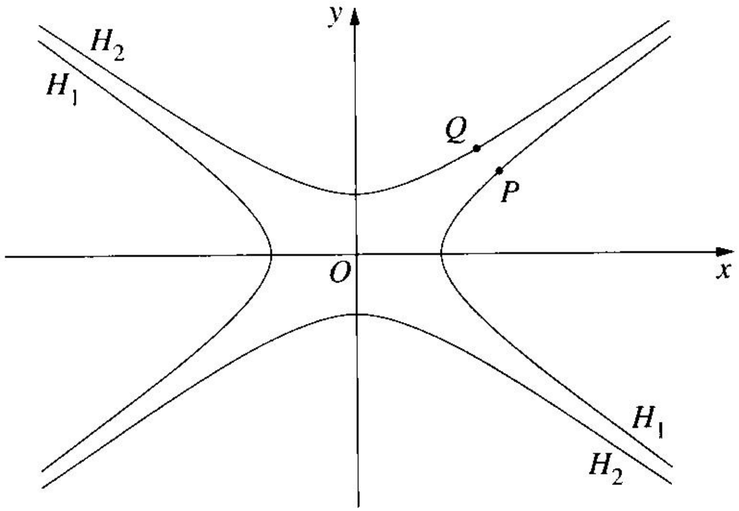
# *Some HSC Conics Questions*

a) **2015 HSC Question 13a)**

The hyperbolas $H_1 : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ and $H_2 : \dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = -1$ are shown in the diagram.




Let $P(a\sec\theta, b\tan\theta)$ lie on $H_1$ as shown in the diagram.

Let $Q$ be the point $(a\tan\theta, b\sec\theta)$

($i$) Verify that the coordinates of $Q(a\tan\theta, b\sec\theta)$ satisfy the equation for $H_2$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = \frac{a^2\tan^2\theta}{a^2} - \frac{b^2\sec^2\theta}{b^2}$$
$$= \tan^2\theta - \sec^2\theta$$
$$= -1$$

$\therefore Q$ lies on $H_2$

($ii$) Show that the equation of the line $P$Q is $bx + ay = ab(\tan\theta + \sec\theta)$

$$m_{PQ} = \frac{b\tan\theta - b\sec\theta}{a\sec\theta - a\tan\theta}$$
$$= -\frac{b}{a}$$

$$y - b\sec\theta = -\frac{b}{a}(x - a\tan\theta)$$
$$ay - ab\sec\theta = -bx + ab\tan\theta$$
$$bx + ay = ab(\tan\theta + \sec\theta)$$

$(iii)$ Prove that the area of $\Delta OPQ$ is independent of $\theta$.

perpendicular distance $O$ to $PQ$

$$= \frac{|-ab(\tan\theta + \sec\theta)|}{\sqrt{a^2 + b^2}}$$

$$= \frac{ab(\tan\theta + \sec\theta)}{\sqrt{a^2 + b^2}}$$

$$d_{PQ} = \sqrt{a^2(\sec \quad - \tan\theta)^2 + b^2(\tan\theta - \sec\theta)^2}$$

$$= \sqrt{(\tan\theta - \sec\theta)^2(a^2 + b^2)}$$

$$= (\tan\theta - \sec\theta)\sqrt{a^2 + b^2}$$

$$A = \frac{1}{2} \times \frac{ab(\tan\theta + \sec\theta)}{\sqrt{a^2 + b^2}} \times (\tan\theta - \sec\theta)\sqrt{a^2 + b^2}$$

$$= \frac{1}{2}ab(\tan^2\theta - \sec^2\theta)$$

$$= \frac{1}{2}ab$$

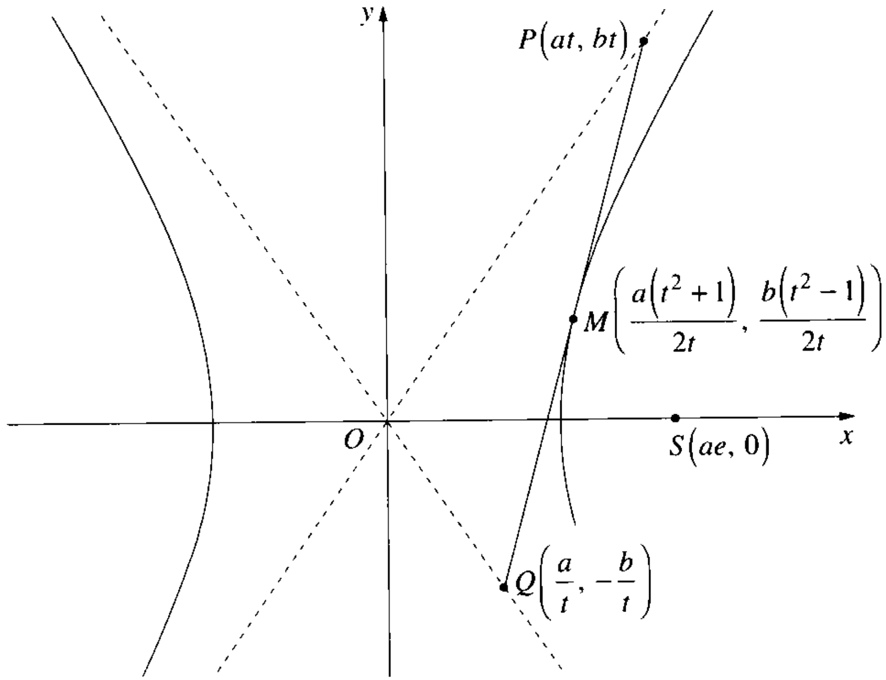
$\therefore$ the area is independent of $\theta$

b) **2014 HSC Question 13c)**

The point $S(ae, 0)$ is the focus of the hyperbola $\dfrac{x^2}{a^2} - \dfrac{y^2}{b^2} = 1$ on the positive $x$-axis.

The points $P(at, bt)$ and $Q\left(\dfrac{a}{t}, -\dfrac{b}{t}\right)$ lie on the asymptotes of the hyperbola, where $t > 0$.

The point $M\left(\dfrac{a\left(t^2+1\right)}{2t}, \dfrac{b\left(t^2-1\right)}{2t}\right)$ is the midpoint of $PQ$



*(i)* Show that $M$ lies on the hyperbola

$$M\left(\frac{a\left(t^2+1\right)}{2t},\frac{b\left(t^2-1\right)}{2t}\right)$$

$$\begin{aligned}\frac{x^2}{a^2}-\frac{y^2}{b^2}&=\frac{a^2\left(t^2+1\right)^2}{4a^2t^2}-\frac{b^2\left(t^2-1\right)^2}{4b^2t^2}\\&=\frac{\left(t^2+1\right)^2-\left(t^2-1\right)^2}{4t^2}\\&=\frac{t^4+2t^2+1-\left(t^4-2t^2+1\right)}{4t^2}\\&=\frac{4t^2}{4t^2}\\&=1\end{aligned}$$

$\therefore$ $M$ lies on the hyperbola

*(ii)* Prove that the line through $P$ and $Q$ is a tangent to the hyperbola at $M$.

$$m_{PQ} = \frac{bt + \frac{b}{t}}{at - \frac{a}{t}} = \frac{b\left(t^2+1\right)}{a\left(t^2-1\right)}$$

$$\frac{x^2}{a^2} - \frac{y^2}{b^2} = 1$$

$$\frac{2x}{a^2} - \frac{2y}{b^2} \cdot \frac{dy}{dx} = 0$$

$$\frac{dy}{dx} = \frac{b^2 x}{a^2 y}$$

$$\text{at } M, \frac{dy}{dx} = \frac{ab^2\left(t^2+1\right)}{2t} \times \frac{2t}{a^2 b\left(t^2-1\right)} = \frac{b\left(t^2+1\right)}{a\left(t^2-1\right)} = m_{PQ}$$

$\therefore PQ$ is parallel to the tangent at $M$

However, as $M$ lies on $PQ$, $PQ$ is the tangent at $M$

(*iii*) Show that $OP \times OQ = OS^2$

$$OP \times OQ = \sqrt{a^2t^2 + b^2t^2} \times \sqrt{\frac{a^2}{t^2} + \frac{b^2}{t^2}}$$
$$= t\sqrt{a^2 + b^2} \times \frac{1}{t}\sqrt{a^2 + b^2}$$
$$= a^2 + b^2$$
$$= a^2e^2 \qquad \left(e^2 = \frac{a^2 + b^2}{a^2}\right)$$
$$= (ae)^2$$
$$\underline{= OS^2}$$

(*iv*) If $P$ and $S$ have the same $x$-coordinate, show that $MS$ is parallel to one of the asymptotes of the hyperbola.

$$at = ae$$
$$t = e$$

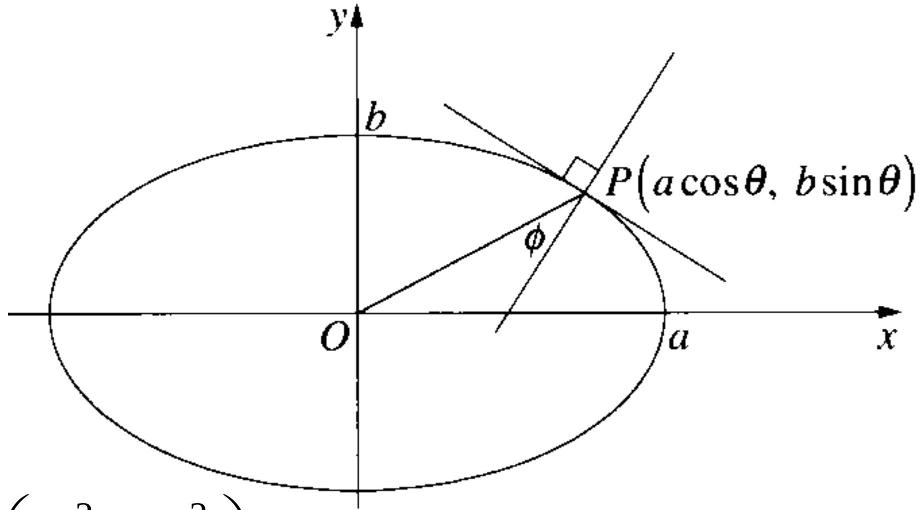
$$m_{MS} = \frac{\frac{b\left(e^2 - 1\right)}{2e}}{\frac{a\left(e^2 + 1\right)}{2e} - ae}$$
$$= \frac{b\left(e^2 - 1\right)}{a\left(1 - e^2\right)} = -\frac{b}{a}$$

$\therefore$ $MS$ is parallel to the asymptote $y = -\frac{b}{a}x$

c) **2014 HSC Question 14b)**

The point $P(a\cos\theta, b\sin\theta)$ lies on the ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$, where $a > b$

The acute angle between $OP$ and the normal to the ellipse at $P$ is $\phi$




(*i*) Show that $\tan\phi=\left(\dfrac{a^2-b^2}{ab}\right)\sin\theta\cos\theta$

$$x=a\cos\theta$$

$$\frac{dx}{d\theta}=-a\sin\theta \qquad \frac{dy}{dx}=-\frac{b\cos\theta}{a\sin\theta}$$

$$y=b\sin\theta$$

$$\frac{dy}{d\theta}=b\cos\theta \qquad \therefore m_{\text{normal}}=\frac{a\sin\theta}{b\cos\theta} \qquad m_{OP}=\frac{b\sin\theta}{a\cos\theta}$$

$$\tan\phi = \left| \frac{\dfrac{b\sin\theta}{a\cos\theta} - \dfrac{a\sin\theta}{b\cos\theta}}{1 + \dfrac{b\sin\theta}{a\cos\theta} \times \dfrac{a\sin\theta}{b\cos\theta}} \right|$$

$$= \left| \frac{\dfrac{a\sin\theta\cos\theta}{b} - \dfrac{b\sin\theta\cos\theta}{a}}{\cos^2\theta + \sin^2\theta} \right|$$

$$= \left| \frac{a^2\sin\theta\cos\theta - b^2\sin\theta\cos\theta}{ab} \right|$$

$$= \left( \frac{a^2 - b^2}{ab} \right) \left| \sin\theta\cos\theta \right|$$

If $P$ is as illustrated, then $\theta$ is acute

$$\Rightarrow \sin\theta > 0 \quad , \quad \cos\theta > 0$$

$$\tan\phi = \left( \frac{a^2 - b^2}{ab} \right) \sin\theta\cos\theta$$

*(ii)* Find a value of $\theta$ for which $\phi$ is a maximum

$$\sin\theta\cos\theta = \frac{1}{2}\sin 2\theta$$

which is a maximum when $2\theta = \dfrac{\pi}{2}$

$$\theta = \frac{\pi}{4}$$

as $\left(\dfrac{a^2 - b^2}{ab}\right)$ is a constant, then

$\tan\phi$ is a maximum when $\theta = \dfrac{\pi}{4}$

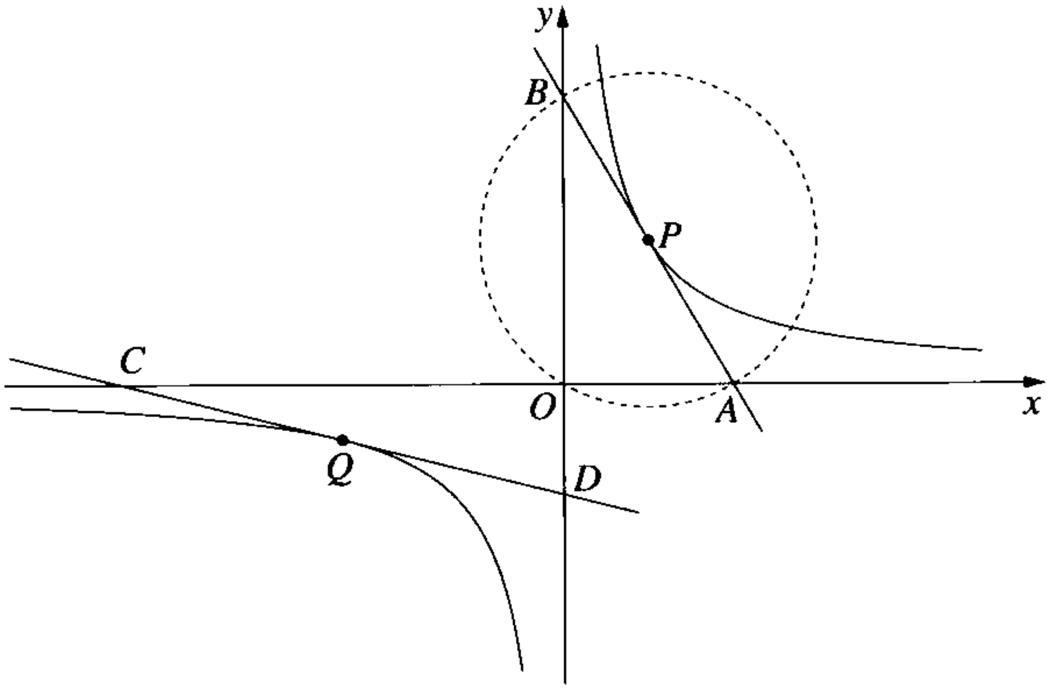
as $\tan\phi$ is an increasing function for $0 < \phi < \dfrac{\pi}{2}$

$\phi$ is a maximum when $\theta = \dfrac{\pi}{4}$

d) **2013 HSC Question 12d)**

The points $P\left(cp, \frac{c}{p}\right)$ and $Q\left(cq, \frac{c}{q}\right)$, where $|p| \neq |q|$, lie on the rectangular hyperbola $xy = c^2$

The tangent to the hyperbola at $P$ intersects the $x$-axis at $A$ and the $y$-axis at $B$. Similarly, the tangent to the hyperbola at $Q$ intersects the $x$-axis at $C$ and the $y$-axis at $D$.



($i$) Show that the equation of the tangent at $P$ is $x + p^2 y = 2cp$

$$y = \frac{c^2}{x}$$
$$\frac{dy}{dx} = \frac{-c^2}{x^2}$$

when $x = cp, \dfrac{dy}{dx} = -\dfrac{c^2}{(cp)^2}$
$$= \frac{-1}{p^2}$$

$$y - \frac{c}{p} = -\frac{1}{p^2}(x - cp)$$
$$p^2 y - cp = -x + cp$$
$$\underline{x + p^2 y = 2cp}$$

($ii$) Show that $A$, $B$ and $O$ are on a circle with centre $P$

$\triangle AOB$ is right angled with $AB$ as hypotenuse

$\therefore A, O, B$ are concyclic, with $AB$ diameter $\quad (\angle \text{ in a semicircle } = 90^\circ)$

centre of circle is the midpoint of $AB$

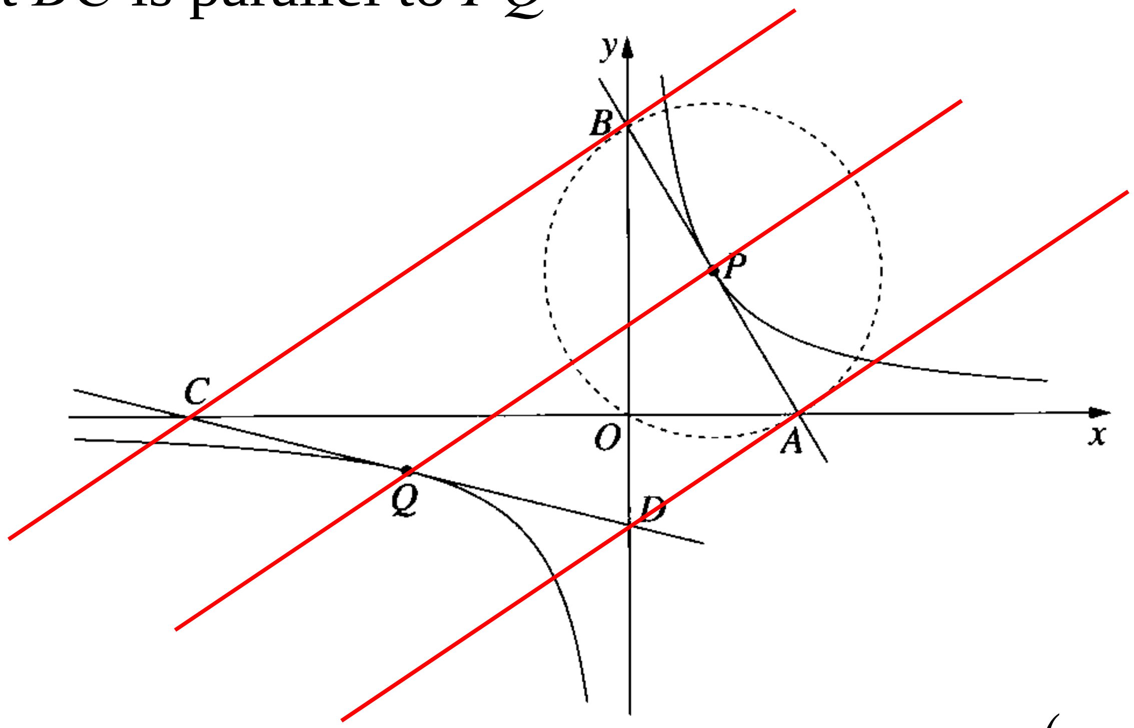
$A; y = 0$
$$x = 2cp$$

$B; x = 0$
$$p^2 y = 2cp$$
$$y = \frac{2c}{p}$$

$$\mathrm{M}_{AB} = \left( \frac{2cp + 0}{2}, \frac{0 + \frac{2c}{p}}{2} \right)$$
$$= \left( cp, \frac{c}{p} \right)$$
$$= P$$

$\underline{\therefore A, O, B \text{ lies on a circle, centre } P}$

*(iii)* Prove that $BC$ is parallel to $PQ$




$P$ is centre of a circle passing through $A, B, O$ (proven in (ii))

Similarly,

$Q$ is centre of a circle passing through $D, C, O$

$$\therefore AP = BP \text{ and } DQ = CQ \qquad (= \text{ radii})$$

$$\frac{AP}{BP} = \frac{DQ}{CQ} = 1$$

$$\therefore BC \parallel PQ \parallel AD \qquad (\text{ratio of intercepts of } \parallel \text{ lines are } =)$$