n!}{(n-k)!}$$

$$= {}^{n}P_{k}$$

e.g. (*i*) From the letters of the word **PROBLEMS** how many 5 letter words are possible if;

a) there are no restrictions?

$$\text{Words} = {}^8P_5$$
$$= 6720$$

b) they must begin with **P**?

the number of ways P can be placed first

$$\text{Words} = 1 \times {}^7P_4$$
$$= 840$$

Question now becomes how many 4 letter words ROBLEMS

c) **P** is included, but not at the beginning, and **M** is excluded?

the number of positions P can be placed in

Question now becomes how many 4 letter words ROBLES

$$\text{Words} = 4 \times {}^6P_4$$
$$= 1440$$

(*ii)* Six people are in a boat with eight seats, for on each side.
What is the probability that Bill and Ted are on the left side and Greg is on the right?

$$\text{Ways (no restrictions)} = {}^8P_6$$
$$= 20160$$

Ways Bill & Ted can go

Ways Greg can go

Ways remaining people can go

$$\text{Ways (restrictions)} = {}^4P_2 \times {}^4P_1 \times {}^5P_3$$
$$= 2880$$

$$P(\text{B \& T left, G right}) = \frac{2880}{20160}$$
$$= \frac{1}{7}$$

2006 Extension 1 HSC Q3c)

Sophia has five coloured blocks: one red, one blue, one green, one yellow and one white.

She stacks two, three, four or five blocks on top of one another to form a vertical tower.

(i) How many different towers are there that she could form that are three blocks high?

$$\text{Towers} = {}^5P_3$$
$$= \underline{60}$$

(ii) How many different towers can she form in total?

$$2\text{ block Towers} = {}^5P_2 \quad = 20$$
$$3\text{ block Towers} = {}^5P_3 \quad = 60$$
$$4\text{ block Towers} = {}^5P_4 \quad = 120$$
$$5\text{ block Towers} = {}^5P_5 \quad = 120$$
$$\underline{\text{Total number of Towers} = 320}$$

Exercise 14A; 2bdfg, 3adfh, 5aceg, 6bdf, 7, 9, 12, 13

Exercise 14B; 2, 4, 6, 8ac, 10, 12, 14, 16, 18, 20, 23, 25

Exercise 14C; 2, 4, 5, 7, 9, 10, 12, 14, 16, 18, 19, 21, 22