Sydney Technical High School

Name: ....

Maths Class: ....

Year 12

MATHEMATICS EXTENSION 2

HSC COURSE

ASSESSMENT 4 – TRIAL HSC

AUGUST, 2018

Time Allowed: 180 minutes

Reading time : 5 minutes

General Instructions:

- Marks for each question are indicated on the question.
- Approved calculators may be used
- All necessary working should be shown
- Full marks may not be awarded for careless work or illegible writing
- ***Begin each question on a new page***
- Write using black or blue pen
- All answers are to be in the writing booklet provided
- A reference sheet is provided. Refer to it if required.

Section 1 Multiple Choice

Questions 1-10

10 Marks

Section II Questions 11-16

90 Marks

Section 1 Use the multiple choice answer sheet for Questions 1 – 10

1. Which of the following complex numbers equals $(\sqrt{3}+i)^4$?

A) $-2+\frac{2}{3}i$ B) $-8+\frac{8}{3}i$

C) $-2+2\sqrt{3}\,i$ D) $-8+8\sqrt{3}\,i$

2. The foci and the directrices of the ellipse with equation $4x^2+y^2=4$ are

A) $(\pm\sqrt{3}, 0)$ and $x=\pm\frac{4\sqrt{3}}{3}$ B) $(0,\pm\sqrt{3})$ and $y=\pm\frac{4\sqrt{3}}{3}$

C) $(0, \pm\sqrt{3})$ and $x=\pm\frac{4\sqrt{3}}{3}$ D) $(\pm\sqrt{3}, 0)$ and $y=\pm\frac{4\sqrt{3}}{3}$

3. The roots of the polynomial $x^3-5x+4=0$ are α, β *and* γ. Find the value of $\alpha^2+\beta^2+\gamma^2$?

A) 25 B) 10 C) 5 D) -5

4. The graph of $y=g(x)$ is a reflection of the graph of $y=f(x)$ across the x axis. Which of the following is true?

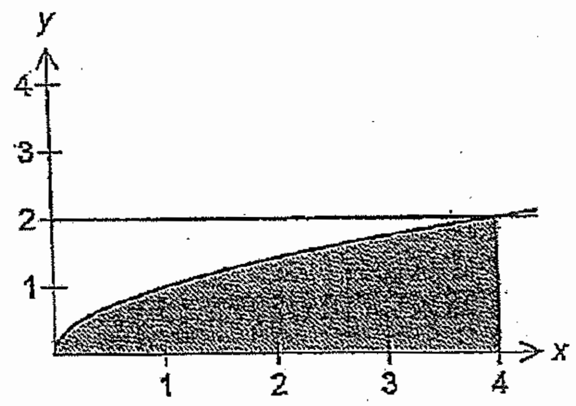
A) $g(x)=|f(x)|$ B) $g(x)=f^{-1}(x)$

C) $g(x)=-f(x)$ D) $g(x)=f(-x)$

5. What is the value of $\displaystyle\int_0^1 \frac{\cos^{-1} x}{\sqrt{1-x^2}}\,dx$

A) $\dfrac{\pi^2}{4}$ B) $-\dfrac{\pi^2}{4}$ C) $\dfrac{\pi^2}{8}$ D) $-\dfrac{\pi^2}{8}$

6. The area bounded by $y = \sqrt{x}$, the x axis and the ordinate $x = 4$ is rotated about the line $y = 2$.



By using the method of cylindrical shells, the volume formed is given by

A) $\displaystyle\pi\int_0^4 4 - x \;dx$

B) $\displaystyle\pi\int_0^2 (2-(2-\sqrt{x})^2)\;dx$

C) $\displaystyle\pi\int_0^4 (4-(2-\sqrt{x})^2)dx$

D) $\displaystyle 2\pi\int_0^2 (2-y)(4-y^2)dy$

7. A particle of mass m is moving horizontally at velocity v in a straight line. Its motion is resisted by a force of magnitude $mk(v+v^2), k > 0$
The displacement x in terms of v is given by

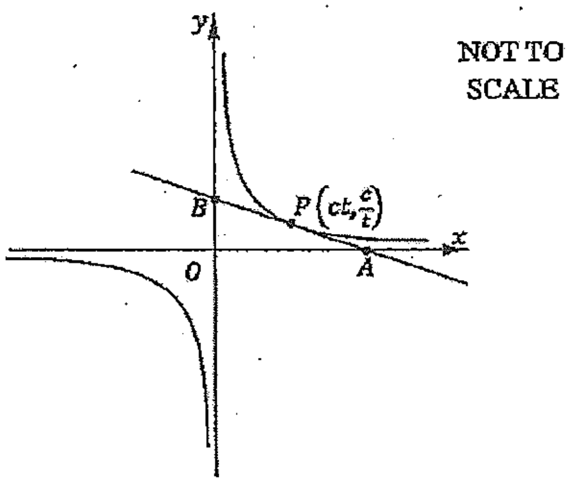
A) $\displaystyle\frac{1}{k}\int \frac{dv}{1+v}$

B) $\displaystyle\frac{1}{k}\int \frac{dv}{v(1+v)}$

C) $\displaystyle-\frac{1}{k}\int \frac{dv}{v(1+v)}$

D) $\displaystyle-\frac{1}{k}\int \frac{dv}{1+v}$

8. The equation of the tangent to the rectangular hyperbola $xy = c^2$ at $P(ct, \frac{c}{t})$ is given by $x + t^2y = 2ct$. The tangent cuts the x and y axes at A and B respectively.



Which of the following statements is false?

A) P is the centre of the circle that passes through O, A and B.

B) The area of ΔAOB is $2c^2$ square units.

C) The distance AB is $\sqrt{4c^2t^2 + \left(\frac{4c^2}{t^2}\right)}$

D) $AP > BP$.

9. Given that $\omega^3 = -1$ and that ω is complex, the value of $(1 + \omega - \omega^2)^3$ is

A) -8 B) 8 C) 1 D) -1

10. Given that $(x-1)p(x) = 16x^5 - 20x^3 + 5x - 1$, then if $p(x) = (4x^2 + ax - 1)^2$, the value of a is:

A) 1 B) 2 C) $\frac{1}{2}$ D) 0

Section II

90 marks
Attempt Questions 11-16
Allow about 2 hours and 45 minutes for this section.

Answer each question on a new page.

In question 11-16, your responses should include relevant mathematical reasoning and/or calculations

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Question 11 (15 marks)

a) Find (i) $\int (\sec x + \tan x)^2 \, dx$ **2**

(ii) $\int \left(\frac{1-x}{1-\sqrt{x}} \right) dx$ **2**

(iii) $\int \frac{e^{2x}}{(e^x+1)^2} \, dx$ **2**

b) Use the substitution $t = \tan\frac{x}{2}$ to evaluate $\int_0^{\frac{\pi}{2}} \frac{1}{\cos x + 2\sin x + 3} dx$ **4**

leaving your answer in exact form.

c) (i) Use De Moivre's theorem to show that **2**

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$ given

$(a+b)^4 = a^4 + 4a^3b + 6a^2b^2 + 4ab^3 + b^4.$

(ii) Find the exact value of $\cos^4\left(\frac{\pi}{12}\right) + \sin^4\left(\frac{\pi}{12}\right)$ **3**

End of Question 11

Question 12 (15 marks) Start a new page

a) The polynomial $P(x) = x^3 - 6x^2 + 9x + c$ has a double zero. **2**

Find the value(s) of c.

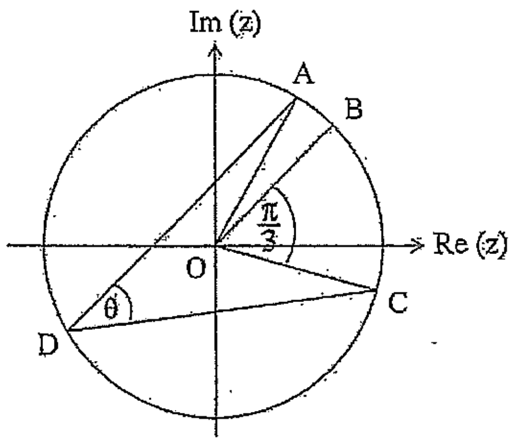
b) Consider the curve defined by the equation $3x^2 + y^2 - 2xy - 8x + 2 = 0$.

(i) Show that $\dfrac{dy}{dx} = \dfrac{3x - y - 4}{x - y}$ **2**

(ii) Find the coordinates of the points on the curve where the tangent to the curve is parallel to the line $y = 2x$. **3**

c) The points A, B, C and D representing the complex numbers $\sqrt{2} + \sqrt{6}i$, $2 + 2i$, ω and ϕ lie on a circle with centre O.

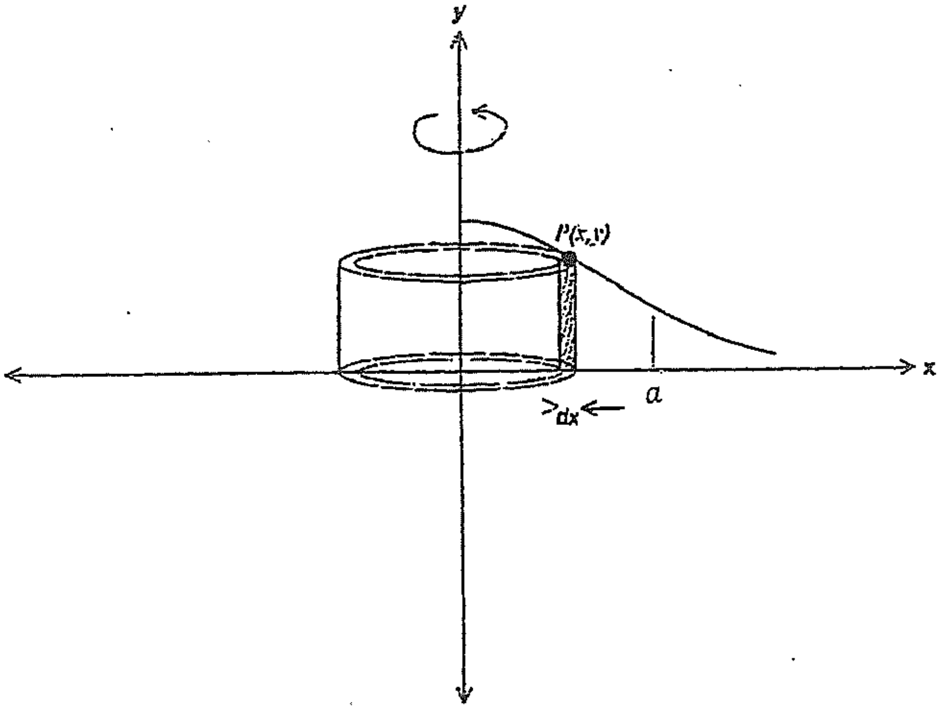
$\angle BOC = \frac{\pi}{3}$ and $\angle ADC = \theta$.



(i) Find ω in the form $x + iy$, where x and y are real numbers. **2**

(ii) Find the value of θ and give a reason. **2**

d) The graph of $y = e^{-x^2}$ is shown below. The graph has a horizontal asymptote at $y = 0$. The region between the curve for $0 \leq x \leq a$, the y axis and the x axis is rotated about the y axis, to form a solid. The volume of this solid is to be found using the cylindrical shell method.



(i) State the volume δV of a typical cylindrical shell shown in the diagram. **1**

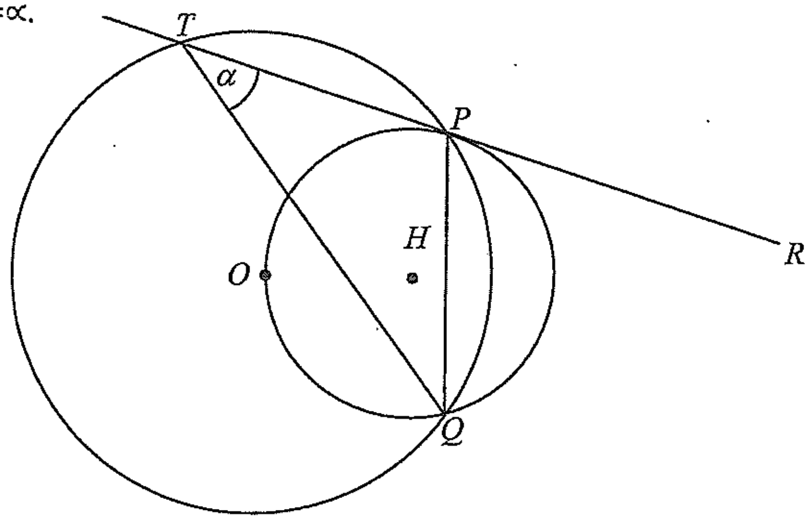
(ii) Find the volume of the solid. **2**

(iii) What is the limiting value of the solid as $a \to \infty$. **1**

End of Question 12

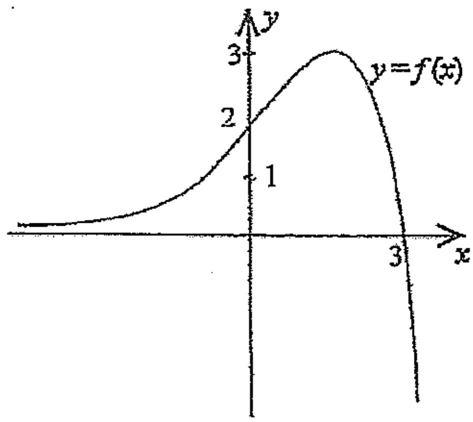
Question 13 (15 marks) **Start a new page**

a) Two circles intersect at P and Q as shown. The centre of the smaller circle is H and the centre of the larger circle is O. The smaller circle passes through the centre of the larger circle. The tangent to the smaller circle at P, RPT, cuts the larger circle at T. The chord in the smaller circle, PQ, bisects $\angle RQO$. Let $\angle PTQ = \alpha$. 3



Prove that ΔPQT is isosceles.

b)



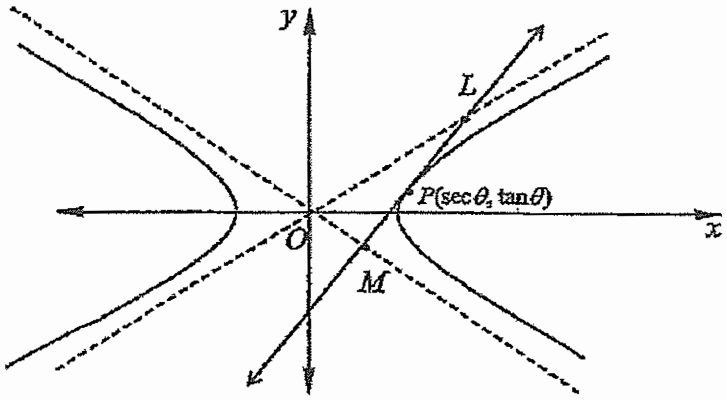
On separate diagrams of approximately one third of a page, sketch showing all important features (on each graph include $y = f(x)$ as a dotted curve):

(i) $y = f(|x|)$ 2

(ii) $y = [f(x)]^3$ 2

(iii) $y = \ln[f(x)]$ 2

c) The diagram shows the hyperbola with equation $x^2 - y^2 = 1$.



NOT TO SCALE

(i) Given that the equation of the tangent to this hyperbola at the point $P(sec\theta, tan\theta)$ is **3**

$$xsec\theta - ytan\theta = 1,$$

find the coordinates of L and M, where the tangent crosses the asymptotes.

(ii) Show that the area of ΔOLM, where O is the origin, is independent of the position of P. **3**

End of Question 13

Question 14 (15 marks) **Start a new page**

a) The equation of a chord of contact drawn from an external point (x_0, y_0) to the ellipse $\frac{x^2}{15} + \frac{y^2}{10} = 1$, is $5x + 6y - 15 = 0$. **2**

Find the coordinates (x_0, y_0).

b) Illustrate the loci below in separate diagrams on the complex plane, showing all important features

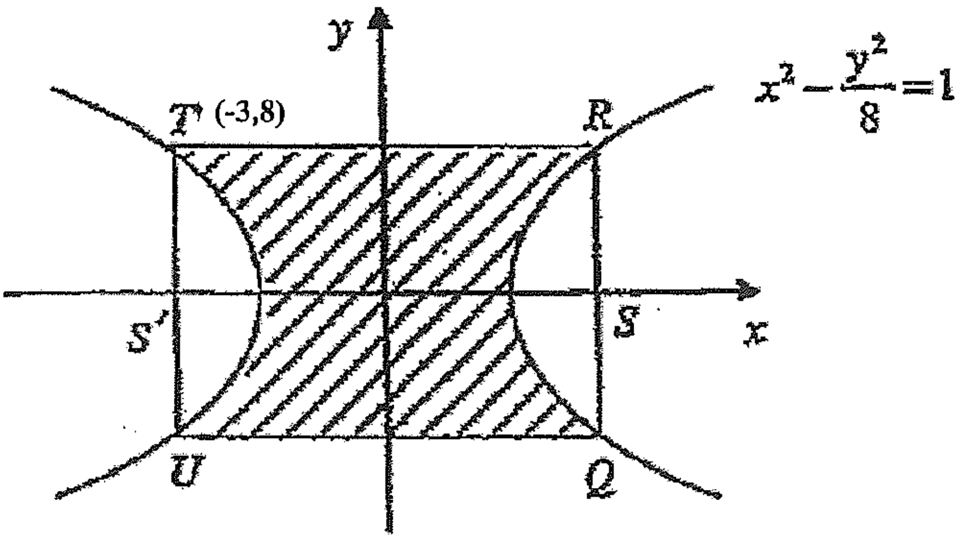
(i) $|z| = |z - 6 - 4i\ |$ **1**

(ii) $\arg(z-1) = \arg(z+1) + \frac{\pi}{4}$ **3**

(iii) $\arg(z+2) = \arg(z - 2 - 5i)$ **3**

c) Solve the equation $x^4 + 2x^3 + x^2 - 1 = 0$, given that one root is $-\frac{1}{2} + i\frac{\sqrt{3}}{2}$. **3**

d)



The base of a particular solid is the shaded region above. Every cross section of the solid perpendicular to the y-axis is an equilateral triangle with one side in the base of the solid. Find the exact volume of the solid. **3**

End of Question 14

Question 15 (15 marks) **Start a new page**

a) (i) Let n be a positive integer and let $I_n = \int_1^2 (\log_e x)^n \; dx$. **3**

Prove that $I_n = 2(\log_e 2)^n - nI_{n-1}$

(ii) Evaluate $\int_1^2 (\log_e x)^3 \; dx$ and leave your answer in exact form as a **3**

polynomial in terms of $\log_e 2$

b) Find the general solution to **1**

$\tan(2\theta - \frac{\pi}{3}) = 1$

c) A particle P of mass $m\ kg$ is projected vertically upwards with speed $U\ m/s$ in a medium in which the resistance to motion has magnitude $\frac{1}{10}mv^2$ when the speed of the particle is $v\ m/s$. After t seconds the particle has height x metres, velocity $v\ m/s$ and acceleration $a\ m/s^2$. Assume $g = 10\ m/s^2$.

(i) Draw a diagram showing forces acting on the particle P, and hence **2**

show that $a = -\left(\frac{v^2+100}{10}\right)$.

(ii) Find, in terms of U, the time taken for the particle to reach the maximum height. **3**

(iii) Find the maximum height in terms of U. **3**

End of Question 15

Question 16 (15 marks) **Start a new page**

a) (i) Prove that $\int_0^a f(x)dx = \int_0^a f(a-x)dx$ **2**

(ii) Show that $\int_0^{\pi} x\cos^2 x dx = \frac{\pi^2}{4}$ **2**

b) If $f(x) = x - \log_e(1 + x + \frac{x^2}{2})$

(i) Show that $f(x)$ is an increasing function for all x. **2**

(ii) Show $e^x < 1 + x + \frac{x^2}{2}$ for $x < 0$ **2**

c) Prove by mathematical induction that for integers $n \geq 1$: **3**

$$1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{n^2} \leq 2 - \frac{1}{n}$$

d) $P(x)$ is a polynomial of degree n with rational coefficients. If the leading coefficient is a_o and $\gamma_1, \gamma_2, \gamma_3, \ldots\ldots \gamma_n$ are the roots of $P(x) = 0$ prove that: **4**

$$P^{1}(x) = \frac{P(x)}{x-\gamma_1} + \frac{P(x)}{x-\gamma_2} + \frac{P(x)}{x-\gamma_3} + \ldots\ldots + \frac{P(x)}{x-\gamma_n}$$

End of Examination

Student Name: STUDENT'S Teacher Name: COPY

2018 Extension 2 Trial Solutions

Section 1 M/C

1. $\sqrt{3}+i = 2\operatorname{cis}\frac{\pi}{6}$

$\left[2\operatorname{cis}\frac{\pi}{6}\right]^4$

$16\operatorname{cis}\frac{2\pi}{3}$

$16\left(\cos\frac{2\pi}{3}+i\sin\frac{2\pi}{3}\right)$

$16\left(\frac{-1}{2}+i\frac{\sqrt{3}}{2}\right)$

$-8+8\sqrt{3}i$

D

2. $4x^2+y^2=4$

$\frac{x^2}{1}+\frac{y^2}{4}=1$

$\Rightarrow a^2=b^2(1-e^2)$

$1=4(1-e^2)$

$\frac{1}{4}=1-e^2$

$e^2=\frac{3}{4}$

$e=\frac{\pm\sqrt{3}}{2}$

$\therefore$ Focii $\left(0, \pm 2\times\frac{\sqrt{3}}{2}\right)$

B

3. $x^3-5x+4=0$

$\alpha^2+\beta^2+\gamma^2=(\alpha+\beta+\alpha)^2-2(\alpha\beta+\alpha\gamma+\beta\gamma)$

$= 0-2\times\frac{-5}{1}$

$= 10$ is B

4. C

5. $\left[\frac{-(\cos^{-1}x)^2}{2}\right]_0^1$

$\frac{-0^2-\left(-\left(\frac{\pi}{2}\right)^2\right)}{2}$

C

6. $\delta V=2\pi(2-y)(4-x)\delta y$

$\delta V=2\pi(2-y)(4-y^2)\delta y$

$V=2\pi\int_0^2(2-y)(4-y^2)\,dy$

7. $m\ddot{x}=-mk(v+v^2)$

$v\frac{dv}{dx}=-k(v+v^2)$

$\frac{-v}{k(v+v^2)}=dx$

$x=-\frac{1}{k}\int\frac{1}{1+v}\,dv$ D

8. $A(2ct,0)$ $B\left(0,\frac{2c}{t}\right)$

P is midpoint of AB

$\therefore$ D is false

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9. $(1+\omega-\omega^2)^3$

$\omega^3+1=0$

$(\omega+1)(\omega^2-\omega+1)=0$

$\therefore \omega^2-\omega+1=0$

$\therefore (1+\omega-\omega^2)^3$

$=(1+\omega-(\omega-1))^3$

$=8 \quad \therefore B$

10. $(x-1)p(x)=16x^5-20x^3+5x-1$

$p(x)=(4x^2+ax-1)^2(x-1)$

Calculate x term by inspection.

$-2ax \times -1 + x = 5$

$2ax + x = 5$

$(2a+1)x = 5$

$a=2 \quad \therefore B$

Section 2

11 a)(i) $\int (\sec x+\tan x)^2\,dx$

$\int \sec^2 x + 2\sec x\tan x + \tan^2 x\,dx$

$\int 2\sec^2 x + 2\sec x\tan x - 1\,dx$

$= 2\tan x + 2\sec x - x + c$

(ii) $\int \frac{1-x}{1-\sqrt{x}}\,dx$

$\int \frac{\cancel{(1-\sqrt{x})}(1+\sqrt{x})}{\cancel{1-\sqrt{x}}}\,dx$

$x+\frac{2}{3}x^{\frac{3}{2}}+c$

b) $\int \frac{e^x \times e^x}{(e^x+1)^2}\,dx$ $\quad u=e^x+1$, $du=e^x\,dx$

$\int \frac{u-1}{u^2}\,du$

$\int \frac{1}{u}-\frac{1}{u^2}\,du$

$=\log_e u+\frac{1}{u}+c$

$=\log_e(e^x+1)+\frac{1}{e^x+1}+c$

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b) $$\int \frac{1}{\frac{1-t^2}{1+t^2} + \frac{2\times 2t}{1+t^2} + 3} \times \frac{2dt}{1+t^2}$$

$$\int \frac{2dt}{1-\cancel{t^2}+4t+3+3t^2}$$

$$\int \frac{2dt}{2t^2+4t+4}$$

$$\int \frac{dt}{t^2+2t+2}$$

$$\int \frac{dt}{(t+1)^2+1}$$

$$\left[\tan^{-1}\left(\tan\frac{x}{2}+1\right)\right]_0^{\frac{\pi}{2}}$$

$$\tan^{-1}(1+1) - \tan^{-1}(0+1)$$

$$\tan^{-1} 2 \quad - \frac{\pi}{4}$$

c)(i) $(\cos\theta + i\sin\theta)^4 = \cos 4\theta + i\sin 4\theta$

$\cos^4\theta + 4a^3 i\sin\theta + 6\cos^2\theta \times -\sin^2\theta + 4\cos\theta\, i^3\sin^3\theta + \sin^4\theta$

Equating real parts

$\cos 4\theta = \cos^4\theta - 6\cos^2\theta\sin^2\theta + \sin^4\theta$

c(ii) $\therefore \cos^4\theta + \sin^4\theta = \cos 4\theta + 6\cos^2\theta\sin^2\theta$

$$\cos^4\left(\frac{\pi}{12}\right) + \sin^4\left(\frac{\pi}{12}\right) = \cos\frac{\pi}{3} + 6\cos^2\frac{\pi}{12}\sin^2\frac{\pi}{12}$$

$$= \frac{1}{2} + \frac{3}{2}\left(\sin\frac{\pi}{6}\right)^2$$

$$= \frac{1}{2} + \frac{3}{2} \times \frac{1}{4}$$

$$= \frac{7}{8}$$

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12 a) $P(x) = x^3 - 6x^2 + 9x + c$

$P'(x) = 3x^2 - 12x + 9 = 0$ for zeros

$x^2 - 4x + 3 = 0$

$(x-1)(x-3) = 0$

$x = 1$ or 3

$\therefore P(1) = 1^3 - 6 + 9 + c = 0 \Rightarrow c = -4$ or

$P(3) = 3^3 - 6\times3^2 + 9\times3 + c = 0 \Rightarrow c = 0$

b) (i) $3x^2 + y^2 - 2xy - 8x + 2 = 0$ differentiating gives

$6x + 2y\frac{dy}{dx} - 2y - 2x\frac{dy}{dx} - 8 = 0$

$3x + \frac{dy}{dx}(y - x) - y - 4 = 0$

$\frac{dy}{dx}(y-x) = \frac{y - 3x + 4}{1}$

$\frac{dy}{dx} = \frac{y - 3x + 4}{y - x}$ or $\frac{3x - y - 4}{x - y}$

(ii) $\frac{dy}{dx} = 2$

$\therefore \frac{3x - y - 4}{x - y} = 2$

$3x - y - 4 = 2x - 2y$

$x = 4 - y$

sub. into original curve

$3(4-y)^2 + y^2 - 2y(4-y) - 8(4-y) + 2 = 0$

$3(16 - 8y + y^2) + y^2 - 8y + 2y^2 - 32 + 8y + 2 = 0$

$48 - 24y + 3y^2 + y^2 - \cancel{8y} + 2y^2 - 32 + \cancel{8y} + 2 = 0$

$6y^2 - 24y + 18 = 0$

$y^2 - 4y + 3 = 0$

$(y-1)(y-3) = 0$

$y = 1$ or 3 $\therefore x = 3$ or 1

$\therefore (3,1)$ and $(1,3)$.

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c)(i) ω is $\text{cis}\left(\frac{-\pi}{3}\right) \times \sqrt{8}\,\text{cis}\frac{\pi}{4}$

$= \sqrt{8}\,\text{cis}\left(\frac{-\pi}{12}\right)$

$= \sqrt{8}\cos\left(\frac{-\pi}{12}\right) + \sqrt{8}\,i\sin\left(\frac{-\pi}{12}\right)$

(ii) Argument of A is $\tan^{-1}\frac{\sqrt{6}}{\sqrt{2}} = \tan^{-1}\sqrt{3} = \frac{\pi}{3}$

$\therefore \angle AOC = \frac{\pi}{3} + \frac{\pi}{12}$

$= \frac{5\pi}{12}$

$\therefore \angle ADC = \frac{5\pi}{24}$ radians (angle at centre of a circle is twice angle at circumference when standing on the same arc)

d.(i) $\delta V = 2\pi r h \times \delta x$

$= 2\pi \times x \times f(x)\,\delta x$

$= 2\pi \times x \times e^{-x^2}\delta x$

(ii) $V = \sum_{x=0}^{a} 2\pi x \times e^{-x^2}\delta x$

$= 2\pi\int_0^a x e^{-x^2}\,dx$

$= -\pi\left[e^{-x^2}\right]_0^a$

$= -\pi\left[e^{-a^2} - 1\right]$

$= \pi\left[1 - \frac{1}{e^{a^2}}\right]$

(iii) As $a \to \infty$

$V \to \pi$ as $\frac{1}{e^{a^2}} \to 0$

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13. a) Join OP and OQ

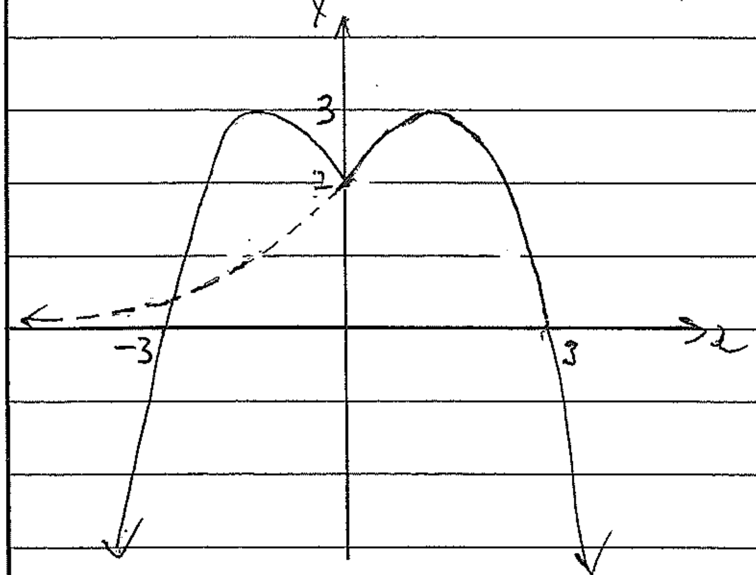
$\angle QOP = 2\alpha$ (Angle at centre equals twice angle at circumference standing on same arc)

$\angle QPR = 2\alpha$ (Alternate segment theorem)

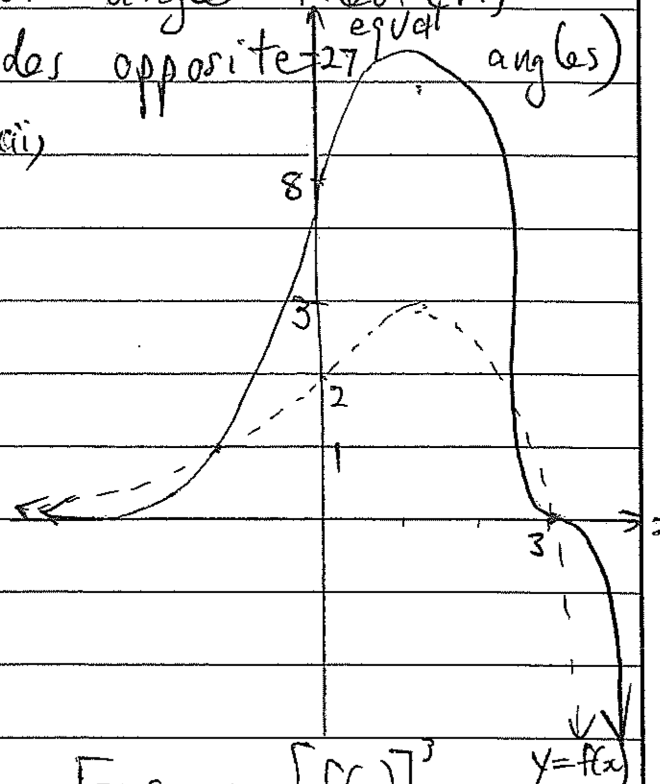
$\angle PQT = \alpha$ (Exterior angle theorem)

$PT = PQ$ (Equal sides opposite equal angles)

b) (i) $y = f(|x|)$



(ii)

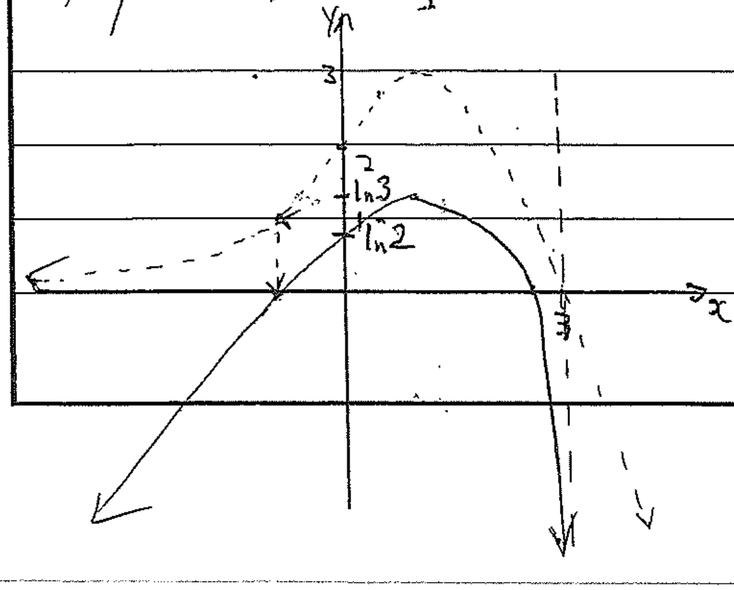


For $y = [f(x)]^3$

$y' = 3[f(x)]^2 f'(x)$

$\therefore$ Stationary pts where $f(x) = f'(x) = 0$

(iii) $y = \ln[f(x)]$



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c)(i) Equations of asymptotes are $y = \pm x$.

Solving simultaneously:

$x\sec\theta - y\tan\theta = 1$ and $x\sec\theta - y\tan\theta = 1$

$y = x$ $\quad$ $y = -x$

$\therefore x\sec\theta - x\tan\theta = 1$ $\quad$ $x\sec\theta + x\tan\theta = 1$

$x(\sec\theta - \tan\theta) = 1$ $\quad$ $x(\sec\theta + \tan\theta) = 1$

$x = \dfrac{1}{\sec\theta - \tan\theta}$ $\quad$ $x = \dfrac{1}{\sec\theta + \tan\theta}$

$\therefore L\left(\dfrac{1}{\sec\theta - \tan\theta}, \dfrac{1}{\sec\theta - \tan\theta}\right)$ $\quad$ $M\left(\dfrac{1}{\sec\theta + \tan\theta}, \dfrac{-1}{\sec\theta + \tan\theta}\right)$

(ii) $A = \frac{1}{2} \times OM \times OL$

$= \frac{1}{2} \times \sqrt{\left(\dfrac{1}{\sec\theta + \tan\theta}\right)^2 + \left(\dfrac{1}{\sec\theta + \tan\theta}\right)^2} \times \sqrt{\dfrac{1}{(\sec\theta - \tan\theta)^2} + \dfrac{1}{(\sec\theta - \tan\theta)^2}}$

$= \dfrac{1}{\sec\theta + \tan\theta} \times \dfrac{1}{\sec\theta - \tan\theta}$

$= \dfrac{1}{\sec^2\theta - \tan^2\theta}$

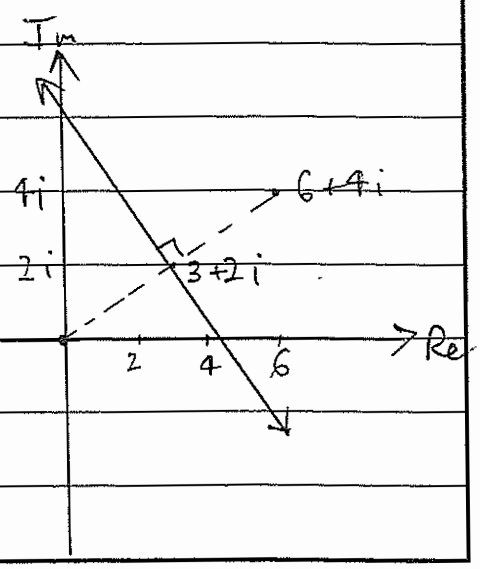
$= \dfrac{1}{1}$

$= 1 \quad \therefore$ independent of P.

14a) $5x + 6y = 15$ $\quad$ b(i)

$\dfrac{5x}{15} + \dfrac{6y}{15} = 1$

$\dfrac{5x}{15} + \dfrac{4y}{10} = 1$



Now as general form of chord of contact is $\dfrac{xx_0}{a^2} + \dfrac{yy_0}{b^2} = 1$ and

$a^2 = 15, \; b^2 = 10$

$\therefore (x_0, y_0)$ is $(5, 4)$

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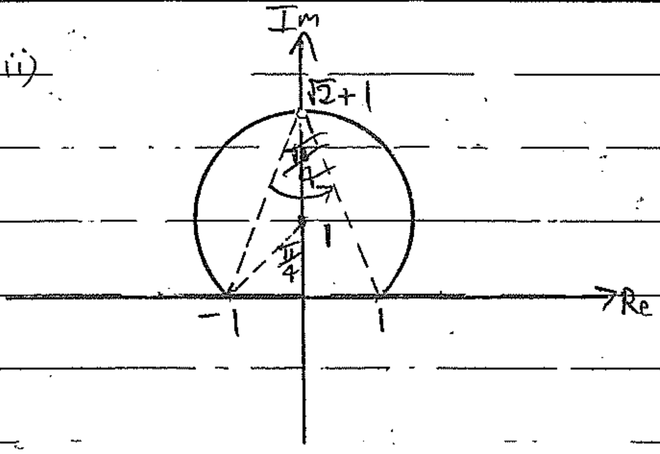
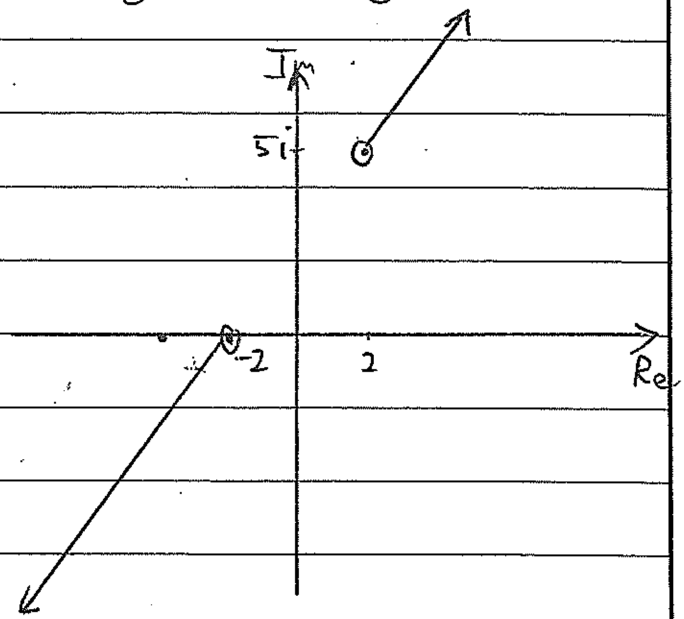
(ii)



c(iii) $\arg(z+2)=\arg(z-(2+5i))$



c) $x^4+2x^3+x^2-1=0$ has real coefficients

$\therefore$ complex roots occur in conjugate pairs

ie: $x=-\frac{1}{2}\pm i\frac{\sqrt{3}}{2}$ are roots

$\alpha+\beta$ sum is -1 $\quad \alpha\beta$ product is $\frac{1}{4}+\frac{3}{4}=1$

Quadratic factor is $x^2-(\alpha+\beta)x+\alpha\beta$

$x^2--1x+1=x^2+x+1$

Now by inspection,

$x^4+2x^3+x^2-1=(x^2+x+1)(x^2+x-1)$

Roots of x^2+x-1 are $x=\frac{-1\pm\sqrt{1-4\times-1}}{2}$

$=\frac{-1\pm\sqrt{5}}{2}$

$\therefore$ Roots are $\frac{-1}{2}\pm i\frac{\sqrt{3}}{2}$ and $\frac{-1\pm\sqrt{5}}{2}$

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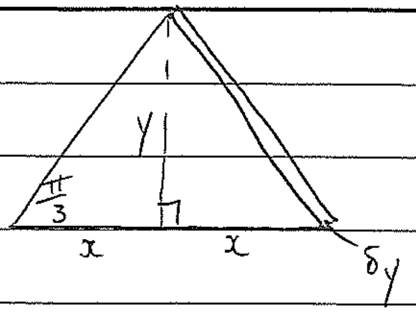
d)



$\tan\frac{\pi}{3} = \frac{y}{x}$

$x\sqrt{3} = y$

$\therefore \delta V = \frac{1}{2}\times 2x \times x\sqrt{3} \times \delta y$

$= \sqrt{3}\,x^2\,\delta y$

$= \sqrt{3}\left(1+\frac{y^2}{8}\right)\delta y$

$V = 2\int_0^8 \sqrt{3}\left(1+\frac{y^2}{8}\right)dy$

$= 2\sqrt{3}\left[y + \frac{y^3}{24}\right]_0^8$

$= 2\sqrt{3}\left[8 + \frac{8^3}{24}\right]$

$= \frac{176\sqrt{3}}{3}$ units^3

15. a (i) $I_n = \int_1^2 (\log_e x)^n\,dx$

$= \int_1^2 \frac{d}{dx}(x)\,(\log_e x)^n\,dx$

$= \left[x(\log_e x)^n\right]_1^2 - \int_1^2 \cancel{x} \times n(\log_e x)^{n-1} \times \frac{1}{\cancel{x}}\,dx$

$= 2(\log_e 2)^n - n\int_1^2 (\log_e x)^{n-1}\,dx$

$I_n = 2(\log_e 2)^n - nI_{n-1}$ as req'd.

(ii) $\int_1^2 (\log_e x)^3\,dx = I_3 = 2(\log_e 2)^3 - 3I_2$

$= 2(\log_e 2)^3 - 3\left[2(\log_e 2)^2 - 2I_1\right]$

$= 2(\log_e 2)^3 - 6(\log_e 2)^2 + 6\left[2\log_e 2 - I_0\right]$

$= 2(\log_e 2)^3 - 6(\log_e 2)^2 + 12\log_e 2 - 6\int_1^2 dx$

$= 2(\log_e 2)^3 - 6(\log_e 2)^2 + 12\log_e 2 - 6$

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b. $\tan\left(2\theta - \frac{\pi}{3}\right) = 1$

$$2\theta - \frac{\pi}{3} = n\pi + \tan^{-1} 1$$

$$2\theta = n\pi + \frac{7\pi}{12}$$

$$\theta = \frac{n\pi}{2} + \frac{7\pi}{24} \quad \text{for } n = 0, \pm 1, \pm 2 \text{ etc.}$$

c. (i) $\downarrow -mg$

$\downarrow -\frac{1}{10} mv^2$

$P \uparrow$

$$\therefore m\ddot{x} = -mg - \frac{1}{10} mv^2$$

$$\ddot{x} = -g - \frac{v^2}{10}$$

Assuming $g = 10$

$$\ddot{x} = -\left(\frac{100 + v^2}{10}\right) \text{ as req'd.}$$

(ii) $\ddot{x} = \frac{dv}{dt} = -\left(\frac{100 + v^2}{10}\right)$

$$\frac{dt}{dv} = -\frac{10}{100 + v^2}$$

$$dt = \frac{-10\,dv}{100 + v^2}$$

$$t = -\tan^{-1}\left(\frac{v}{10}\right) + C$$

when $t = 0$, $v = U$

$$\therefore C = \tan^{-1}\frac{U}{10}$$

$$\therefore t = \tan^{-1}\frac{U}{10} - \tan^{-1}\left(\frac{v}{10}\right)$$

max. ht. occurs when

$v = 0$

$$\therefore t = \tan^{-1}\frac{U}{10}$$

c(iii) Need x in terms of t ie: use $v = \frac{dx}{dt}$

$$\therefore t = \tan^{-1}\left(\frac{U}{10}\right) - \tan^{-1}\left(\frac{\frac{dx}{dt}}{10}\right)$$

$$\tan^{-1}\left(\frac{\frac{dx}{dt}}{10}\right) = \tan^{-1}\frac{U}{10} - t$$

$$\frac{\frac{dx}{dt}}{10} = \tan\left[\tan^{-1}\frac{U}{10} - t\right]$$

$$\frac{dx}{dt} = 10\tan\left[\tan^{-1}\frac{U}{10} - t\right]$$

$$x = 10\log_e \cos\left(\tan^{-1}\frac{U}{10} - t\right) + C$$

When $t = 0$, $x = 0$

$$\therefore C = -10\log_e \cos\left(\tan^{-1}\frac{U}{10}\right)$$

Now sub. in $t = \tan^{-1}\frac{U}{10}$

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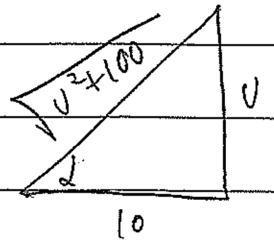
$$\therefore x = 10\log_e \cos\left(\tan^{-1}\frac{v}{10} - \tan^{-1}\frac{v}{10}\right) - 10\log_e \cos\left(\tan^{-1}\frac{v}{10}\right)$$

$$= \cancel{10\log_e 1} - 10\log_e \cos\left(\tan^{-1}\frac{v}{10}\right)$$



$\therefore x = 10\log_e \frac{\sqrt{v^2+100}}{10}$ is maximum height.

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16a) (i) $I_n \quad \int_0^a f(a-x)\,dx$

Let $u = a - x$ when $x = 0$ $u = a$

$\therefore du = -dx$ when $x = a$ $u = 0$

$\Rightarrow \int_0^a f(a-x)\,dx$

$= \int_a^0 f(u) \times -du$

$= \int_0^a f(u)\,du = \int_0^a f(x)\,dx$ (variable irrelevant)

(ii) $\int_0^{\pi} x\cos^2 x\,dx = \int_0^{\pi} (\pi - x)\cos^2(\pi - x)\,dx$

$= \int_0^{\pi} (\pi - x)\cos^2 x\,dx$

$= \pi\int_0^{\pi} \cos^2 x\,dx - \int_0^{\pi} x\cos^2 x\,dx$

$\therefore 2\int_0^{\pi} x\cos^2 x\,dx = \pi\int_0^{\pi} \cos^2 x\,dx$

$= \pi\int_0^{\pi} \frac{\cos 2x + 1}{2}\,dx$

$= \frac{\pi}{2}\left[\frac{1}{2}\sin 2x + x\right]_0^{\pi}$

$= \frac{\pi}{2}\left[\pi\right]$

$\therefore \int_0^{\pi} x\cos^2 x\,dx = \frac{\pi^2}{4}$

b) (i) $f(x) = x - \log_e\left(1 + x + \frac{x^2}{2}\right)$

$f'(x) = 1 - \frac{1 + x}{1 + x + \frac{x^2}{2}}$

$= \frac{\cancel{1} + x + \frac{x^2}{2} - \cancel{1} - x}{1 + x + \frac{x^2}{2}} \times \frac{2}{2}$

$= \frac{x^2}{x^2 + 2x + 2} = \frac{x^2}{(x+1)^2 + 1} > 0$ for all x.

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$\therefore$ increasing for $x < 0$.

(ii) $f(0) = 0 \quad \therefore$ for $x < 0$, $f(x) < 0$

$$x - \log_e\left(1 + x + \frac{x^2}{2}\right) < 0$$

$$\log_e\left(1 + x + \frac{x^2}{2}\right) > x$$

$$1 + x + \frac{x^2}{2} > e^x \text{ as required.}$$

c) Show true for $n = 1$

$$\frac{1}{1^2} \leq 2 - \frac{1}{1} \text{ is true}$$

Assume true for $n = k$, k integral ≥ 1.

ie: $1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{k^2} \leq 2 - \frac{1}{k}$

Need to show true for $n = k+1$

From assumption,

$$\begin{aligned}
1 + \frac{1}{2^2} + \frac{1}{3^2} + \ldots\ldots + \frac{1}{k^2} + \frac{1}{(k+1)^2} &\leq 2 - \frac{1}{k} + \frac{1}{(k+1)^2} \\
&\leq 2 - \left(\frac{1}{k} - \frac{1}{(k+1)^2}\right) \\
&\leq 2 - \left(\frac{(k+1)^2 - k}{k(k+1)^2}\right) \\
&\leq 2 - \left(\frac{k^2 + 2k + 1 - k}{k(k+1)^2}\right) \\
&\leq 2 - \left(\frac{k^2 + k + 1}{k(k+1)^2}\right) \\
&\leq 2 - \left(\frac{k^2 + k}{k(k+1)^2} + \frac{1}{k(k+1)^2}\right) \\
&\leq 2 - \frac{\cancel{k(k+1)}}{\cancel{k(k+1)^2}} \text{ since } \cancel{\frac{1}{k(k+1)^2}} \\
&\leq 2 - \frac{1}{k+1}
\end{aligned}$$

as required

Since result is true for $n = 1$, it must be true for $n = 1+1 = 2$, $n = 2+1 = 3$ and hence all positive integral values of n.

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d) $P(x) = a_0(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_n)$

$\log_e P(x) = \log_e[a(x-\alpha_1)(x-\alpha_2)\ldots\ldots(x-\alpha_n)]$

$\log_e P(x) = \log_e a + \log_e(x-\alpha_1) + \log_e(x-\alpha_2)\ldots\ldots + \log_e(x-\alpha_n)$

Differentiating both sides gives

$$\frac{P'(x)}{P(x)} = 0 + \frac{1}{x-\alpha_1} + \frac{1}{x-\alpha_2} + \ldots + \frac{1}{x-\alpha_n}$$

$$\therefore P'(x) = \frac{P(x)}{x-\alpha_1} + \frac{P(x)}{x-\alpha_2} + \ldots + \frac{P(x)}{x-\alpha_n}$$

as required.

2018 Extension 2 Examiner's Comments

GENERAL COMMENTS:

- *Poor writing needs to be addressed – it may very well cost you at the HSC*
- *Notation **IS** maths – it needs to be done properly -* $\int x^2\, dx$ *not* $\int x^2$

Question 11:

(a) (iii) various answers (only 1 in your solutions) – depending upon the substitution made and the method of integration.

(c) (i) Needed to state that $(cosx + isinx)^4 = cos4x + isin4x$. You cannot just throw it into the solution stating "by comparing Real values" when you haven't said where the *cos4x* came from in the first place.

(ii) The intention for this question was that you had to use part (i). In exams parts of a question (ie (c) (i) and (c) (ii)) are nearly always related to each other. There were, of course, other ways of doing it.

Question 12

a. care with algebra and showing both values of C were the only issues

b. parallel if dy/dx = 2 then needed to generate a relationship that once substituted back into the CURVE could be solved many students generated more than the two answers

c. Using a diagram would have helped find the arg of the point relating to C

ii. Using a diagram would have helped find the correct angle <AOC then find < ADC with a reason

d. i. Poor understanding of ΔV. The need to have the width of the shell using Δx was the most common issue here

ii. Poor integration skills in some cases but overall it was fine (Note some students spent time doing integration by substitution and incorrectly changing the limits even though this was not the best method)

Question 13

(a) A lot of people stated that RQ was a tangent to the smaller circle. This was never mentioned and made some proofs untrue. The actual proof was quite easy. People should NOT expect to have to do extra constructions on their diagrams (ie "invent" other points). If you do, you need to have an accompanying diagram otherwise the marker has no idea!

(b) These were much improved from previous efforts. You need to show the values of important turning points which came from the original. *ln*(small numbers) becomes very negative (and quickly) (cf. the graph of *y=lnx*)

(c) (ii) Recognising that the two asymptotes in this case were mutually perpendicular meant that the area of the triangle was $\frac{1}{2}\ OL.OM$, and the value was 1 (ie no Θ)
Finding the perpendicular distance of O from LM, just made for complicated algebra.

Question 14

a) Poor algebra skills resulted in students not obtaining full marks.

b) Graph Questions

i) Candidates needed to show the perpendicular line to obtain the one mark.

ii) open circles required, needed to show that an angle of $\frac{\pi}{4}$.
Common error made by students was showing the wrong centre of the circle.
Centre of the circle is (0,1).

iii) Students graphed a straight line which was incorrect.
Two open circle were required at (2,5) and (-2,0) and rays on either side.

c) The easiest way to solve the equation was to use the conjugate theorem then factorise. Use of long division would allow the students to find the other 2 roots. Many different ways were used however unsuccessful in obtaining full marks.

d) Students needed to draw a diagram to assist with the question.
Many found the height of the triangle and then struggled finding the volume.
Other students found the volume parallel to the y-axis. Reading the question and highlighting what is required, is highly recommended.

Question 15

a. i. please use the correct notation esp. in a show that question – the limits need to be on the integral to show I_{n-1}

ii. starting with I_3 and then working to I_2 etc was best as students doing this made less errors with coefficients and signs. Poor understanding of log laws was lao something that appeared – $2(\ln 2)^3$ is NOT equal to $6(\ln 2)$

b. Use the rule that is on the reference sheet and understand that the general solution formula is used as soon as the inverse tan is applied (see solution)

c. interestingly a lot of students did part iii accidentally before part ii – becareful you label the questions.

Show the steps and ensure that what you are writing can be read.

Question 16

a(i) To those who forgot, learn this simple technique!

(ii) A number of students did integration by parts, which was twice the work. All multiple part questions are related so use part (i).

b(i) To score 2 marks you needed to explicitly show why quadratic denominator of derivative is positive, either by completing the square or using the discriminant.

(ii) Show that the y intercept is (0,0). Then you can state that since the function is increasing, $f(x)<0$ for $x<0$.

c Don't start with required result and work backwards. Start with assumption and add (k+1)th term to both sides and deduce RHS with reasoning(preferable). Also when simplifying keep the 2 separate to the algebraic fraction as the required result contains 2- …… Better organisation of the Induction proof by most students is required.

d One mark for writing polynomial in factorised form including the leading coefficient a. Marks 2,3 and 4 for taking the log of both sides, differentiating then multiplying both sides by P(x). Product rule attempts make no mathematical sense as you then need to use the quotient rule, clutching straws.