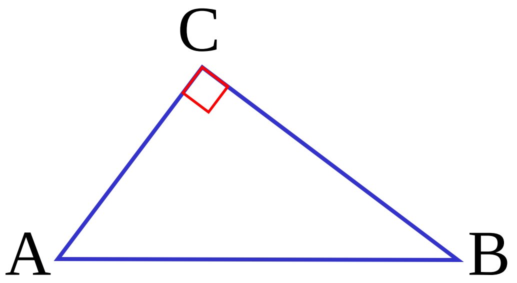
Harder Extension 1

Circle Geometry

Converse Circle Theorems

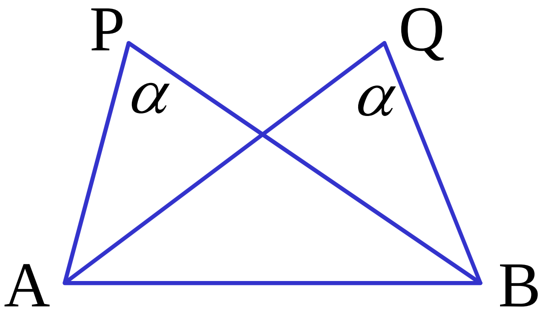
(1) The circle whose diameter is the hypotenuse of a right angled triangle passes through the third vertex.



ABC are concyclic with AB diameter

$(\angle \text{ in a semicircle} = 90^\circ)$

(2) If an interval AB subtends the same angle at two points P and Q on the same side of AB, then A,B,P,Q are concyclic.



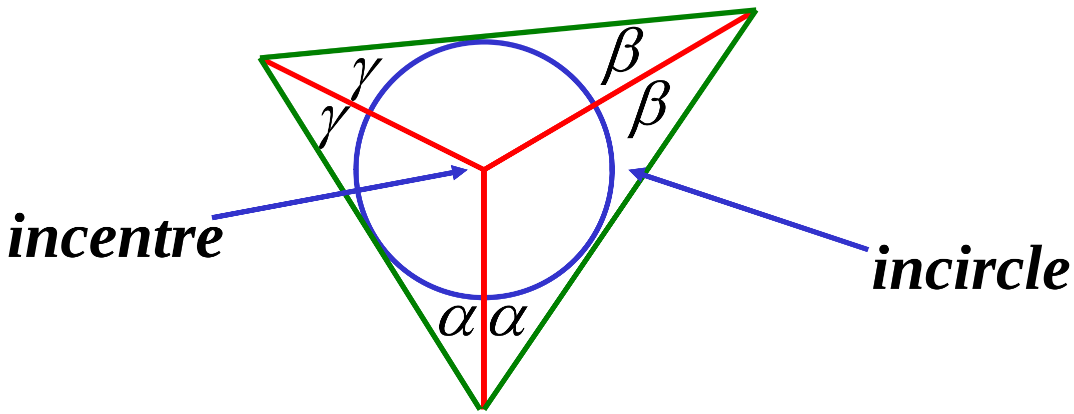
ABQP is a cyclic quadrilateral

$(\angle' \text{s in same segment are} =)$

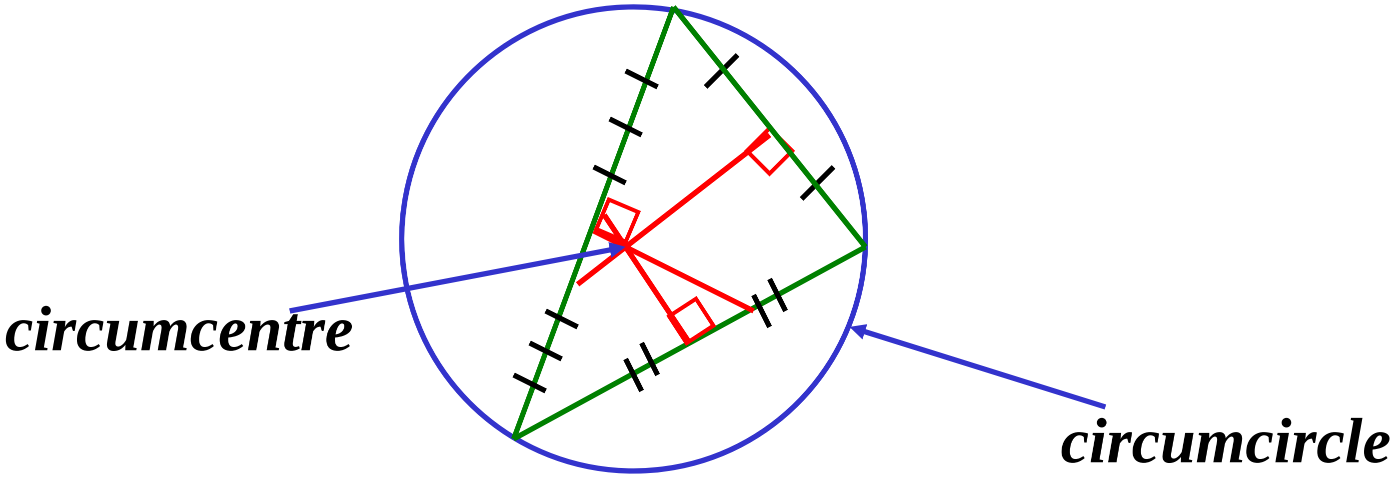
(3) If a pair of opposite angles in a quadrilateral are supplementary (or if an exterior angle equals the opposite interior angle) then the quadrilateral is cyclic.

The Four Centres Of A Triangle

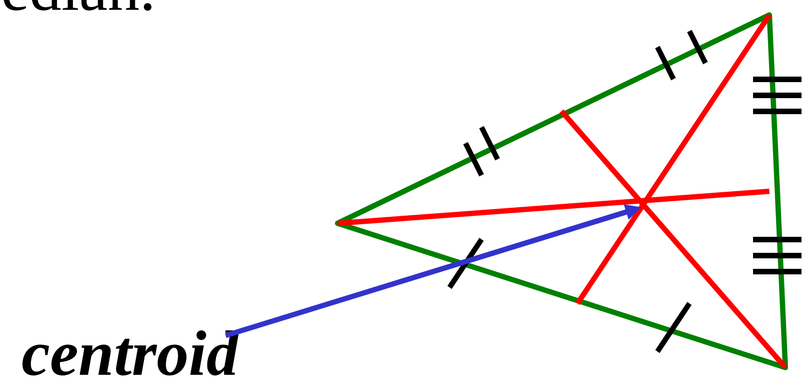
(1) The angle bisectors of the vertices are concurrent at the ***incentre*** which is the centre of the ***incircle***, tangent to all three sides.



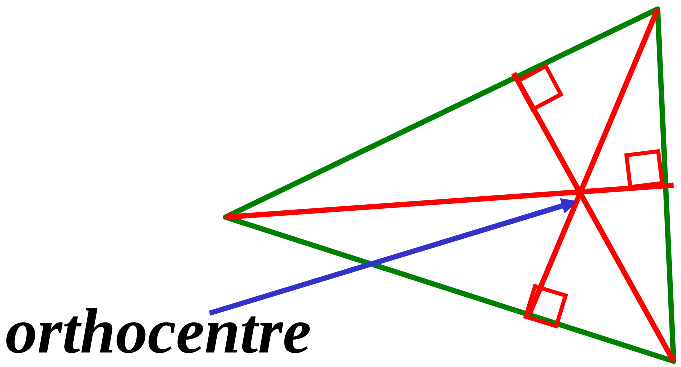
(2) The perpendicular bisectors of the sides are concurrent at the ***circumcentre*** which is the centre of the ***circumcircle***, passing through all three vertices.



(3) The medians are concurrent at the ***centroid***, and the centroid trisects each median.



(4) The altitudes are concurrent at the ***orthocentre***.

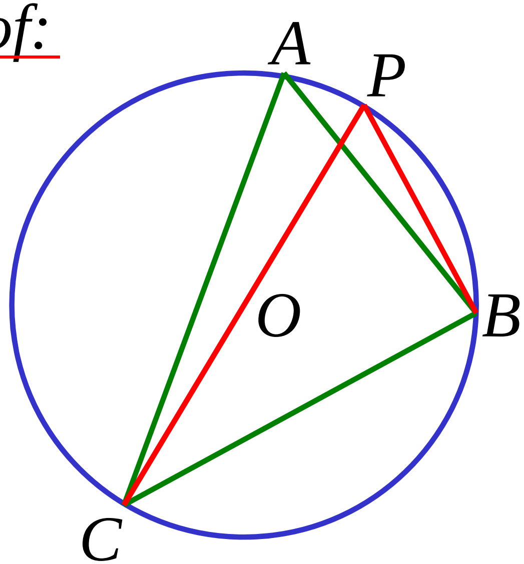


Note: the circumcentre, centroid and orthocentre are ***collinear*** *on Euler's line*

Interaction Between Geometry & Trigonometry

$$\frac{a}{\sin A} = \frac{b}{\sin B} = \frac{c}{\sin C} = \text{diameter if circumcircle}$$

Proof:



$\angle A = \angle P$ $\quad (\angle \text{ in same segment} =)$

$\therefore \sin A = \sin P$

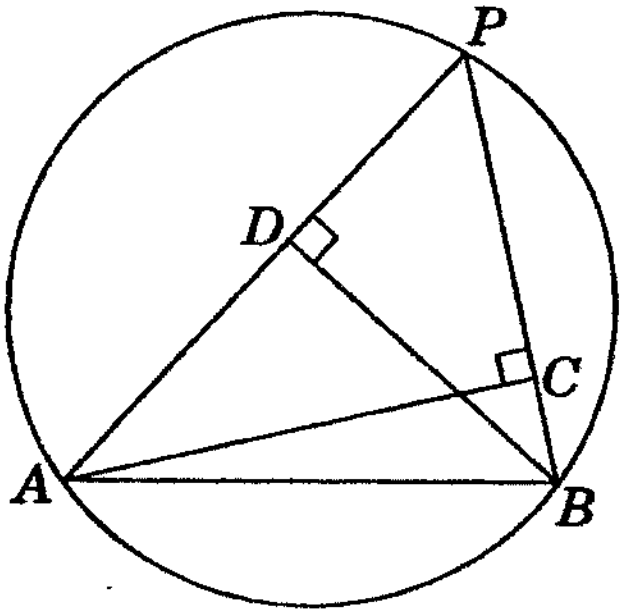
$\angle PBC = 90^\circ$ $\quad (\angle \text{ in semicircle} = 90^\circ)$

$$\frac{BC}{PC} = \sin P$$

$$\frac{BC}{\sin P} = PC$$

$$\therefore PC = \frac{BC}{\sin A}$$

e.g. (1990)



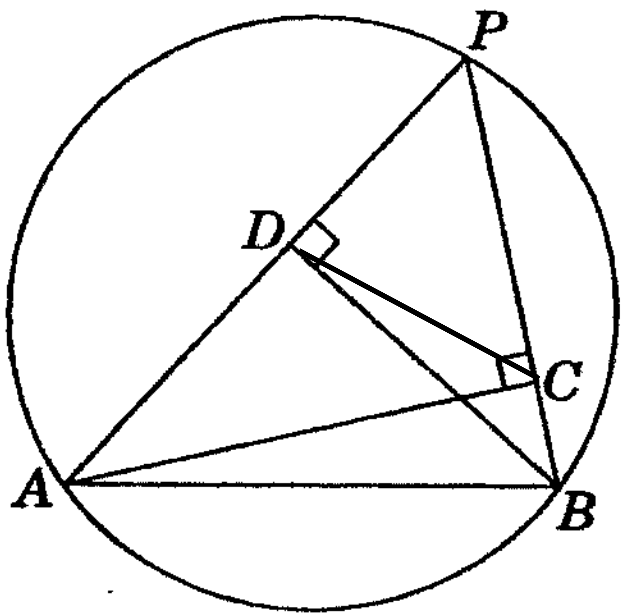
In the diagram, *AB* is a fixed chord of a circle, *P* a variable point in the circle and *AC* and *BD* are perpendicular to *BP* and *AP* respectively.

(*i*) Show that *ABCD* is a cyclic quadrilateral on a circle with *AB* as diameter.

$\angle BDA = \angle ACB = 90^\circ$ (given)

$\therefore$ *ABCD* is a cyclic quadrilateral ($\angle$'s in same segment are $=$)

AB is diameter as $\angle$ in semicircle $= 90^\circ$

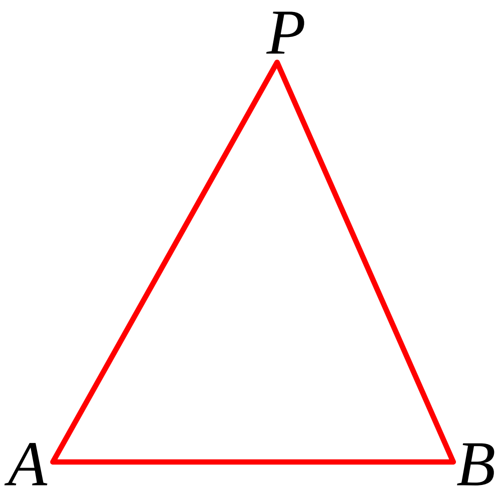


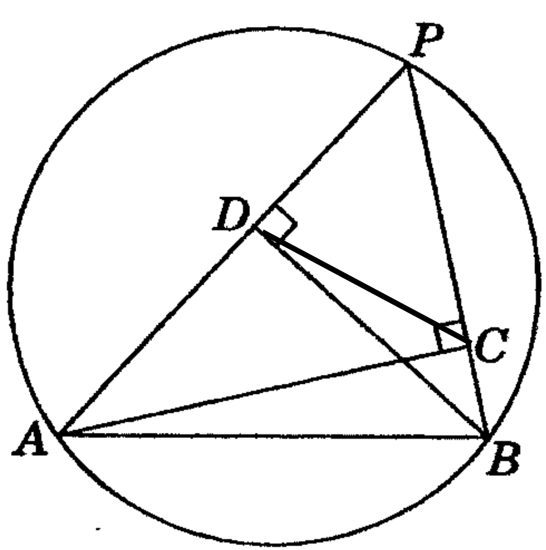
(ii) Show that triangles PCD and APB are similar

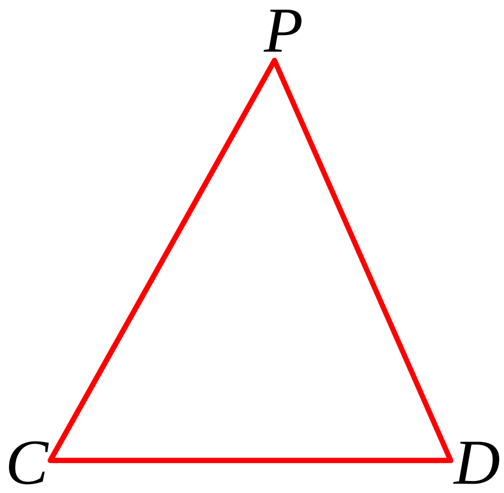
$\angle APB = \angle DPC$ (common $\angle$'s)

$\angle PDC = \angle PBA$ (exterior$\angle$ cyclic quadrilateral)

$\therefore \Delta PDC \,|||\, \Delta PBA$ (equiangular)







(iii) Show that as P varies, the segment CD has constant length.

$$\frac{CD}{AB} = \frac{PC}{AP} \qquad \text{(ratio of sides in } ||| \ \Delta'\text{s)}$$

$$\text{In } \Delta PCA, \frac{PC}{AP} = \cos P$$

$$\therefore \frac{CD}{AB} = \cos P$$

$$CD = AB \cos P$$

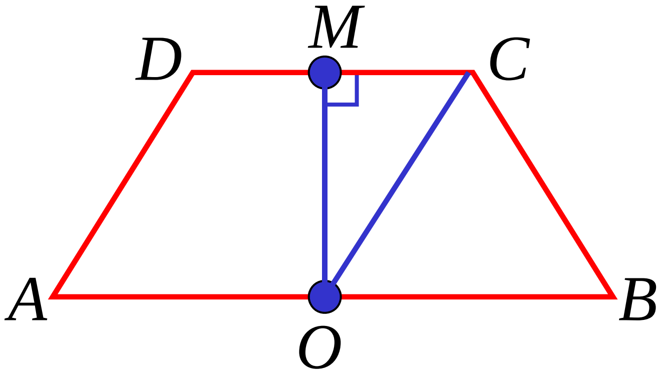
Now, $\angle P$ is constant ($\angle$'s in same segment are =)

and AB is fixed (given)

$\therefore CD$ is constant

(iv) Find the locus of the midpoint of CD.

$ABCD$ is a cyclic quadrilateral with AB diameter.



Let M be the midpoint of CD

O is the midpoint of AB

OM is constant

(= chords are equidistant from the centre)

$\therefore M$ is a fixed distance from O

$$OM^2 = OC^2 - MC^2$$

$$= \left(\frac{1}{2}AB\right)^2 - \left(\frac{1}{2}AB\cos P\right)^2$$

$$= \frac{1}{4}AB^2 - \frac{1}{4}AB^2\cos^2 P$$

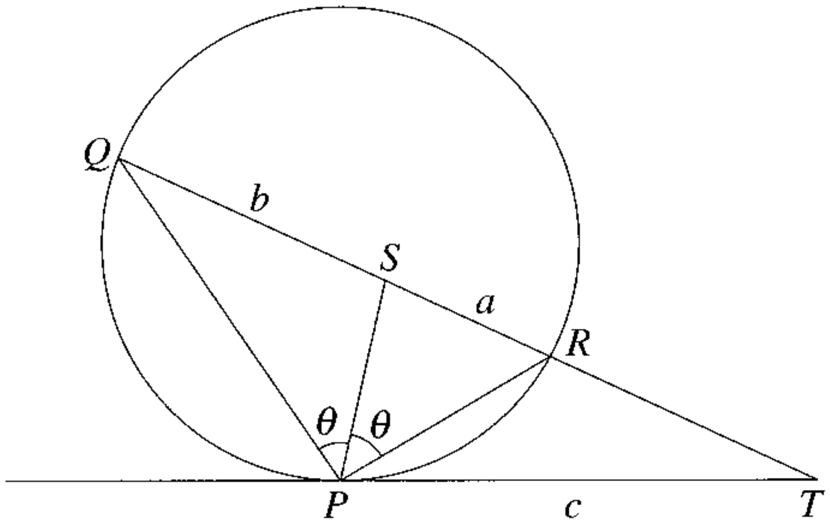
$$= \frac{1}{4}AB^2\sin^2 P$$

$$OM = \frac{1}{2}AB\sin P$$

$\therefore$ locus is circle, centre O

and radius $= \frac{1}{2}AB\sin P$

2008 Extension 2 Question 7b)



In the diagram, the points P, Q and R lie on a circle. The tangent at P and the secant QR intersect at T. The bisector of $\angle PQR$ meets QR at S so that $\angle QPS = \angle RPS = \theta$. The intervals RS, SQ and PT have lengths a, b and c respectively.

(i) Show that $\angle TSP = \angle TPS$

$\angle RQP = \angle RPT$ (alternate segment theorem)

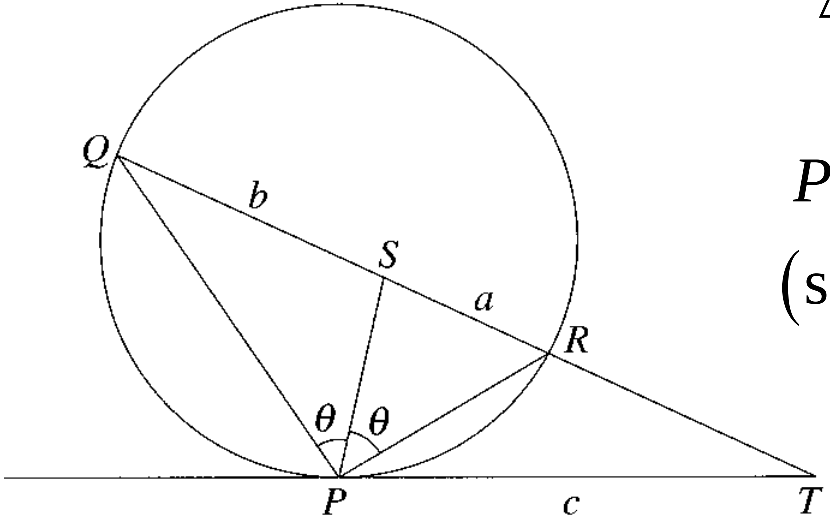
$\angle TSP = \angle RQP + \angle SPQ$ (exterior $\angle$,ΔSPQ)

$\angle TSP = \angle RPT + \theta$

$\angle SPT = \angle RPT + \theta$ (common $\angle$)

$\therefore \angle SPT = \angle TSP$

(ii) Hence show that $\dfrac{1}{a} = \dfrac{1}{b} + \dfrac{1}{c}$



ΔTPS is isosceles (2 = sides)

$\therefore ST = c$ (= sides in isosceles Δ)

$PT^2 = QT \times RT$

(square of tangents=products of intercepts)

$$c^2 = (c+b)(c-a)$$

$$c^2 = c^2 - ac + bc - ab$$

$$bc = ac + ab$$

$$\frac{1}{a} = \frac{1}{b} + \frac{1}{c}$$

Past HSC Papers
"Cambridge" **: Exercise 10A**
Patel : Exercise 10C*