

Multiple Roots

If $P(x)$, has a root, $x = a$, of multiplicity m ,
then $P'(x)$ has a root, $x = a$, of multiplicity $m - 1$

Proof:

$$P(x) = (x - a)^m Q(x) \quad (m > 1, x = a \text{ is not a root of } Q(x))$$

$$P'(x) = (x - a)^m Q'(x) + Q(x)m(x - a)^{m-1} (1)$$

$$= (x - a)^{m-1} [(x - a)Q'(x) + mQ(x)]$$

$$= (x - a)^{m-1} R(x) \quad (\text{where } x = a \text{ is not a root of } R(x))$$

$\therefore P'(x)$ has a root, $x = a$, of multiplicity $m - 1$

e.g. (i) Solve the equation $x^3 - 4x^2 - 3x + 18 = 0$, given that it has a double root

$$P(x) = x^3 - 4x^2 - 3x + 18$$

$$P'(x) = 3x^2 - 8x - 3$$
$$= (3x + 1)(x - 3)$$

\therefore double root is $x = -\frac{1}{3}$ or $x = 3$

NOT LIKELY

As $(3x + 1)$ is not a factor

$$x^3 - 4x^2 - 3x + 18 = 0$$

$$(x - 3)^2(x + 2) = 0$$

$$\underline{x = -2 \text{ or } x = 3}$$

(ii) (1991 Extension 2 HSC)

Let $x = \alpha$ be a root of the quartic polynomial;

$$P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1$$

where $(2 + B)^2 \neq 4A^2$

a) show that α cannot be 0, 1 or -1

$$P(0) = 1 \neq 0, \quad \therefore \underline{\alpha \neq 0}$$

$$\begin{aligned} P(1) &= 1 + A + B + A + 1 \\ &= 2A + B + 2 \end{aligned}$$

$$\begin{aligned} P(-1) &= 1 - A + B - A + 1 \\ &= -2A + B + 2 \end{aligned}$$

BUT

$$(2 + B)^2 \neq 4A^2$$

$$2 + B \neq \pm 2A$$

$$\pm 2A + B + 2 \neq 0$$

$$\therefore P(1) \neq 0, P(-1) \neq 0$$

hence $\alpha \neq \pm 1$

b) Show that $\frac{1}{\alpha}$ is a root

$$\begin{aligned} P\left(\frac{1}{\alpha}\right) &= \frac{1}{\alpha^4} + \frac{A}{\alpha^3} + \frac{B}{\alpha^2} + \frac{A}{\alpha} + 1 \\ &= \frac{1 + A\alpha + B\alpha^2 + A\alpha^3 + \alpha^4}{\alpha^4} \\ &= \frac{P(\alpha)}{\alpha^4} \\ &= \frac{0}{\alpha^4} \quad (\because P(\alpha) = 0 \text{ as } \alpha \text{ is a root}) \\ &= 0 \end{aligned}$$

$\therefore \frac{1}{\alpha}$ is a root of $P(x)$

**This result always holds
when the coefficients are
palindromic**

c) Deduce that if α is a multiple root, then its multiplicity is 2 and

$$4B = 8 + A^2$$

If α is a double root of $P(x)$, then so is $\frac{1}{\alpha}$, which accounts for 4 roots

However $P(x)$ is a quartic which has a maximum of 4 roots

Thus no roots can have a multiplicity > 2

$$P'(x) = 4x^3 + 3Ax^2 + 2Bx + A$$

let the roots be $\alpha, \frac{1}{\alpha}$ and β

$$\alpha + \frac{1}{\alpha} + \beta = -\frac{3}{4}A$$

(sum of roots)...(1)

$$1 + \alpha\beta + \frac{\beta}{\alpha} = \frac{1}{2}B$$

($\sum \alpha\beta$)...(2)

$$\beta = -\frac{1}{4}A$$

($\sum \alpha\beta\gamma$)...(3)

Substitute (3) into (1)

$$\alpha + \frac{1}{\alpha} - \frac{1}{4}A = -\frac{3}{4}A$$

$$\alpha + \frac{1}{\alpha} = -\frac{1}{2}A$$

Substitute (3) into (2)

$$1 - \frac{1}{4}A\alpha - \frac{1}{4}A\frac{1}{\alpha} = \frac{1}{2}B$$

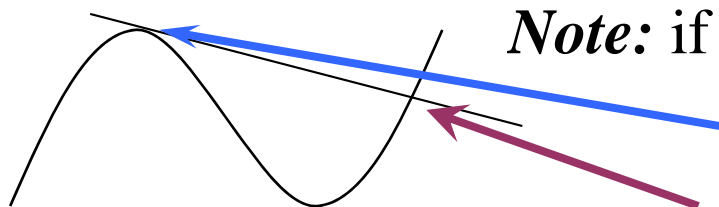
$$1 - \frac{1}{4}A\left(\alpha + \frac{1}{\alpha}\right) = \frac{1}{2}B$$

$$1 + \frac{1}{8}A^2 = \frac{1}{2}B$$

$$8 + A^2 = 4B$$

Exercise 10G; 1, 2, 5, 6, 7, 8, 11 to 16

Exercise 10H; 1 to 10



Note: if a tangent to a cubic has two solutions only.

A double root

and a single root