NORTH SYDNEY GIRLS HIGH SCHOOL

2018 | HSC Trial Examination

Mathematics Extension 2

General Instructions

- Reading Time – 5 minutes
- Working Time – 3 hours
- Write using black pen
- NESA approved calculators may be used
- A reference sheet is provided
- In Questions 11 – 16, show relevant mathematical reasoning and/or calculations

Total marks – 100

Section I – 10 marks (pages 2 – 6)

- Attempt Questions 1 – 10
- Allow about 15 minutes for this section

Section II – 90 marks (pages 7 – 15)

- Attempt Questions 11 – 16
- Allow about 2 hours 45 minutes for this section

NAME:_______________________________ **TEACHER:**____________________________

STUDENT NUMBER: | | | | | | | | |

Question	1-10	11	12	13	14	15	16	Total
Mark	/10	/15	/15	/15	/15	/15	/15	/100

Section I

10 marks
Attempt Questions 1–10
Allow about 15 minutes for this section
Use the multiple-choice answer sheet for Questions 1–10.

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1 What is the value of $(1+i)^{12}$?

A. 64

B. -64

C. $64i$

D. $-64i$

2 If $|z-3+4i|=5$, what is the maximum value of $|z|$?

A. $\sqrt{5}$

B. 5

C. $\sqrt{10}$

D. 10

3 Which of the following is the solution to $\displaystyle\int \frac{dx}{(x-1)(x+2)}$?

A. $\dfrac{1}{3}\ln\left|\dfrac{x-1}{x+2}\right| + c$

B. $\dfrac{1}{3}\ln\left|(x-1)(x+2)\right| + c$

C. $\dfrac{1}{3}\ln\left|\dfrac{x+2}{x-1}\right| + c$

D. $3\ln\left|(x-1)(x+2)\right| + c$

4 For what values of k does the equation $|z+1|+|z-1|=k$ describe an ellipse?

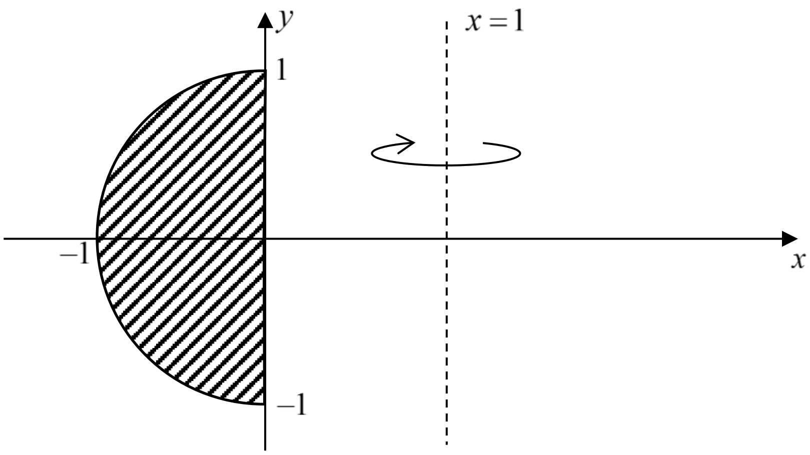
A. $0 < k < 2$

B. $0 < k \leq 2$

C. $k > 2$

D. $k \geq 2$

5 The region bounded by the circle $x^2+y^2=1$ for $-1 \leq x \leq 0$ is rotated about the line $x=1$.



By using the method of cylindrical shells, which integral gives the volume of the solid formed?

A. $2\pi \int_{-1}^{0} (1+x)\sqrt{1-x^2}\, dx$

B. $2\pi \int_{-1}^{0} (1-x)\sqrt{1-x^2}\, dx$

C. $4\pi \int_{-1}^{0} (1+x)\sqrt{1-x^2}\, dx$

D. $4\pi \int_{-1}^{0} (1-x)\sqrt{1-x^2}\, dx$

6 Given that the ellipse $\frac{x^2}{a^2}+\frac{y^2}{b^2}=1$ $\left(\text{where } a>b\right)$ has eccentricity e, what is the eccentricity of the hyperbola $\frac{x^2}{a^2}-\frac{y^2}{b^2}=1$?

A. $\sqrt{1-e^2}$

B. $\sqrt{2-e^2}$

C. $\sqrt{2}-e$

D. $2-e$

7 If $I=\int_0^{\ln 2}\frac{e^x\,dx}{e^x+e^{-x}}$ and $J=\int_0^{\ln 2}\frac{e^{-x}\,dx}{e^x+e^{-x}}$, what is the value of $I-J$?

A. $\ln\frac{5}{4}$

B. $\ln\frac{3}{2}$

C. $\ln 2$

D. $\ln 5$

8 What is the value of $\sum_{n=1}^{99}\left(\sqrt{n}-\sqrt{n+2}\right)$?

A. $1-\sqrt{101}$

B. $\sqrt{99}-\sqrt{101}$

C. $\sqrt{2}-\sqrt{101}-9$

D. $\sqrt{2}-\sqrt{99}-9$

9 The graph of the curve $y=\frac{x^3+16}{x}$ has a stationary point at $(2, 12)$.

Let $P(x)=x^3-bx+16$.

For what values of b will the equation $P(x)=0$ have three distinct roots?

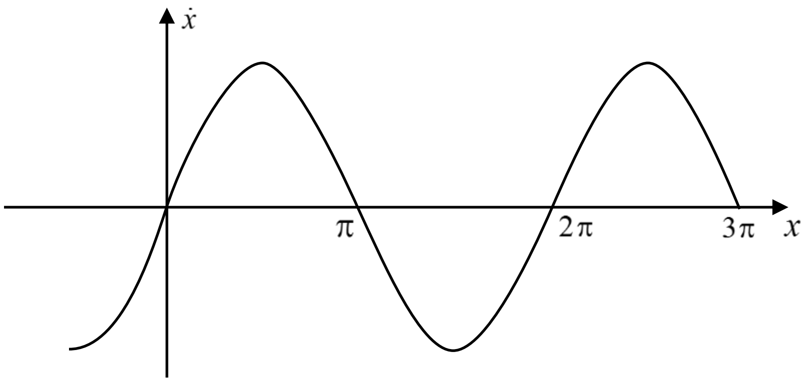
A. $b<0$

B. $0<b<2$

C. $2<b<12$

D. $b>12$

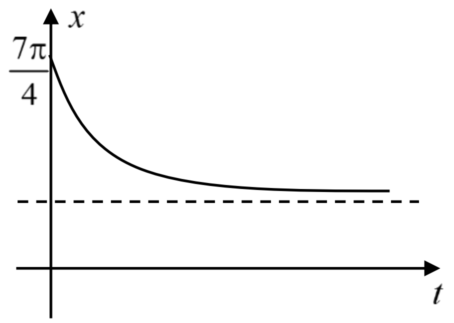
10 The velocity of a particle moving along the x-axis is given by $\dot{x} = \sin x$.



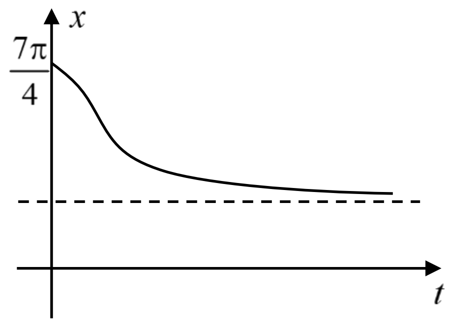
At $t = 0$, the particle is located at $x = \frac{7\pi}{4}$.

Which of the following graphs best illustrates the motion of the particle for $t \geq 0$?
(Graphs not drawn to scale.)

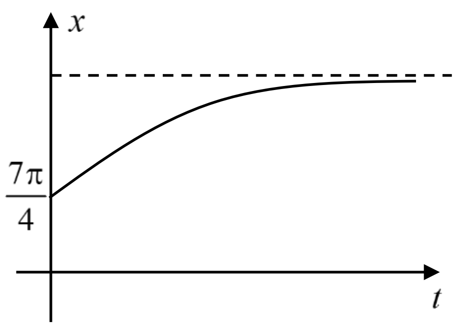
A.



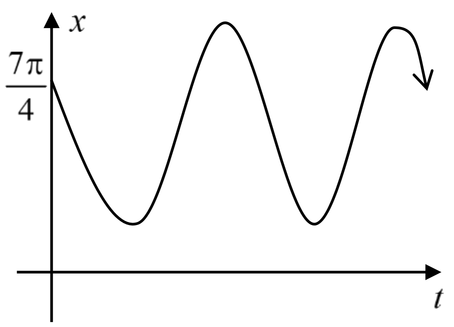
B.



C.



D.



Section II

Total marks − 90
Attempt Questions 11−16
Allow about 2 hour 45 minutes for this section.
Answer each question in a SEPARATE writing booklet. Extra writing booklets are available.

In Questions 11 to 16, your responses should include relevant mathematical reasoning and/or calculations.

Question 11 (15 marks) Use a SEPARATE writing booklet.

(a) Find

(i) $\int \cos^3 x\,dx$ **2**

(ii) $\int x\sec^2 x\,dx$ **2**

(b) Evaluate $\displaystyle\int_2^5 \frac{2dx}{x^2-4x+13}$ **3**

(c) Let $\alpha = 2-3i$. Find, in the form $a+ib$, the value of

(i) $\alpha - 2\overline{\alpha}$ **1**

(ii) $\dfrac{1}{\alpha+2}$ **2**

(d) On an Argand diagram, sketch the set of all complex numbers which satisfy **2**

$$|z| \le 1 \text{ and } \left|\arg(z-1)\right| \le \frac{3\pi}{4}$$

(e) (i) If $P(x)$ and $Q(x)$ are distinct polynomials which share a factor $(x-a)$, show that $R(x) = P(x)-Q(x)$ will have the same factor. **1**

(ii) Hence or otherwise find the two zeros that $P(x) = 6x^3+7x^2-x-2$ and $Q(x) = 6x^3-5x^2-3x+2$ have in common. **2**

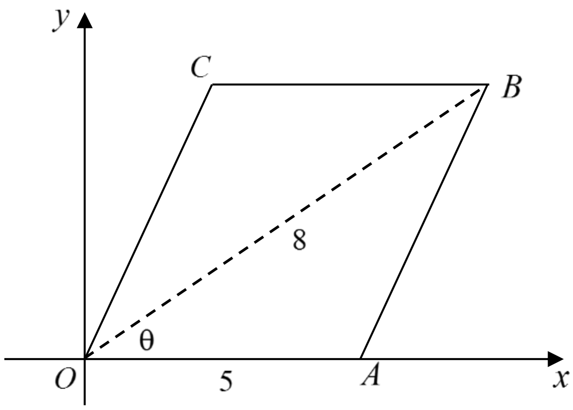
Question 12 (15 marks) Use a SEPARATE writing booklet.

(a) The polynomial $P(x) = x^3 - x - 6$ has zeros α, β and γ.

(i) Find the polynomial equation with roots $\alpha^2, \beta^2, \gamma^2$. **2**

(ii) Find the value of $(\alpha^2+1)(\beta^2+1)(\gamma^2+1)$ **2**

(b) The diagram shows a rhombus $OABC$ in the first quadrant of the Argand diagram.



Let $z = \cos\theta + i\sin\theta$, where $\theta = \angle AOB$.

(i) Explain why OC represents the complex number $5z^2$. **2**

(ii) Show that $5z^2 = 8z - 5$. **1**

(iii) Hence, or otherwise, find the complex numbers represented by B and C. **3**

(c) Sketch graphs of the following functions for $-\pi \le x \le \pi$:

(i) $y = \sin|x|$ **1**

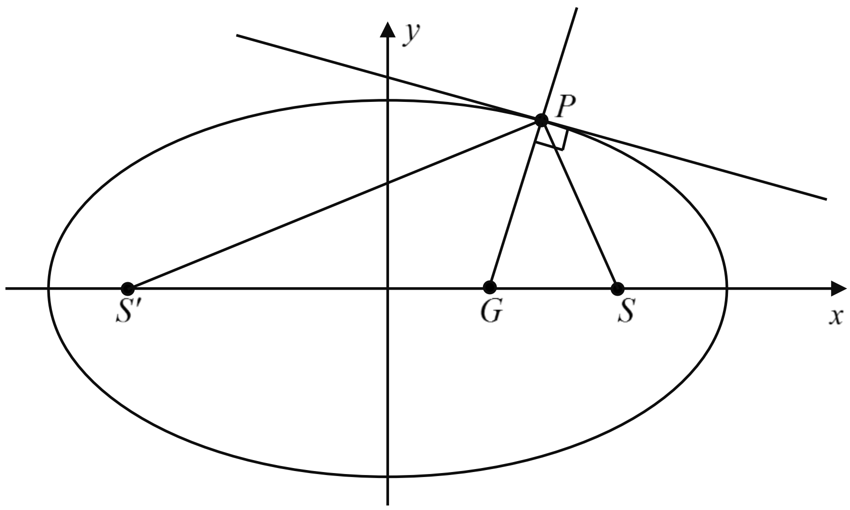
(ii) $y = \sin x \cdot \sin 2x$ (Do not attempt to find coordinates of stationary points) **2**

(iii) $y = e^{\sin x}$ **2**

Question 13 (15 marks) Use a SEPARATE writing booklet.

(a) Sketch the hyperbola $5x^2 - 4y^2 = 20$, showing the coordinates of the vertices and foci, and the equations of the directrices and asymptotes. **2**

(b) The ellipse $\frac{x^2}{a^2} + \frac{y^2}{b^2} = 1$ has foci $S(ae, 0)$ and $S'(-ae, 0)$, and directrices $x = \pm\frac{a}{e}$. $P(a\cos\theta, b\sin\theta)$ is a point on the ellipse, with the normal at P meeting the x-axis at G.



(i) Use the focus-directrix definition of an ellipse to show that

$$\frac{PS}{PS'} = \frac{1 - e\cos\theta}{1 + e\cos\theta}.$$ **2**

(ii) The equation of the normal at P is given by $\frac{ax}{\cos\theta} - \frac{by}{\sin\theta} = a^2 - b^2$. **3**

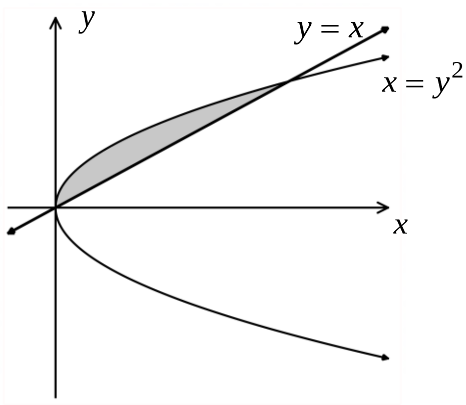
(Do NOT show this)

Show that $\frac{GS}{GS'} = \frac{PS}{PS'}$.

Question 13 continues on page 10

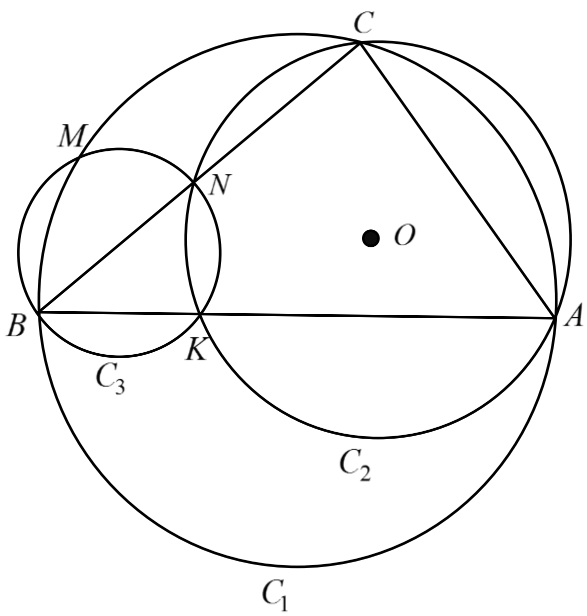
Question 13 (continued)

(c) The region bounded by $x = y^2$ and $y = x$ is to be rotated about the x-axis to form a solid. **3**



Use the method of cylindrical shells to find the volume of this solid.

(d) A triangle ABC has its vertices on the circle C_1. Another circle C_2 has its centre O lying inside ΔABC. This circle passes through A and C, and cuts AB and BC at K and N respectively.
A third circle C_3, passing through B, K and N, cuts circle C_1 at M.



Let $\angle BMK = \alpha$.

(i) Show that $\angle KAC = \alpha$. **2**

(ii) State why $\angle BMC = 180° - \alpha$. **1**

(iii) Show that $MKOC$ is a cyclic quadrilateral. **2**

End of Question 13

Question 14 (15 marks) Use a SEPARATE writing booklet

(a) Consider the polynomial $P(z) = z^3 + \beta z^2 + k$.

The equation $P(z) = 0$ has two non-real roots ω and $\overline{\omega}$, and a real root β.

Let $\omega = a + ib$.

(i) Show that $\beta = -a$. **1**

(ii) Show that $|\omega|^2 = 2\beta^2$. **2**

(iii) Show that ω and $\overline{\omega}$ lie on the lines $y = \pm x$. **1**

(iv) The three roots are represented as points A, B, C in the Argand plane. **2**

Find the area of ΔABC, expressed in terms of ω only.

(b) (i) Evaluate $\int_1^e (1 - \ln x)\, dx$. **2**

(ii) You are given that $I_n = \int_1^e (1 - \ln x)^n\, dx$ for $n = 1, 2, 3, \ldots$ **2**

Show that $I_n = -1 + nI_{n-1}$ for $n = 2, 3, 4, \ldots$

(iii) Hence find the value of I_3. **1**

(c) (i) Given that x is a positive real number and that n is a positive integer, **1**

show that $\dfrac{1}{1 + x^n} < 1$.

(ii) Let $I_n = \int_0^1 \dfrac{dx}{1 + x^n}$, where $n = 2, 3, 4, \ldots$ **3**

Show that $\dfrac{\pi}{4} \le I_n < 1$.

Question 15 (15 marks) Use a SEPARATE writing booklet.

(a) (i) Prove that, if $c \geq a$ and $d \geq b$, then $ab + cd \geq bc + ad$. **1**

(ii) Use part (i) to show that, if $x \geq y$, then $x^2 + y^2 \geq 2xy$. **1**

(iii) Without the use of further algebra, explain why the result of part (ii) is in fact true for all values of x and y. **1**

(b) Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos x \, dx}{1 + \sin^2 x}$.

(i) Evaluate I. **2**

(ii) Use the substitution $t = \tan \frac{x}{2}$ to show that $I = 2\int_0^1 \frac{1 - t^2}{1 + 6t^2 + t^4} dt$. **2**

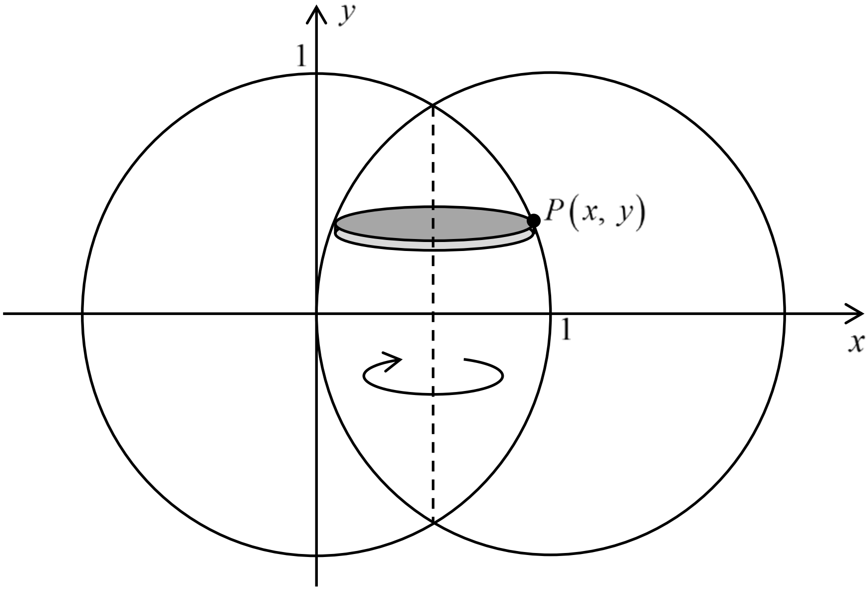
(iii) By considering $J = \int_0^{\frac{\pi}{2}} \frac{\cos x \, dx}{1 + k \sin^2 x}$ for an appropriate value of k, **3**

evaluate $\int_0^1 \frac{1 - t^2}{1 + 14t^2 + t^4} dt$.

Question 15 continues on page 13

(c) The diagram shows two circles of radius 1, centred at $(0, 0)$ and $(1, 0)$.

The region common to the two circles is to be rotated about its axis of symmetry to create a solid.



A typical slice of width δy is shown, intersecting the first circle at $P(x, y)$.

(i) Show that the volume of the slice is $\pi\left(\frac{5}{4} - y^2 - \sqrt{1-y^2}\right)\delta y$. **2**

(ii) Hence find the volume of the solid. **3**

End of Question 15

Question 16 (15 marks) Use a SEPARATE writing booklet.

(a) You are given that $(x+1)$ is a factor of $P(x) = 4x^3 + 2x^2 - 3x - 1$.

(i) Solve $P(x) = 0$. **1**

(ii) It can be shown that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. (Do NOT prove this.) **3**
Use this result and the result of part (i) to show that

$$\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}.$$

(b) A series of functions $f_0(x)$, $f_1(x)$, $f_2(x)$, ... is defined by

$$f_n(x) = \begin{cases} \dfrac{1}{1-x} & \text{if} \quad n = 0 \\ f_0\left[\, f_{n-1}(x)\right] & \text{if} \quad n = 1, 2, 3, \ldots \end{cases}$$

(i) Show that $f_3(x) = f_0(x)$. **2**

(ii) Hence explain why $f_{n+3}(x) = f_n(x)$. **1**

(iii) Hence evaluate $f_{100}(100)$. **1**

(c) A sequence a_k is defined by the following rules: **3**

$a_1 = 1$.

a_{k+1} is equal to one more than the product of all terms that precede it.

$\left(\text{That is } a_{k+1} = 1 + a_k \cdot a_{k-1} \cdot \ldots \cdot a_3 \cdot a_2 \cdot a_1\right)$

Prove by mathematical induction that

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n} = 2 - \frac{1}{a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n}$$

Question 16 continues on page 15

Question 16 (continued)

(d) Given that x is an angle in the first quadrant:

(i) Show that $\sqrt{\cot x} - \sqrt{\tan x} = \dfrac{\cos x - \sin x}{\sqrt{\sin x \cos x}}$. **1**

(ii) By using the substitution $\sec\theta = \sin x + \cos x$, or otherwise, find

$$\int_0^{\frac{\pi}{4}} \left(\sqrt{\cot x} - \sqrt{\tan x}\right) dx.$$ **3**

End of paper

Extension 2 2018 Trial HSC Solutions

Multiple Choice

Summary of Answers:

1.	B	**2.**	D	**3.**	A	**4.**	C	**5.**	D
6.	B	**7.**	A	**8.**	C	**9.**	D	**10.**	B

Solutions

1. $$\begin{aligned}(1+i)^{12} &= \left(\sqrt{2}\,\text{cis}\frac{\pi}{4}\right)^{12}\\ &= 2^6\,\text{cis}\,3\pi\\ &= -64\end{aligned}$$

2. The equation describes a circle of radius 5 units, centred on $(3,-4)$. and passing through the origin.
The complex number of maximum modulus is the point on this locus which is furthest from the origin.
That is, the point diametrically opposite the origin, which is 10 units from the origin.

3. $$\begin{aligned}\frac{1}{(x-1)(x+2)} &= \frac{A}{x-1}+\frac{B}{x+2}\\ 1 &= A(x+2)+B(x-1)\end{aligned}$$

$$(x=1)\quad 1=3A \quad\Rightarrow\quad A=\frac{1}{3}$$

$$(x=-2)\quad 1=-3B \quad\Rightarrow\quad B=-\frac{1}{3}$$

$$\begin{aligned}\int\frac{dx}{(x-1)(x+2)} &= \frac{1}{3}\int\left(\frac{1}{x-1}-\frac{1}{x+2}\right)dx\\ &= \frac{1}{3}\Big[\ln|x-1|-\ln|x+2|\Big]+c\\ &= \frac{1}{3}\ln\left|\frac{x-1}{x+2}\right|+c\end{aligned}$$

4. Equation is of the form $|z+s|+|z-s| = 2a$ where s is the focal length and a is the semi-major axis.
For an ellipse, $s<a$. That is $1<\frac{k}{2} \quad\Rightarrow\quad k>2$.

5. Let $P(x,y)$ be a point on the boundary of the circle, where $-1<x<0$.
Radius of shell $=1-x$.
Height of shell $=2y = 2\sqrt{1-x^2}$.
Volume of shell $=2\pi rh\delta r = 2\pi(1-x)\cdot 2\sqrt{1-x^2}\cdot\delta x$

$$\therefore\ V = 4\pi\int_{-1}^{0}(1-x)\sqrt{1-x^2}\,dx$$

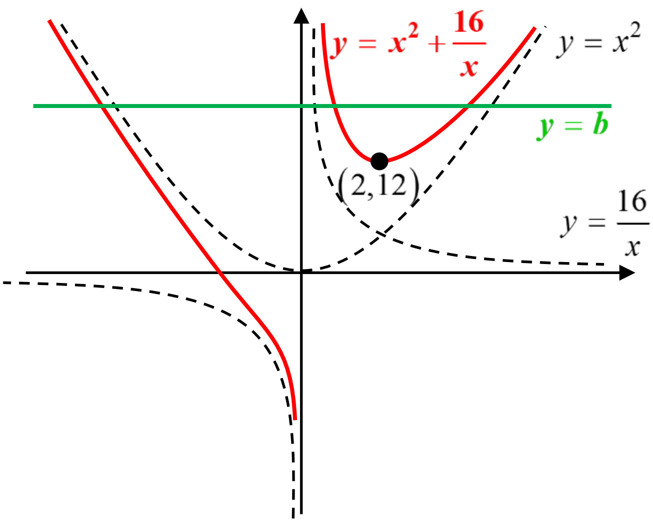
6. Ellipse: $$\begin{aligned}b^2 &= a^2\left(1-e^2\right)\\ \frac{b^2}{a^2} &= 1-e^2\end{aligned}$$

Let eccentricity of hyperbola be E: $$\begin{aligned}b^2 &= a^2\left(E^2-1\right)\\ E^2-1 &= \frac{b^2}{a^2}\\ &= 1-e^2\\ E^2 &= 2-e^2\\ E &= \sqrt{2-e^2}\end{aligned}$$

7. $$\begin{aligned} I - J &= \int_0^{\ln 2} \frac{e^x\,dx}{e^x+e^{-x}} - \int_0^{\ln 2} \frac{e^{-x}\,dx}{e^x+e^{-x}} \\ &= \int_0^{\ln 2} \frac{e^x - e^{-x}}{e^x+e^{-x}}\,dx \\ &= \Big[\ln\left(e^x+e^{-x}\right)\Big]_0^{\ln 2} \\ &= \ln\left(e^{\ln 2}+e^{-\ln 2}\right) - \ln(1+1) \\ &= \ln\frac{2+\frac{1}{2}}{2} \\ &= \ln\frac{5}{4} \end{aligned}$$

8. $$\begin{aligned} \sum_{n=1}^{99}\left(\sqrt{n}-\sqrt{n+2}\right) &= \left(\sqrt{1}-\cancel{\sqrt{3}}\right)+\left(\sqrt{2}-\cancel{\sqrt{4}}\right)+\left(\cancel{\sqrt{3}}-\cancel{\sqrt{5}}\right)+\left(\cancel{\sqrt{4}}-\cancel{\sqrt{6}}\right)+\dots \\ &\quad \dots+\left(\cancel{\sqrt{96}}-\cancel{\sqrt{98}}\right)+\left(\cancel{\sqrt{97}}-\cancel{\sqrt{99}}\right)+\left(\cancel{\sqrt{98}}-\sqrt{100}\right)+\left(\cancel{\sqrt{99}}-\sqrt{101}\right) \\ &= \sqrt{1}+\sqrt{2}-\sqrt{100}-\sqrt{101} \\ &= \sqrt{2}-\sqrt{101}-9 \end{aligned}$$

9. $$\begin{aligned} x^3 - bx + 16 &= 0 \\ x^3 + 16 &= bx \\ \frac{x^3+16}{x} &= b \end{aligned}$$



From the graph, the curve and the line have three points of intersection when $b > 12$.

10. The graph shows that at $x = \frac{7\pi}{4}$, the velocity is negative, so the particle moves towards the left (x decreases). As x decreases, at first the velocity becomes more negative (the particle moves at a faster rate towards the left), then less negative (the particle slows down). As the particle moves towards $x = \pi$, the particle moves at an aver slower rate towards the left.

Section II

Question 11

(a) Find (i) $\int \cos^3 x\,dx$ **2**

$$\begin{aligned}\int \cos^3 x\,dx &= \int \left(1-\sin^2 x\right)\cos x\ dx\\ &= \int \left(1-\sin^2 x\right)\cdot d\left(\sin x\right)\\ &= \sin x - \frac{1}{3}\sin^3 x + c\end{aligned}$$

(ii) $\int x\sec^2 x\,dx$ **2**

$$\begin{aligned}\int x\sec^2 x\,dx &= \int x\cdot d\left(\tan x\right)\\ &= x\tan x - \int \tan x\cdot d\left(x\right)\\ &= x\tan x + \int \frac{-\sin x}{\cos x}dx\\ &= x\tan x + \ln\left(\cos x\right) + c\end{aligned}$$

(b) Evaluate $\int_2^5 \frac{2dx}{x^2-4x+13}$ **3**

$$\begin{aligned}\int_2^5 \frac{2dx}{x^2-4x+13} &= 2\int_2^5 \frac{dx}{\left(x-2\right)^2+9}\\ &= \frac{2}{3}\left[\tan^{-1}\frac{x-2}{3}\right]_2^5\\ &= \frac{2}{3}\left(\tan^{-1}1-\tan^{-1}0\right)\\ &= \frac{2}{3}\cdot\frac{\pi}{4}\\ &= \frac{\pi}{6}\end{aligned}$$

(c) Let $\alpha = 2-3i$. Find, in the form $a+ib$, the value of

(i) $\alpha - 2\bar{\alpha}$ **1**

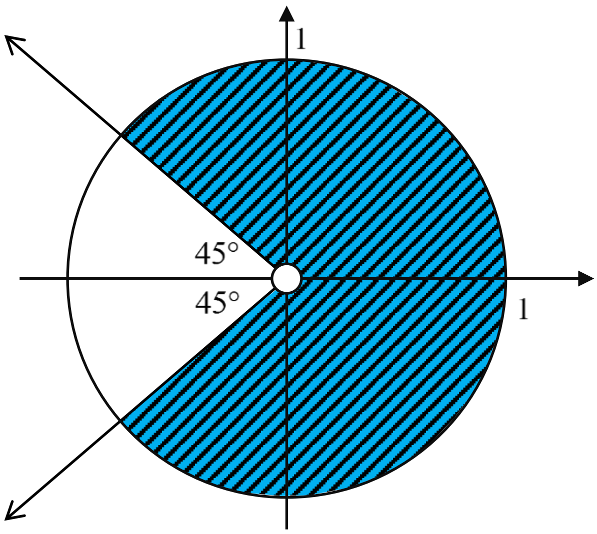
$$\begin{aligned}\alpha - 2\bar{\alpha} &= (2-3i) - 2\left(\overline{2-3i}\right)\\ &= 2-3i-2(2+3i)\\ &= 2-3i-4-6i\\ &= -2-9i\end{aligned}$$

(ii) $\dfrac{1}{\alpha+2}$ **2**

$$\begin{aligned}\frac{1}{\alpha+2} &= \frac{1}{4-3i}\times\frac{4+3i}{4+3i}\\ &= \frac{4+3i}{16+9}\\ &= \frac{4}{25}+\frac{3}{25}i\end{aligned}$$

(d) On an Argand diagram, sketch the set of all complex numbers which satisfy **2**

$$|z| \le 1 \quad \text{and} \quad |\arg(z-1)| \le \frac{3\pi}{4}$$



(e) (i) If $P(x)$ and $Q(x)$ are distinct polynomials which share a factor $(x-a)$, show that $R(x) = P(x) - Q(x)$ will have the same factor. **1**

Let $P(x) = (x-a)P_1(x)$ and $Q(x) = (x-a)Q_1(x)$

$$\begin{aligned}R(x) &= (x-a)P_1(x) - (x-a)Q_1(x)\\ &= (x-a)\left[P_1(x) - Q_1(x)\right]\end{aligned}$$

$\therefore$ $R(x)$ has the same factor of $(x-a)$.

(ii) Hence or otherwise find the two zeros that $P(x) = 6x^3 + 7x^2 - x - 2$ and $Q(x) = 6x^3 - 5x^2 - 3x + 2$ have in common. **2**

If $P(x)$ and $Q(x)$ have common roots, then they are also roots of $P(x) - Q(x)$ from (i)

$$\begin{aligned} P(x) - Q(x) &= 12x^2 + 2x - 4 \\ &= 2\left(6x^2 + x - 2\right) \\ &= 2(3x+2)(2x-1) \end{aligned}$$

The roots of $P(x) - Q(x) = 0$ are $-\frac{2}{3}$ and $\frac{1}{2}$

We can verify that $-\frac{2}{3}$ and $\frac{1}{2}$ are roots of both $P(x)$ and $Q(x)$.

$\therefore$ shared roots are $-\frac{2}{3}$ and $\frac{1}{2}$

Question 12

(a) The polynomial $P(x) = x^3 - x - 6$ has zeros α, β and γ.

(i) Find the polynomial equation with roots $\alpha^2, \beta^2, \gamma^2$. **2**

$$\begin{aligned} P\left(\sqrt{x}\right) &= 0 \\ \left(\sqrt{x}\right)^3 - \left(\sqrt{x}\right) - 6 &= 0 \\ \sqrt{x}\left(x-1\right) &= 6 \\ x\left(x-1\right)^2 &= 36 \\ x^3 - 2x^2 + x - 36 &= 0 \end{aligned}$$

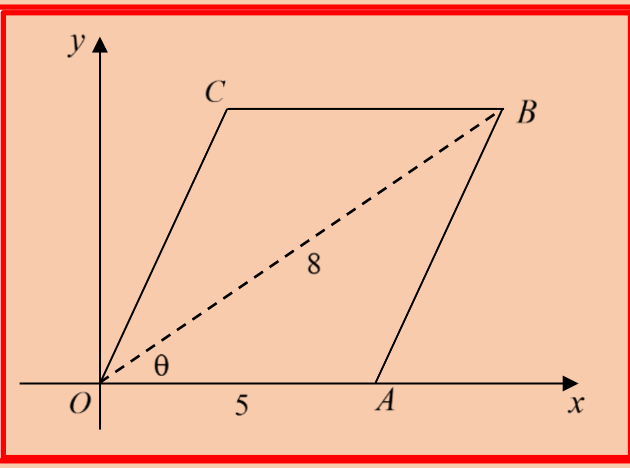
(ii) Find the value of $\left(\alpha^2+1\right)\left(\beta^2+1\right)\left(\gamma^2+1\right)$ **2**

Polynomial with roots $\alpha^2+1, \beta^2+1, \gamma^2+1$:

$$\begin{aligned} \left(x-1\right)^3 - 2\left(x-1\right)^2 + \left(x-1\right) - 36 &= 0 \\ x^3 - 3x^2 + 3x - 1 - 2x^2 + 4x - 2 + x - 1 - 36 &= 0 \\ x^3 - 5x^2 + 8x - 4 &= 0 \end{aligned}$$

$$\begin{aligned} \left(\alpha^2+1\right)\left(\beta^2+1\right)\left(\gamma^2+1\right) &= \text{product of roots} \\ &= 4 \end{aligned}$$

(b) The diagram shows a rhombus $OABC$, lying in the first quadrant of the Argand diagram.



Let $z = \cos\theta + i\sin\theta$, where $\theta = \angle AOB$.

(i) Explain why OC represents the complex number $5z^2$. **2**

$\angle COB = \angle BOA = \theta$ (angle of rhombus bisected by diagonal)

$\arg\overrightarrow{OC} = \angle AOC = 2\theta$

$\left|\overrightarrow{OC}\right| = OC = OA = 5$ (sides of rhombus)

$$\begin{aligned} \therefore\ \overrightarrow{OC} &= 5\,\text{cis}\,2\theta \\ &= 5\left(\text{cis}\,\theta\right)^2 \quad \left(\text{de Moivre's theorem}\right) \\ &= 5z^2 \end{aligned}$$

(ii) Show that $5z^2 = 8z - 5$. **1**

$$\overrightarrow{OB} = 8\operatorname{cis}\theta = 8z$$

$$\begin{aligned}\overrightarrow{OC} &= \overrightarrow{AB} \quad (\text{sides of rhombus})\\ &= \overrightarrow{OB} - \overrightarrow{OA}\\ 5z^2 &= 8z - 5\end{aligned}$$

(iii) Hence, or otherwise, find the complex numbers represented by B and C. **3**

$$\begin{aligned}5z^2 - 8z + 5 &= 0\\ z^2 - \frac{8}{5}z &= -1\\ z^2 - \frac{8}{5}z + \frac{16}{25} &= \frac{16}{25} - 1\\ \left(z - \frac{4}{5}\right)^2 &= -\frac{9}{25} = \frac{9}{25}i^2\\ z - \frac{4}{5} &= \pm\frac{3}{5}i\\ z &= \frac{4 \pm 3i}{5}\end{aligned}$$

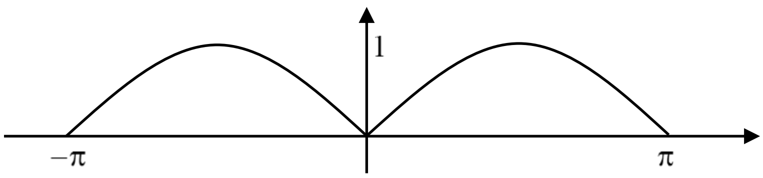
But z in 1st quadrant, so $z = \dfrac{4+3i}{5}$

$$\overrightarrow{OB} = 8z = \frac{32+24i}{5}$$

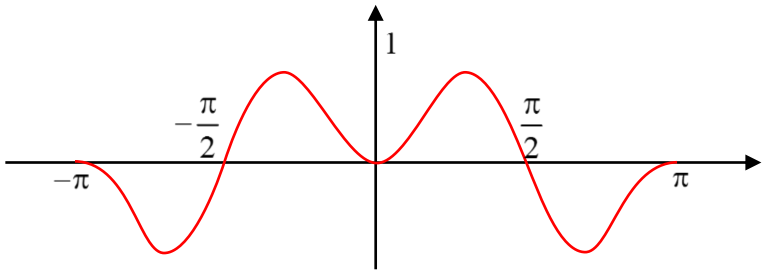
$$\begin{aligned}\overrightarrow{OC} &= \frac{32+24i}{5} - 5\\ &= \frac{7+24i}{5}\end{aligned}$$

(c) Sketch graphs of the following functions for $-\pi \le x \le \pi$:

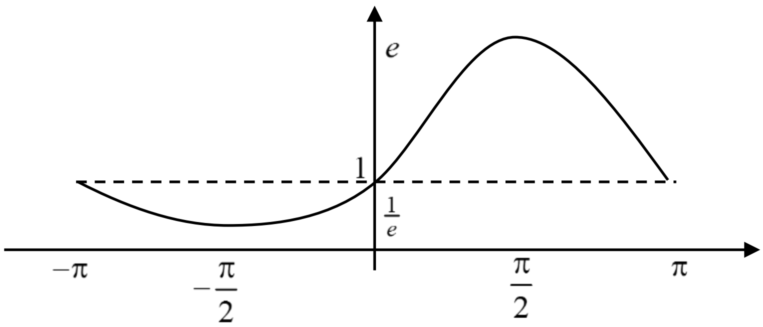
(i) $y = \sin|x|$ **1**



(ii)	$y = \sin x \cdot \sin 2x$	(Do not attempt to find coordinates of stationary points)	**2**



(iii)	$y = e^{\sin x}$	**2**



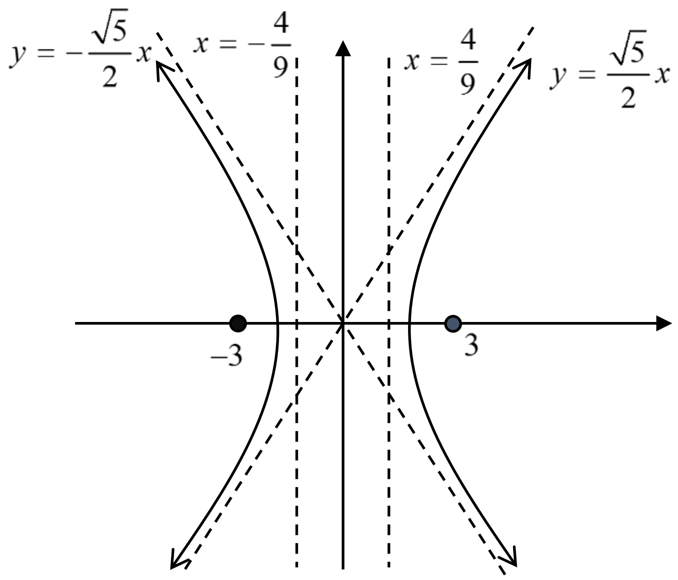
Question 13

(a) Sketch the hyperbola $5x^2 - 4y^2 = 20$, showing the coordinates of the vertices and foci, and the equations of the directrices and asymptotes. **2**

$$5x^2 - 4y^2 = 20$$
$$\frac{x^2}{4} - \frac{y^2}{5} = 1$$

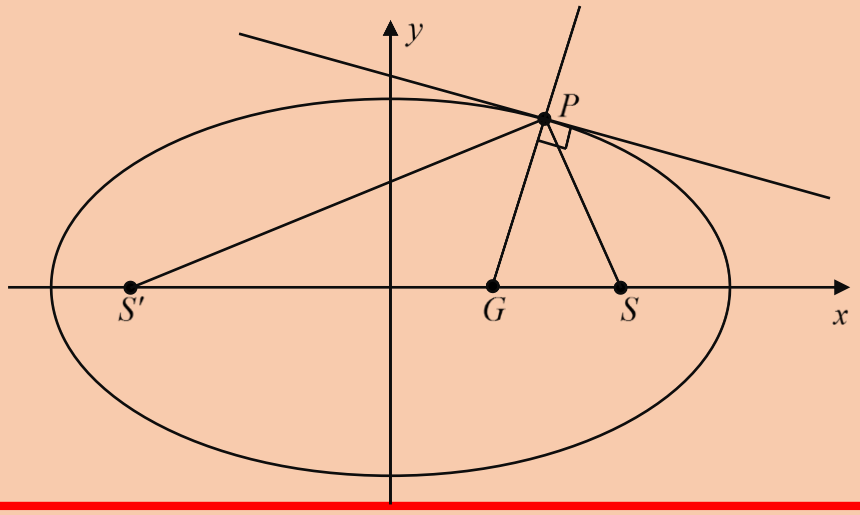
$$b^2 = a^2\left(e^2 - 1\right)$$
$$5 = 4\left(e^2 - 1\right)$$
$$e^2 - 1 = \frac{5}{4}$$
$$e^2 = \frac{9}{4}$$
$$e = \frac{3}{2}$$

$$ae = 2\times\frac{3}{2} = 3, \quad \frac{a}{e} = 2\times\frac{2}{3} = \frac{4}{9}, \quad \frac{b}{a} = \frac{\sqrt{5}}{2}$$



(b) The ellipse $\dfrac{x^2}{a^2}+\dfrac{y^2}{b^2}=1$ has foci $S(ae,0)$ and $S'(-ae,0)$, and directrices $x=\pm\dfrac{a}{e}$.

$P(a\cos\theta, b\sin\theta)$ is a point on the ellipse, with the normal at P meeting the x-axis at G.



(i) Use the focus-directrix definition of an ellipse to show that

$$\frac{PS}{PS'}=\frac{1-e\cos\theta}{1+e\cos\theta}.$$ **2**

Let M and M' be the feet of the perpendiculars from P to the directrices corresponding to S and S' respectively.

$$\begin{aligned}\frac{PS}{PS'}&=\frac{\cancel{e}\cdot PM}{\cancel{e}\cdot PM'}\\&=\frac{\frac{a}{e}-a\cos\theta}{a\cos\theta-\left(-\frac{a}{e}\right)}\times\frac{e/a}{e/a}\\&=\frac{1-e\cos\theta}{1+e\cos\theta}\end{aligned}$$

(ii) The equation of the normal at P is given by $\dfrac{ax}{\cos\theta}-\dfrac{by}{\sin\theta}=a^2-b^2$. **3**

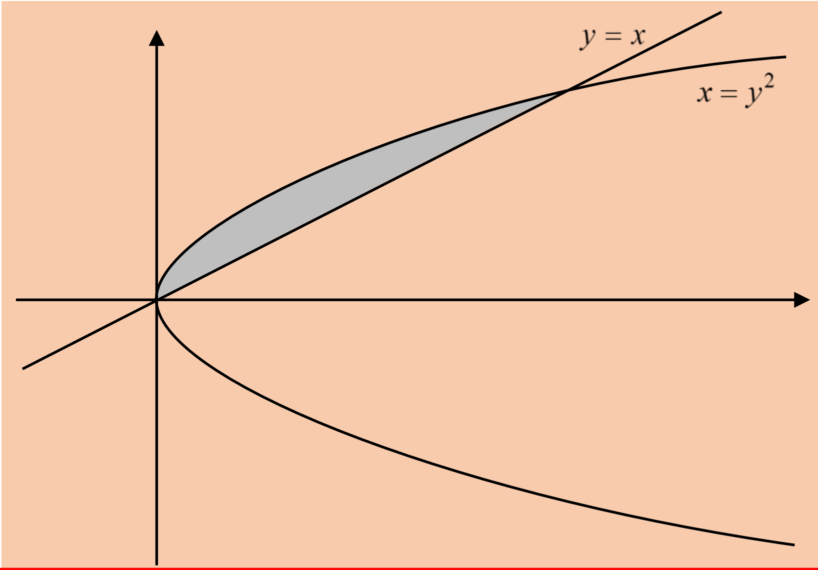
(Do NOT show this)

Show that $\dfrac{GS}{GS'}=\dfrac{PS}{PS'}$.

$$\begin{aligned}G:\quad \frac{ax}{\cos\theta}-\frac{b(0)}{\sin\theta}&=a^2-b^2\\ \frac{ax}{\cos\theta}&=a^2-a^2\left(1-e^2\right)\\&=a^2e^2\\ x_G&=ae^2\cos\theta\end{aligned}$$

$$\begin{aligned}\frac{GS}{GS'}&=\frac{ae-ae^2\cos\theta}{ae^2\cos\theta-(-ae)}\\&=\frac{1-e\cos\theta}{1+e\cos\theta}\\&=\frac{PS}{PS'}\end{aligned}$$

(b) The region bounded by $x = y^2$ and $y = x$ is to be rotated about the x-axis to form a solid. **3**



Use the method of cylindrical shells to find the volume of this solid.

Take a typical slice within the shaded region, parallel to the x-axis, meeting the parabola at (x_1, y) and the line at (x_2, y), and with width δy.

Rotate this slice about the x-axis to form a cylindrical shell.

Volume of slice $\delta V \approx 2\pi y(x_2 - x_2)\delta y$

$$= 2\pi y\left(y - y^2\right)\delta y$$

Volume of solid $V = 2\pi\int_0^1 \left(y^2 - y^3\right) dy$

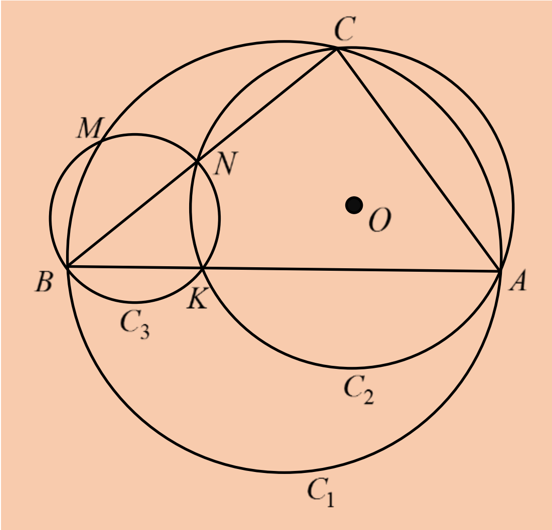
$$= 2\pi\left[\frac{y^3}{3} - \frac{y^4}{4}\right]_0^1$$

$$= 2\pi\left(\frac{1}{3} - \frac{1}{4}\right)$$

$$= \frac{\pi}{6} \text{ u}^3$$

(c) A triangle ABC has its vertices on the circle C_1. Another circle C_2 has its centre O lying inside ΔABC. This circle passes through A and C, and cuts AB and BC at K and N respectively.

A third circle C_3, passing through B, K and N, cuts circle C_1 at M.



Let $\angle BMK = \alpha$.

(i) Show that $\angle KAC = \alpha$. **2**

$$\begin{aligned} \angle KAC &= \angle BNK && \text{(exterior angle of cyclic quad } ACNK \text{ equals opposite interior angle)} \\ &= \angle BMK && \text{(angles subtended by same arc } BK \text{ in circle } C_3\text{)} \\ &= \alpha \end{aligned}$$

(ii) State why $\angle BMC = 180° - \alpha$. **1**

$$\begin{aligned} \angle BMC + \angle KAC &= 180° && \text{(opposite angles of cyclic quad } ABMC\text{)} \\ \angle BMC &= 180° - \angle KAC \\ &= 180° - \alpha \end{aligned}$$

(iii) Show that $MKOC$ is a cyclic quadrilateral. **2**

$$\begin{aligned} \angle KOC &= 2\angle KAC && \text{(angle at centre is twice angle at circumference in circle } C_2\text{)} \\ &= 2\alpha \end{aligned}$$

$$\begin{aligned} \angle MKC &= \angle BMC - \angle BMK \\ &= (180 - \alpha) - \alpha \\ &= 180 - 2\alpha \end{aligned}$$

$\therefore$ $MKOC$ is cyclic (opposite angles supplementary)

Question 14

(a) Consider the polynomial $P(z) = z^3 + \beta z^2 + k$.

The equation $P(z) = 0$ has two non-real roots ω and $\overline{\omega}$, and a real root β. Let $\omega = a + ib$.

(i) Show that $\beta = -a$. **1**

Sum of roots:

$$\begin{aligned} \omega + \overline{\omega} + \beta &= -\beta \\ a + ib + a - ib + \beta &= -\beta \\ 2a &= -2\beta \\ \beta &= -a \end{aligned}$$

(ii) Show that $|\omega|^2 = 2\beta^2$. **2**

Sum of roots in pairs:

$$\begin{aligned} \omega\overline{\omega} + \omega\beta + \overline{\omega}\beta &= 0 \\ |\omega|^2 + \beta(\omega + \overline{\omega}) &= 0 \\ |\omega|^2 + \beta(2a) &= 0 \\ |\omega|^2 + \beta(-2\beta) &= 0 \\ |\omega|^2 &= 2\beta^2 \end{aligned}$$

(iii) Show that ω and $\overline{\omega}$ lie on the lines $y = \pm x$. **1**

$$\begin{aligned} |\omega|^2 &= 2\beta^2 \\ &= 2a^2 \quad (\text{from part i}) \\ a^2 + b^2 &= 2a^2 \\ b^2 &= a^2 \\ b &= \pm a \end{aligned}$$

ie. (a, b) and $(a, -b)$ lie on $y = \pm x$

(iv) The three roots are represented as points A, B, C in the Argand plane. **2**
Find the area of ΔABC, expressed in terms of ω only.

Roots are represented by points $A(a, a)$, $B(a, -a)$ and $C(-a, 0)$, and let D be $(a, 0)$.

$$\begin{aligned} \text{Area} &= \frac{1}{2} \times AB \times CD \\ &= \frac{1}{2} \times 2|a| \times 2|a| \\ &= 2a^2 \\ &= |\omega|^2 \end{aligned}$$

(b) (i) Evaluate $\int_1^e (1-\ln x)\,dx$. **2**

$$\begin{aligned}
\int_1^e (1-\ln x)\,d(x) &= \left[x(1-\ln x)\right]_1^e - \int_1^e x\cdot d(1-\ln x)\\
&= 0-1-\int_1^e x\cdot\left(-\frac{1}{x}\right)dx\\
&= -1+\int_1^e dx\\
&= -1+[x]_1^e\\
&= -1+(e-1)\\
&= e-2
\end{aligned}$$

(ii) You are given that $I_n = \int_1^e (1-\ln x)^n\,dx$ for $n = 1, 2, 3, \ldots$ **2**

Show that $I_n = -1+nI_{n-1}$ for $n = 2, 3, 4, \ldots$

$$\begin{aligned}
I_n &= \int_1^e (1-\ln x)^n\,d(x)\\
&= \left[x(1-\ln x)^n\right]_1^e - \int_1^e x\cdot d(1-\ln x)^n\\
&= 0-1-\int_1^e x\cdot n(1-\ln x)^{n-1}\cdot\left(-\frac{1}{x}\right)dx\\
&= -1+n\int_1^e x(1-\ln x)^{n-1}\,dx\\
&= -1+nI_{n-1}
\end{aligned}$$

(iii) Hence find the value of I_3. **1**

$$\begin{aligned}
I_3 &= -1+3I_2\\
&= -1+3(-1+2I_1)\\
&= -4+6I_1\\
&= -4+6(e-2)\\
&= 6e-16
\end{aligned}$$

(c)	(i)	Given that x is a positive real number and that n is a positive integer, show that $\dfrac{1}{1+x^n} < 1$.	**1**

$$
\begin{aligned}
1-\frac{1}{1+x^n} &= \frac{\left(1+x^n\right)-1}{1+x^n} \\
&= \frac{x^n}{1+x^n} > 0 \quad (\text{as } x>0)
\end{aligned}
$$

$$\frac{1}{1+x^n} < 1$$

	(ii)	Let $I_n = \displaystyle\int_0^1 \frac{dx}{1+x^n}$, where $n = 2, 3, 4, \ldots$, show that $\dfrac{\pi}{4} \le I_n < 1$.	**3**

Given $0 \le x \le 1$:

$$
\begin{aligned}
\frac{1}{1+x^n}-\frac{1}{1+x^2} &= \frac{\left(1+x^2\right)-\left(1+x^n\right)}{\left(1+x^2\right)\left(1+x^n\right)} \\
&= \frac{x^2-x^n}{\left(1+x^2\right)\left(1+x^n\right)} \\
&= \frac{x^2\left(1-x^{n-2}\right)}{\left(1+x^2\right)\left(1+x^n\right)} \\
&\ge 0 \quad \left(\text{As } x^2, 1+x^2 \text{ and } 1+x^n \ge 0\ \forall x \ge 0,\ 1-x^{n-2} > 0 \text{ for } 0<x<1\right)
\end{aligned}
$$

$$\frac{1}{1+x^n} \ge \frac{1}{1+x^2}$$

So $\dfrac{1}{1+x^2} \le \dfrac{1}{1+x^n} < 1$ for $0 \le x \le 1$

$$
\begin{aligned}
\therefore \int_0^1 \frac{1}{1+x^2}\,dx &\le \int_0^1 \frac{1}{1+x^n}\,dx < \int_0^1 1\,dx \\
\left[\tan^{-1} x\right]_0^1 &\le \int_0^1 \frac{1}{1+x^n}\,dx < \left[x\right]_0^1 \\
\frac{\pi}{4} &\le \int_0^1 \frac{1}{1+x^n}\,dx < 1
\end{aligned}
$$

Question 15

(a)	(i)	Prove that, if $c \geq a$ and $d \geq b$, then $ab + cd \geq bc + ad$.	**1**

$$\begin{aligned}(ab+cd)-(bc+ad) &= (ab-bc)-(ad-cd)\\ &= b(a-c)+d(a-c)\\ &= (a-c)(b-d)\\ &\geq 0 \quad (\text{as } a \leq c \text{ so } a-c \leq 0,\ \text{and } b \leq d \text{ so } b-d \leq 0)\\ ab+cd &\geq bc+ad\end{aligned}$$

	(ii)	Use part (i) to show that, if $x \geq y$, then $x^2 + y^2 \geq 2xy$.	**1**

As $x \geq y$, replace c and d by x, and replace a and b by y.

$$\begin{aligned}ab+cd &\geq bc+ad\\ y\cdot y + x\cdot x &\geq y\cdot x + y\cdot x\\ x^2+y^2 &\geq 2xy\end{aligned}$$

	(ii)	Without the use of further algebra, explain why the result of part (ii) is in fact true for all values of x and y.	**1**

The result is symmetric in x and y, so it doesn't matter which is larger.

(b)	Let $I = \int_0^{\frac{\pi}{2}} \frac{\cos x \, dx}{1+\sin^2 x}$.		
	(i)	Evaluate I.	**2**

$$\begin{aligned}I &= \int_0^{\frac{\pi}{2}} \frac{\cos x \, dx}{1+\sin^2 x}\\ &= \int_0^{\frac{\pi}{2}} \frac{d(\sin x)}{1+(\sin x)^2}\\ &= \left[\tan^{-1}(\sin x)\right]_0^{\frac{\pi}{2}}\\ &= \tan^{-1} 1 - \tan^{-1} 0\\ &= \frac{\pi}{4}\end{aligned}$$

(ii) Use the substitution $t = \tan\frac{x}{2}$ to show that $I = 2\int_0^1 \frac{1-t^2}{1+6t^2+t^4}\,dt$. **2**

$$t = \tan\frac{x}{2}$$
$$x = 2\tan^{-1} t$$
$$dx = \frac{2dt}{1+t^2}$$

$$
\begin{aligned}
I &= \int_0^{\frac{\pi}{2}} \frac{\cos x\; dx}{1+\sin^2 x} \\
&= \int_0^1 \frac{\frac{1-t^2}{1+t^2}}{1+\left(\frac{2t}{1+t^2}\right)^2} \cdot \frac{2dt}{1+t^2} \times \frac{1+t^2}{1+t^2} \\
&= \int_0^1 \frac{2\left(1-t^2\right)dt}{\left(1+t^2\right)^2+4t^2} \\
&= 2\int_0^1 \frac{1-t^2}{1+6t^2+t^4}\,dt
\end{aligned}
$$

(iii) By considering $J = \int_0^{\frac{\pi}{2}} \frac{\cos x\; dx}{1+k\sin^2 x}$ for an appropriate value of k, **3**

evaluate $\int_0^1 \frac{1-t^2}{1+14t^2+t^4}\,dt$.

Using the t-substitution, $\frac{\cos x dx}{1+k\sin^2 x} = \frac{2\left(1-t^2\right)dt}{\left(1+t^2\right)^2+4kt^2} = \frac{2\left(1-t^2\right)dt}{1+(4k+2)t^2+t^4}$

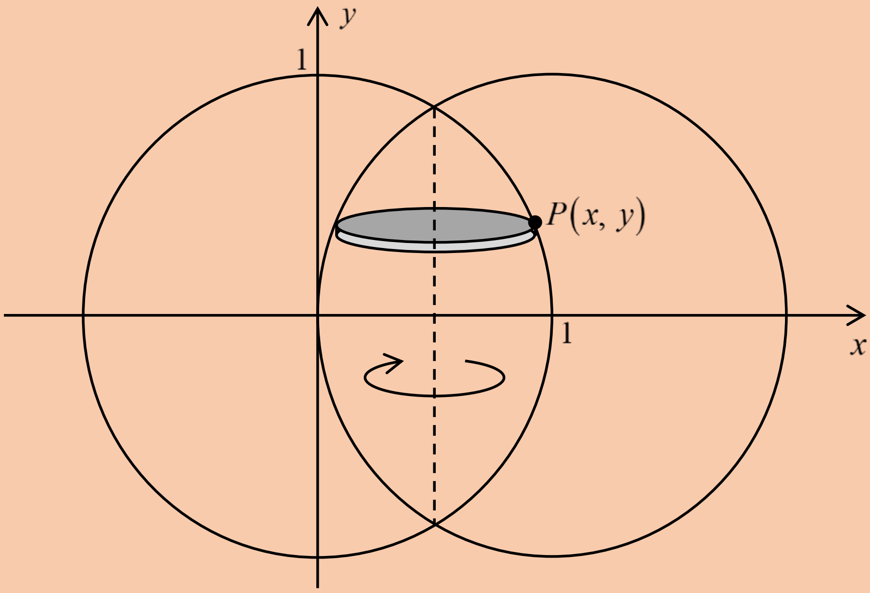
To get our integrand, $4k+2=14$

$$k=3$$

$$
\begin{aligned}
\int_0^1 \frac{1-t^2}{1+14t^2+t^4}\,dt &= \int_0^{\frac{\pi}{2}} \frac{\cos x\, dx}{1+3\sin^2 x} \\
&= \frac{1}{3}\int_0^{\frac{\pi}{2}} \frac{d(\sin x)}{\frac{1}{3}+\sin^2 x} \\
&= \frac{1}{3}\cdot\sqrt{3}\left[\tan^{-1}\left(\sqrt{3}\sin x\right)\right]_0^{\frac{\pi}{2}} \\
&= \frac{1}{\sqrt{3}}\left(\tan^{-1}\sqrt{3}-\tan^{-1} 0\right) \\
&= \frac{\pi}{3\sqrt{3}}
\end{aligned}
$$

(c) The diagram shows two circles of radius 1, centred at $(0, 0)$ and $(1, 0)$.

The region common to the two circles is to be rotated about its axis of symmetry to create a solid.



A typical slice of width δy is shown, intersecting the first circle at $P(x, y)$.

(i) Show that the volume of the slice is $\pi\left(\frac{5}{4} - y^2 - \sqrt{1-y^2}\right)\delta y$. **2**

Radius of slice: $r = x - \frac{1}{2}$

$= \sqrt{1-y^2} - \frac{1}{2}$

Volume of slice $\delta V = \pi r^2 h$

$= \pi\left(\sqrt{1-y^2} - \frac{1}{2}\right)^2 \delta y$

$= \pi\left(1 - y^2 - \sqrt{1-y^2} + \frac{1}{4}\right)\delta y$

$= \pi\left(\frac{5}{4} - y^2 - \sqrt{1-y^2}\right)$

(ii) Hence find the volume of the solid. **3**

When $x = \frac{1}{2}$, $y = \pm\sqrt{1 - \left(\frac{1}{2}\right)^2} = \pm\frac{\sqrt{3}}{2}$

$$V = \pi\int_{-\frac{\sqrt{3}}{2}}^{\frac{\sqrt{3}}{2}} \left(\frac{5}{4} - y^2 - \sqrt{1-y^2}\right) dy$$

$$= 2\pi\int_0^{\frac{\sqrt{3}}{2}} \left(\frac{5}{4} - y^2\right) dy - 2\pi\int_0^{\frac{\sqrt{3}}{2}} \sqrt{1-y^2}\, dy$$

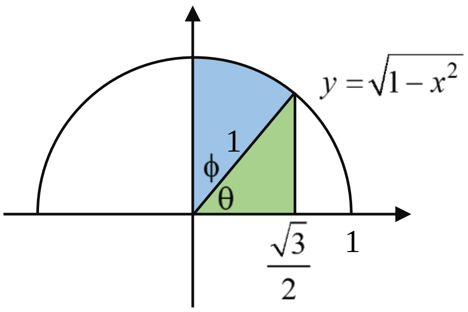
$$= 2\pi(I - J)$$

$$I = \left[\frac{5y}{4} - \frac{y^3}{3}\right]_0^{\frac{\sqrt{3}}{2}}$$

$$= \frac{5\sqrt{3}}{8} - \frac{3\sqrt{3}}{24}$$

$$= \frac{\sqrt{3}}{2}$$

J : Method 1



$$\cos\theta = \frac{\sqrt{3}}{2}$$

$$\theta = \frac{\pi}{6}$$

$$\phi = \frac{\pi}{2} - \frac{\pi}{6} = \frac{\pi}{3}$$

J = Area of Triangle + Area of Sector

$$= \frac{1}{2}\cdot\frac{\sqrt{3}}{2}\cdot\frac{1}{2} + \frac{1}{2}\cdot 1^2\cdot\frac{\pi}{3}$$

$$= \frac{\sqrt{3}}{8} + \frac{\pi}{6}$$

Method 2

$$J = \int_0^{\frac{\pi}{3}} \sqrt{1-\sin^2\theta}\cdot\cos\theta\, d\theta$$

$$= \int_0^{\frac{\pi}{3}} \cos^2\theta\ d\theta$$

$$= \frac{1}{2}\int_0^{\frac{\pi}{3}} (1+\cos 2\theta)\, d\theta$$

$$= \frac{1}{2}\left[\theta + \frac{1}{2}\sin 2\theta\right]_0^{\frac{\pi}{3}}$$

$$= \frac{1}{2}\left(\frac{\pi}{3} + \frac{\sqrt{3}}{4}\right)$$

Let $y = \sin\theta$

$dy = \cos\theta\, d\theta$

$\left(x = \frac{\sqrt{3}}{2}\right) \quad y = \frac{\pi}{3}$

$(x = 0) \quad y = 0$

$$\therefore\ V = 2\pi\left[\frac{\sqrt{3}}{2} - \left(\frac{\sqrt{3}}{8} + \frac{\pi}{6}\right)\right]$$

$$= 2\pi\left(\frac{3\sqrt{3}}{8} - \frac{\pi}{6}\right)$$

$$= \frac{\pi}{12}\left(9\sqrt{3} - 4\pi\right) \text{ units}^3$$

Question 16

(a) You are given that $(x+1)$ is a factor of $P(x) = 4x^3 + 2x^2 - 3x - 1$.

(i) Solve $P(x) = 0$. **1**

Let roots be $-1, \alpha, \beta$.

Sum of roots: $\alpha + \beta - 1 = -\frac{1}{2}$

$\alpha + \beta = \frac{1}{2}$

Product of roots: $-\alpha\beta = \frac{1}{4}$

$\alpha\beta = -\frac{1}{4}$

Quadratic equation with roots α, β:

$$x^2 - \frac{1}{2}x - \frac{1}{4} = 0$$

$$4x^2 - 2x - 1 = 0$$

$$x = \frac{2 \pm \sqrt{20}}{8}$$

$$x = \frac{1 \pm \sqrt{5}}{4},\ -1$$

(ii) It can be shown that $\cos 3\theta = 4\cos^3\theta - 3\cos\theta$. (Do NOT prove this.) **3**

Use this result and the result of part (i) to show that

$$\cos\frac{3\pi}{5} = \frac{1-\sqrt{5}}{4}.$$

Let $x = \cos\theta$

$$4x^3 + 2x^2 - 3x - 1 = 0$$

$$4\cos^3\theta - 3\cos\theta + 2\cos^2\theta - 1 = 0$$

$$\cos 3\theta + \cos 2\theta = 0$$

$$\cos 3\theta = -\cos 2\theta$$

$$3\theta = \pi - 2\theta + 2k\pi \qquad 3\theta = \pi + 2\theta + 2k\pi$$

$$5\theta = (2k+1)\pi \qquad \theta = (2k+1)\pi$$

$$\theta = \frac{2k+1}{5}\pi$$

For $0 \le \theta \le 2\pi$: $\theta = \frac{\pi}{5}, \frac{3\pi}{5}, \pi, \frac{7\pi}{5}, \frac{9\pi}{5}$

$$x = \cos\frac{\pi}{5},\ \cos\frac{3\pi}{5}, -1,\ \cos\frac{7\pi}{5},\ \cos\frac{9\pi}{5}$$

But $\cos\frac{\pi}{5} = \cos\frac{9\pi}{5}$ and $\cos\frac{3\pi}{5} = \cos\frac{7\pi}{5}$

So roots of polynomial are $-1, \cos\frac{\pi}{5}, \cos\frac{3\pi}{5}$

But $\cos\frac{3\pi}{5} < \cos\frac{\pi}{5}$ as $\cos x$ decreases in first quadrant.

So $\cos\frac{3\pi}{5}$ is the lesser of the two irrational roots, ie. $\frac{1-\sqrt{5}}{3}$

(b) A series of functions $f_0(x)$, $f_1(x)$, $f_2(x)$, ... is defined by

$$f_n(x) = \begin{cases} \dfrac{1}{1-x} & \text{if} \quad n = 0 \\ f_0\left[f_{n-1}(x)\right] & \text{if} \quad n = 1, 2, 3, \dots \end{cases}$$

(i) Show that $f_3(x) = f_0(x)$. **2**

$$\begin{aligned} f_1(x) &= f_0\left[f_0(x)\right] \\ &= \frac{1}{1-\frac{1}{1-x}} \times \frac{1-x}{1-x} \\ &= \frac{1-x}{(1-x)-1} \\ &= \frac{x-1}{x} \end{aligned}$$

$$\begin{aligned} f_2(x) &= f_0\left[f_1(x)\right] \\ &= \frac{1}{1-\frac{x-1}{x}} \times \frac{x}{x} \\ &= \frac{x}{x-(x-1)} \\ &= x \end{aligned}$$

$$\begin{aligned} f_3(x) &= f_0\left[f_2(x)\right] \\ &= \frac{1}{1-x} \\ &= f_0(x) \end{aligned}$$

(ii) Hence explain why $f_{n+3}(x) = f_n(x)$. **1**

Continuing the algebra, we would get $f_4(x) = \dfrac{x-1}{x} = f_1(x)$, $f_5(x) = x = f_2(x)$, etc.

So we cycle through a set of three functions.

(iii) Hence evaluate $f_{100}(100)$. **1**

$$\begin{aligned} f_{100}(100) &= f_{3(33)+1}(100) \\ &= f_1(100) \\ &= \frac{100-1}{100} \\ &= \frac{99}{100} \end{aligned}$$

(c) A sequence a_k is defined by the following rules: **3**

$a_1 = 1$.

a_{k+1} is equal to one more than the product of all terms that precede it.

(That is $a_{k+1} = 1 + a_k \cdot a_{k-1} \cdot \ldots \cdot a_3 \cdot a_2 \cdot a_1$)

Prove by mathematical induction that

$$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n} = 2 - \frac{1}{a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n}$$

RTP: $$\frac{1}{a_1} + \frac{1}{a_2} + \frac{1}{a_3} + \ldots + \frac{1}{a_n} = 2 - \frac{1}{a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_n}$$

$$\text{where } a_{k+1} = \begin{cases} 1 & \text{if } k = 0 \\ 1 + a_k \cdot a_{k-1} \cdot \ldots \cdot a_3 \cdot a_2 \cdot a_1 & \text{if } k > 0 \end{cases}$$

Test $n = 1$: $\text{LHS} = \dfrac{1}{a_1}$ $\quad$ $\text{RHS} = 2 - \dfrac{1}{a_1}$

$$= \frac{1}{1} \qquad = 2 - \frac{1}{1}$$

$$= 1 \qquad = \text{LHS}$$

Assume true for $n = k$: ie. $\dfrac{1}{a_1} + \dfrac{1}{a_2} + \dfrac{1}{a_3} + \ldots + \dfrac{1}{a_k} = 2 - \dfrac{1}{a_1 \cdot a_2 \cdot a_3 \cdot \ldots \cdot a_k}$

Prove true for $n = k + 1$:

$$\left(\frac{1}{a_1} + \frac{1}{a_2} + \ldots + \frac{1}{a_k}\right) + \frac{1}{a_{k+1}}$$

$$= 2 - \frac{1}{a_1 \cdot a_2 \cdot \ldots \cdot a_k} + \frac{1}{a_1 \cdot a_2 \cdot \ldots \cdot a_k + 1} \qquad \left(\text{by assumption and by definition of } a_{k+1}\right)$$

$$= 2 - \frac{1}{u} + \frac{1}{u+1} \qquad \left(\text{where } u = a_1 \cdot a_2 \cdot \ldots \cdot a_k = a_{k+1} - 1\right)$$

$$= 2 - \left(\frac{1}{u} - \frac{1}{u+1}\right)$$

$$= 2 - \frac{(u+1) - u}{u(u+1)}$$

$$= 2 - \frac{1}{(u)(u+1)}$$

$$= 2 - \frac{1}{(a_1 \cdot a_2 \cdot \ldots \cdot a_k)(a_{k+1})}$$

(d) (You may assume for both parts that x is an angle that lies in the first quadrant.)

(i) Show that $\sqrt{\cot x} - \sqrt{\tan x} = \dfrac{\cos x - \sin x}{\sqrt{\sin x \cos x}}$. **1**

$$\sqrt{\cot x} - \sqrt{\tan x} = \sqrt{\frac{\cos x}{\sin x}} - \sqrt{\frac{\sin x}{\cos x}}$$

$$= \frac{\sqrt{\cos^2 x} - \sqrt{\sin^2 x}}{\sqrt{\sin x} \cdot \sqrt{\cos x}}$$

$$= \frac{\cos x - \sin x}{\sqrt{\sin x \cos x}}$$

(ii) By using the substitution $\sec\theta = \sin x + \cos x$, or otherwise, find

$$\int_0^{\frac{\pi}{4}} \left(\sqrt{\cot x} - \sqrt{\tan x}\right) dx.$$ **3**

$$\int_0^{\frac{\pi}{4}}\left(\sqrt{\cot x}-\sqrt{\tan x}\right)dx = \int_0^{\frac{\pi}{4}}\frac{\cos x-\sin x}{\sqrt{\sin x\cos x}}dx$$

$$= \int_0^{\frac{\pi}{4}}\frac{\sec\theta\tan\theta\ d\theta}{\frac{1}{\sqrt{2}}\tan\theta}$$

$$= \sqrt{2}\int_0^{\frac{\pi}{4}}\sec\theta\ d\theta$$

$$= \sqrt{2}\int_0^{\frac{\pi}{4}}\frac{\sec\theta\left(\tan\theta+\sec\theta\right)}{\sec\theta+\tan\theta}d\theta$$

$$= \sqrt{2}\left[\ln\left(\sec\theta+\tan\theta\right)\right]_0^{\frac{\pi}{4}}$$

$$= \sqrt{2}\left[\ln\left(\sqrt{2}+1\right)-\ln\left(1+0\right)\right]$$

$$= \sqrt{2}\ln\left(\sqrt{2}+1\right)$$

Let $\sec\theta = \sin x+\cos x$

$$\sec\theta\tan\theta d\theta = \left(\cos x-\sin x\right)dx$$

$$\sec^2\theta = \left(\sin x+\cos x\right)^2$$

$$= \sin^2 x+\cos^2 x+2\sin x\cos x$$

$$= 1+2\sin x\cos x$$

$$\sin x\cos x = \frac{\sec^2\theta-1}{2}$$

$$= \frac{1}{2}\tan^2\theta$$

$$\sqrt{\sin x\cos x} = \frac{1}{\sqrt{2}}\tan\theta$$

$\left(x=0\right)$

$$\sec\theta = \sin 0+\cos 0$$

$$= 1$$

$$\cos\theta = 1$$

$$\theta = 0$$

$\left(x=\frac{\pi}{4}\right)$

$$\sec\theta = \sin\frac{\pi}{4}+\cos\frac{\pi}{4}$$

$$= \frac{1}{\sqrt{2}}+\frac{1}{\sqrt{2}}$$

$$= \sqrt{2}$$

$$\cos\theta = \frac{1}{\sqrt{2}}$$

$$\theta = \frac{\pi}{4}$$