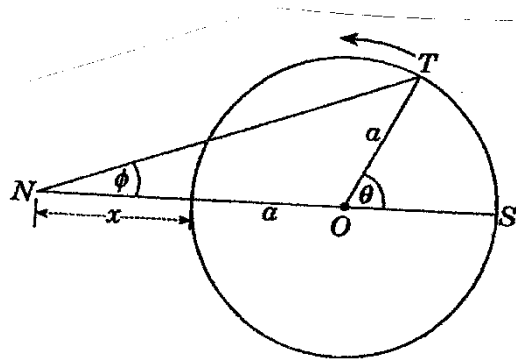


Miscellaneous Dynamics

Questions

e.g. (i) (1992) – *variable angular velocity*



The diagram shows a model train T that is moving around a circular track, centre O and radius a metres.

The train is travelling at a constant speed of u m/s. The point N is in the same plane as the track and is x metres from the nearest point on the track. The line NO produced meets the track at S .

Let $\angle TNS = \phi$ and $\angle TOS = \theta$ as in the diagram

a) Express $\frac{d\theta}{dt}$ in terms of a and u

$$l = a\theta$$
$$\frac{dl}{dt} = a \frac{d\theta}{dt}$$

$$u = a \frac{d\theta}{dt}$$

$$\underline{\frac{d\theta}{dt} = \frac{u}{a}}$$

b) Show that $a \sin(\theta - \phi) - (x + a) \sin \phi = 0$ and deduce that;

$$\frac{d\phi}{dt} = \frac{u \cos(\theta - \phi)}{(x + a) \cos \phi + a \cos(\theta - \phi)}$$

$$\angle NTO + \angle TNO = \angle TOS \quad (\text{exterior } \angle, \Delta OTN)$$

$$\angle NTO + \phi = \theta$$

$$\angle NTO = \theta - \phi$$

$$\text{In } \Delta NTO; \frac{a}{\sin \phi} = \frac{a + x}{\sin(\theta - \phi)}$$

$$a \sin(\theta - \phi) = (a + x) \sin \phi$$

$$\underline{a \sin(\theta - \phi) - (a + x) \sin \phi = 0}$$

differentiate with respect to t

$$a \cos(\theta - \phi) \left(\frac{d\theta}{dt} - \frac{d\phi}{dt} \right) - (a + x) \cos \phi \cdot \frac{d\phi}{dt} = 0$$

$$a \cos(\theta - \phi) \cdot \frac{d\theta}{dt} - [a \cos(\theta - \phi) + (a + x) \cos \phi] \cdot \frac{d\phi}{dt} = 0$$

$$[a \cos(\theta - \phi) + (a + x) \cos \phi] \cdot \frac{d\phi}{dt} = a \cos(\theta - \phi) \cdot \frac{u}{a}$$

$$\frac{d\phi}{dt} = \frac{u \cos(\theta - \phi)}{a \cos(\theta - \phi) + (a + x) \cos \phi}$$

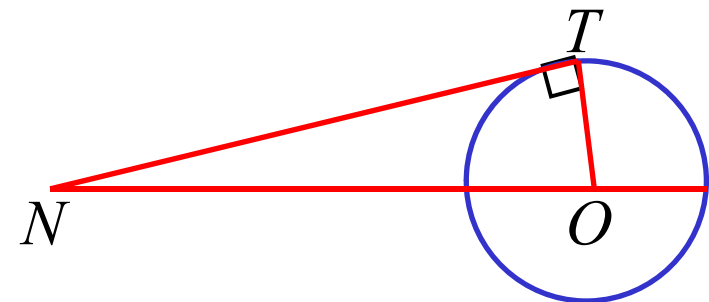
c) Show that $\frac{d\phi}{dt} = 0$ when NT is tangential to the track.

when NT is a tangent;

$$\angle NTO = 90^\circ \quad (\text{tangent} \perp \text{radius})$$

$$\therefore \theta - \phi = 90^\circ$$

$$\frac{d\phi}{dt} = \frac{u \cos 90^\circ}{a \cos 90^\circ + (a + x) \cos \phi}$$

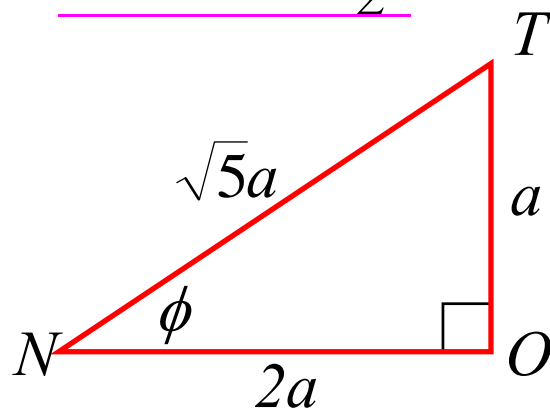


$$\therefore \frac{d\phi}{dt} = 0$$

d) Suppose that $x = a$

Show that the train's angular velocity about N when $\theta = \frac{\pi}{2}$ is $\frac{3}{5}$ times the angular velocity about N when $\theta = 0$

when $\theta = \frac{\pi}{2}$



$$\cos \phi = \frac{2}{\sqrt{5}}$$

$$\cos\left(\frac{\pi}{2} - \phi\right) = \frac{1}{\sqrt{5}}$$

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{u \cos\left(\frac{\pi}{2} - \phi\right)}{a \cos\left(\frac{\pi}{2} - \phi\right) + 2a \cos \phi} \\ &= \frac{u \left(\frac{1}{\sqrt{5}}\right)}{a \left(\frac{1}{\sqrt{5}}\right) + (2a) \left(\frac{2}{\sqrt{5}}\right)} \\ &= \frac{u}{5a} \end{aligned}$$

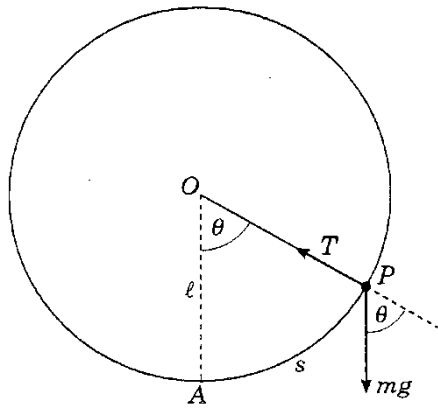
when $\theta = 0$

$$\begin{aligned} \frac{d\phi}{dt} &= \frac{u \cos 0}{a \cos 0 + 2a \cos 0} \\ &= \frac{u}{3a} \\ &= \frac{5}{3} \cdot \frac{u}{5a} \end{aligned}$$

Thus the angular velocity when $\theta = \frac{\pi}{2}$ is $\frac{3}{5}$ times

the angular velocity when $\theta = 0$

(ii) (2000)



A string of length l is initially vertical and has a mass P of m kg attached to it. The mass P is given a horizontal velocity of magnitude V and begins to move along the arc of a circle in a counterclockwise direction.

Let O be the centre of this circle and A the initial position of P . Let s denote the arc length AP , $v = \frac{ds}{dt}$,

$\theta = \angle AOP$ and let the tension in the string be T . The acceleration due to gravity is g and there are no frictional forces acting on P .

For parts a) to d), assume the mass is moving along the circle.

a) Show that the tangential acceleration of P is given by $\frac{d^2s}{dt^2} = \frac{1}{l} \frac{d}{d\theta} \left(\frac{1}{2} v^2 \right)$

$$s = l\theta$$

$$v = \frac{ds}{dt}$$

$$= l \frac{d\theta}{dt}$$

$$\frac{d^2s}{dt^2} = \frac{dv}{dt}$$

$$= \frac{dv}{d\theta} \cdot \frac{d\theta}{dt}$$

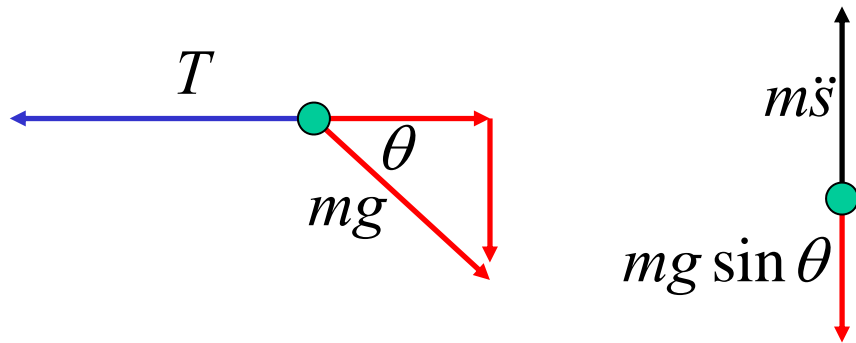
$$= \frac{dv}{d\theta} \cdot \frac{v}{l}$$

$$= \frac{1}{l} \cdot \frac{dv}{d\theta} \cdot v$$

$$= \frac{1}{l} \cdot \frac{dv}{d\theta} \cdot \frac{d}{dv} \left(\frac{1}{2} v^2 \right)$$

$$= \frac{1}{l} \frac{d}{d\theta} \left(\frac{1}{2} v^2 \right)$$

b) Show that the equation of motion of P is $\frac{1}{l} \frac{d}{d\theta} \left(\frac{1}{2} v^2 \right) = -g \sin \theta$



$$m\ddot{s} = -mg \sin \theta$$

$$\ddot{s} = -g \sin \theta$$

$$\frac{1}{l} \frac{d}{d\theta} \left(\frac{1}{2} v^2 \right) = -g \sin \theta$$

c) Deduce that $V^2 = v^2 + 2gl(1 - \cos \theta)$

$$\frac{1}{l} \frac{d}{d\theta} \left(\frac{1}{2} v^2 \right) = -g \sin \theta$$

$$\frac{d}{d\theta} \left(\frac{1}{2} v^2 \right) = -gl \sin \theta$$

$$\frac{1}{2} v^2 = gl \cos \theta + c$$

$$v^2 = 2gl \cos \theta + c$$

when $\theta = 0, v = V$

$$\therefore V^2 = 2gl + c$$

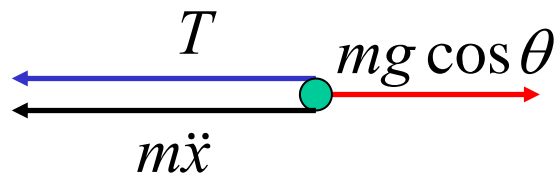
$$c = V^2 - 2gl$$

$$v^2 = 2gl \cos \theta + V^2 - 2gl$$

$$V^2 = v^2 + 2gl - 2gl \cos \theta$$

$$\underline{V^2 = v^2 + 2gl(1 - \cos \theta)}$$

d) Explain why $T - mg \cos \theta = \frac{1}{l}mv^2$



$$m\ddot{x} = T - mg \cos \theta$$

But, the resultant force towards the centre is centripetal force.

$$\frac{mv^2}{l} = T - mg \cos \theta$$

$$\underline{T - mg \cos \theta = \frac{1}{l}mv^2}$$

e) Suppose that $V^2 = 3gl$. Find the value of θ at which $T = 0$

$$T - mg \cos \theta = \frac{1}{l}m[V^2 - 2gl(1 - \cos \theta)]$$

$$0 - mg \cos \theta = \frac{1}{l}m[3gl - 2gl(1 - \cos \theta)]$$

$$-mg \cos \theta = m(g + 2g \cos \theta)$$

$$3mg \cos \theta = -mg$$

$$\cos \theta = -\frac{1}{3}$$

$$\underline{\theta = 1.91 \text{ radians}}$$

f) Consider the situation in part e). Briefly describe, in words, the path of P after the tension T becomes zero.

When $T = 0$, the particle would undergo projectile motion, i.e. it would follow a parabolic arc.

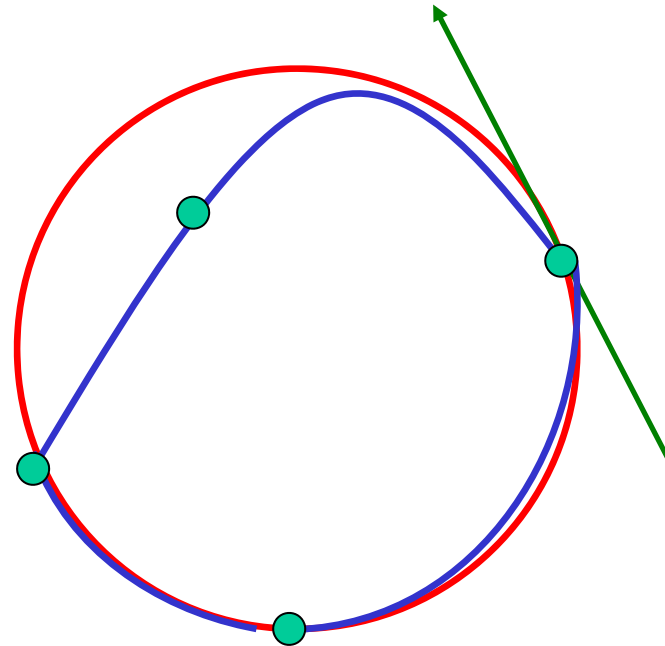
Its initial velocity would be tangential to the circle with magnitude;

$$T - mg \cos \theta = \frac{1}{l} mv^2$$

$$-mg \left(-\frac{1}{3} \right) = \frac{1}{l} mv^2$$

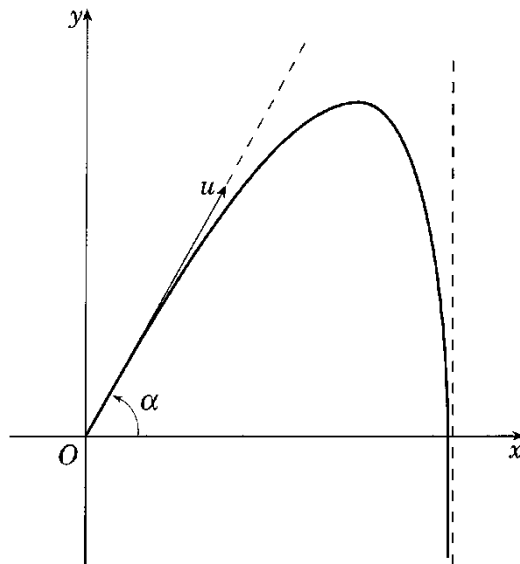
$$v^2 = \frac{gl}{3}$$

$$v = \sqrt{\frac{gl}{3}}$$



(iii) (2003)

A particle of mass m is thrown from the top, O , of a very tall building with an initial velocity u at an angle of α to the horizontal. The particle experiences the effect of gravity, and a resistance proportional to its velocity in both directions.



The equations of motion in the horizontal and vertical directions are given respectively by

$$\ddot{x} = -k\dot{x} \quad \text{and} \quad \ddot{y} = -k\dot{y} - g$$

where k is a constant and the acceleration due to gravity is g .

(You are NOT required to show these)

a) Derive the result $\dot{x} = ue^{-kt} \cos \alpha$

$$\frac{d\dot{x}}{dt} = -k\dot{x}$$

$$t = -\frac{1}{k} \int_{u \cos \alpha}^{\dot{x}} \frac{d\dot{x}}{\dot{x}}$$

$$t = -\frac{1}{k} [\log \dot{x}]_{u \cos \alpha}^{\dot{x}}$$

$$t = -\frac{1}{k} [\log \dot{x} - \log(u \cos \alpha)]$$

$$t = -\frac{1}{k} \log \left(\frac{\dot{x}}{u \cos \alpha} \right)$$

$$-kt = \log \left(\frac{\dot{x}}{u \cos \alpha} \right)$$

$$\frac{\dot{x}}{u \cos \alpha} = e^{-kt}$$

$$\underline{\dot{x} = ue^{-kt} \cos \alpha}$$

b) Verify that $\dot{y} = \frac{1}{k} [(ku \sin \alpha + g)e^{-kt} - g]$ satisfies the appropriate

equation of motion and initial condition

$$\frac{d\dot{y}}{dt} = -k\dot{y} - g$$

$$t = -\int_{u \sin \alpha}^{\dot{y}} \frac{d\dot{y}}{k\dot{y} + g}$$

$$t = -\frac{1}{k} [\log(k\dot{y} + g)]_{u \sin \alpha}^{\dot{y}}$$

$$-kt = \log(k\dot{y} + g) - \log(ku \sin \alpha + g)$$

$$-kt = \log \left(\frac{k\dot{y} + g}{ku \sin \alpha + g} \right)$$

$$\frac{k\dot{y} + g}{ku \sin \alpha + g} = e^{-kt}$$

$$\underline{\dot{y} = \frac{1}{k} [(ku \sin \alpha + g)e^{-kt} - g]}$$

c) Find the value of t when the particle reaches its maximum height

Maximum height occurs when $\dot{y} = 0$

$$t = -\frac{1}{k} [\log(k\dot{y} + g)]_{u \sin \alpha}^0$$

$$t = -\frac{1}{k} [\log(g) - \log(ku \sin \alpha + g)]$$

$$t = \frac{1}{k} \log\left(\frac{ku \sin \alpha + g}{g}\right)$$

d) What is the limiting value of the horizontal displacement of the particle?

$$\dot{x} = ue^{-kt} \cos \alpha$$

$$\frac{dx}{dt} = ue^{-kt} \cos \alpha$$

$$x = \lim_{t \rightarrow \infty} u \cos \alpha \int_0^{\infty} e^{-kt} dt$$

$$x = \lim_{t \rightarrow \infty} u \cos \alpha \left[-\frac{1}{k} e^{-kt} \right]_0^t$$

$$x = \lim_{t \rightarrow \infty} \frac{u \cos \alpha}{k} (-e^{-kt} + 1)$$

$$x = \frac{u \cos \alpha}{k}$$

“Cambridge” Exercise 9D; 1, 2, 4, 6

Patel Exercise 9E; 1 to 4, 7

Patel Exercise 9F; 1, 2, 4, 7, 9, 12, 14, 16, 20, 22, 25