# *Multiple Roots*

If $P(x)$, has a root, $x = a$, of multiplicity $m$,

then $P'(x)$ has a root, $x = a$, of multiplicity $m - 1$

***Proof***:

$$P(x) = (x-a)^m Q(x) \qquad (m > 1,\ x = a \text{ is not a root of } Q(x))$$

$$P'(x) = (x-a)^m Q'(x) + Q(x)m(x-a)^{m-1}(1)$$

$$= (x-a)^{m-1}\left[(x-a)Q'(x) + mQ(x)\right]$$

$$= (x-a)^{m-1} R(x) \qquad (\text{where } x = a \text{ is not a root of } R(x))$$

$\therefore$ <u>$P'(x)$ has a root, $x = a$, of multiplicity $m - 1$</u>

e.g. $(i)$ Solve the equation $x^3 - 4x^2 - 3x + 18 = 0$ , given that it has a double root

$$P(x) = x^3 - 4x^2 - 3x + 18$$

$$P'(x) = 3x^2 - 8x - 3$$

$$= (3x+1)(x-3)$$

$\because$ double root is $x = -\frac{1}{3}$ or $x = 3$

**NOT LIKELY**

As (3*x* + 1) is not a factor

$$x^3 - 4x^2 - 3x + 18 = 0$$

$$(x-3)^2(x+2) = 0$$

$$\underline{x = -2 \text{ or } x = 3}$$

($ii$) (1991 Extension 2 HSC)

Let $x = \alpha$ be a root of the quartic polynomial;

$$P(x) = x^4 + Ax^3 + Bx^2 + Ax + 1$$

where $(2+B)^2 \neq 4A^2$

a) show that $\alpha$ cannot be 0, 1 or -1

$P(0) = 1 \neq 0, \quad \therefore \alpha \neq 0$

$P(1) = 1 + A + B + A + 1$

$= 2A + B + 2$

$P(-1) = 1 - A + B - A + 1$

$= -2A + B + 2$

**BUT**

$(2+B)^2 \neq 4A^2$

$2 + B \neq \pm 2A$

$\pm 2A + B + 2 \neq 0$

$\therefore P(1) \neq 0, P(-1) \neq 0$

hence $\alpha \neq \pm 1$

b) Show that $\frac{1}{\alpha}$ is a root

$$P\left(\frac{1}{\alpha}\right) = \frac{1}{\alpha^4} + \frac{A}{\alpha^3} + \frac{B}{\alpha^2} + \frac{A}{\alpha} + 1$$

$$= \frac{1 + A\alpha + B\alpha^2 + A\alpha^3 + \alpha^4}{\alpha^4}$$

$$= \frac{P(\alpha)}{\alpha^4}$$

$$= \frac{0}{\alpha^4} \qquad (\because P(\alpha) = 0 \text{ as } \alpha \text{ is a root})$$

$$= 0$$

$\therefore \frac{1}{\alpha}$ is a root of $P(x)$

**This result always holds when the coefficients are palandromic**

c) Deduce that if $\alpha$ is a multiple root, then its multiplicity is 2 and

$4B = 8 + A^2$

If $\alpha$ is a double root of $P(x)$, then so is $\frac{1}{\alpha}$, which accounts for 4 roots

However $P(x)$ is a quartic which has a maximum of 4 roots

Thus no roots can have a multiplicity > 2

$$P'(x) = 4x^3 + 3Ax^2 + 2Bx + A$$

let the roots be $\alpha, \frac{1}{\alpha}$ and $\beta$

$$\alpha + \frac{1}{\alpha} + \beta = -\frac{3}{4}A \qquad \text{(sum of roots)} \ldots (1)$$

$$1 + \alpha\beta + \frac{\beta}{\alpha} = \frac{1}{2}B \qquad \left(\sum \alpha\beta\right) \ldots (2)$$

$$\beta = -\frac{1}{4}A \qquad \left(\sum \alpha\beta\gamma\right) \ldots (3)$$

Substitute (3) into (1)

$$\alpha + \frac{1}{\alpha} - \frac{1}{4}A = -\frac{3}{4}A$$

$$\alpha + \frac{1}{\alpha} = -\frac{1}{2}A$$

Substitute (3) into (2)

$$1 - \frac{1}{4}A\alpha - \frac{1}{4}A\frac{1}{\alpha} = \frac{1}{2}B$$

$$1 - \frac{1}{4}A\left(\alpha + \frac{1}{\alpha}\right) = \frac{1}{2}B$$

$$1 + \frac{1}{8}A^2 = \frac{1}{2}B$$

$$8 + A^2 = 4B$$

**Exercise 10G; 1, 2, 5, 6, 7, 8, 11 to 16**

**Exercise 10H; 1 to 10**

***Note:*** if a tangent to a cubic has two solutions only.