Calculating Coefficients without Pascal's Triangle

$$^nC_k = \frac{n!}{k!(n-k)!}$$

Proof

Step 1: Prove true for $k = 0$

$$LHS = {}^nC_0$$
$$= 1$$

$$RHS = \frac{n!}{0!n!}$$
$$= \frac{n!}{n!}$$
$$= 1$$

$$\therefore LHS = RHS$$

Hence the result is true for $k = 0$

Step 2 : Assume the result is true for $k = r$ where r is an integer ≥ 0

$$i.e. {}^{n}C_{r} = \frac{n!}{r!(n-r)!}$$

Step 3 : Prove the result is true for $k = r+1$

$$i.e. {}^{n}C_{r+1} = \frac{n!}{(r+1)!(n-r-1)!}$$

Proof

$${}^{n+1}C_{r+1} = {}^{n}C_{r} + {}^{n}C_{r+1}$$

$${}^{n}C_{r+1} = {}^{n+1}C_{r+1} - {}^{n}C_{r}$$

$$= \frac{(n+1)!}{(r+1)!(n-r)!} - \frac{n!}{r!(n-r!)}$$

$$= \frac{(n+1)! - n!(r+1)}{(r+1)!(n-r)!}$$

$$= \frac{n!(n+1-r-1)}{(r+1)!(n-r)!}$$

$$= \frac{n!(n+1-r-1)}{(r+1)!(n-r)!}$$

$$= \frac{n!(n-r)}{(r+1)!(n-r)!}$$

$$= \frac{n!}{(r+1)!(n-r-1)!}$$

Hence the result is true for $k = r + 1$ if it is also true for $k = r$

Step 4: Hence the result is true for all positive integral values of n by induction

The Binomial Theorem

$$(1+x)^n = {}^nC_0 + {}^nC_1 x + {}^nC_2 x^2 + \ldots + {}^nC_k x^k + \ldots + {}^nC_n x^n$$

$$= \sum_{k=0}^{n} {}^nC_k x^k$$

where ${}^nC_k = \dfrac{n!}{k!(n-k)!}$ and n is a positive integer

NOTE: there are (n + 1) terms

This extends to;

$$(a+b)^n = \sum_{k=0}^{n} {}^nC_k a^{n-k} b^k$$

$e.g.$ Evaluate ${}^{11}C_4$

$${}^{11}C_4 = \frac{11!}{4!7!}$$

$$= \frac{11 \times 10 \times \overset{3}{\cancel{9}} \times \cancel{8}}{\cancel{4} \times \cancel{3} \times \cancel{2} \times 1}$$

$$= 330$$

(ii) Find the value of n so that;

a) ${}^{n}C_{5} = {}^{n}C_{8}$

$${}^{n}C_{k} = {}^{n}C_{n-k}$$

$$\therefore 8 = n - 5$$

$$\underline{n = 13}$$

b) ${}^{n}C_{7} + {}^{n}C_{8} = {}^{20}C_{8}$

$${}^{n-1}C_{k} + {}^{n-1}C_{k-1} = {}^{n}C_{k}$$

$$\therefore \underline{n = 19}$$

(iii) Find the 5th term in the expansion of $\left(5a - \frac{3}{b}\right)^{11}$

$$T_{k+1} = {}^{11}C_{k}(5a)^{11-k}\left(-\frac{3}{b}\right)^{k}$$

$$T_{5} = {}^{11}C_{4}(5a)^{7}\left(-\frac{3}{b}\right)^{4}$$

$$= \frac{{}^{11}C_{4}5^{7}3^{4}a^{7}}{b^{4}}$$ ← *unsimplified*

$$= \frac{330 \times 78125 \times 81a^{7}}{b^{4}}$$

$$= \frac{2088281250a^{7}}{b^{4}}$$

(iv) Obtain the term independent of x in $\left(3x^2 - \frac{1}{2x}\right)^9$

$$T_{k+1} = {}^9C_k \left(3x^2\right)^{9-k} \left(-\frac{1}{2x}\right)^k$$

term independent of x means term with x^0

$$\left(x^2\right)^{9-k}\left(x^{-1}\right)^k = x^0$$

$$x^{18-2k} \times x^{-k} = x^0$$

$$18 - 3k = 0$$

$$k = 6$$

$$T_7 = {}^9C_6 \left(3x^2\right)^3 \left(-\frac{1}{2x}\right)^6$$

$$= \frac{{}^9C_6 3^3}{2^6}$$

$$= \frac{{}^9C_6 3^3}{2^6}$$

Exercise 5D; 2, 3, 5, 7, 9, 10ac, 12ac, 13, 14, 15ac, 19, 25