Mathematical Induction

e.g. (i) Prove $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{n^2}\leq 2-\frac{1}{n}$

Test: $n=1$

$$L.H.S=\frac{1}{1^2}=1 \qquad R.H.S=2-\frac{1}{1}=1$$

$$\therefore L.H.S\leq R.H.S$$

Ⓐ $(n=k)$ $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{k^2}\leq 2-\frac{1}{k}$

Ⓟ $(n=k+1)$ $1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(k+1)^2}\leq 2-\frac{1}{k+1}$

Proof:

$$1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(k+1)^2}=1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{k^2}+\frac{1}{(k+1)^2}$$

$$\le 2-\frac{1}{k}+\frac{1}{(k+1)^2}$$

$$=2-\frac{(k+1)^2-k}{k(k+1)^2}$$

$$=2-\frac{k^2+k+1}{k(k+1)^2}$$

$$=2-\frac{k^2+k}{k(k+1)^2}-\frac{1}{k(k+1)^2}$$

$$<2-\frac{k(k+1)}{k(k+1)^2}$$

$$=2-\frac{1}{k+1}$$

$$\therefore 1+\frac{1}{2^2}+\frac{1}{3^2}+\cdots+\frac{1}{(k+1)^2}\le 2-\frac{1}{k+1}$$

Induction & Recursive Formulae

A recursive formula is when one term is defined in terms of one or more preceding terms

If a recursive formula is defined in terms of m preceding terms, you will need to;

1. Prove true for the first m terms

2. Assume true for $n = k$, $n = k - 1$, … , $n = k - (m - 1)$

Note: *the recursive formula is given to be true, do not try to prove it*

(ii) A sequence is defined by;

$$a_1 = \sqrt{2} \qquad a_{n+1} = \sqrt{2 + a_n} \text{ for } n \geq 1$$

Show that $a_n < 2$ for $n \geq 1$

Test: $n = 1$ $\quad a_1 = \sqrt{2} < 2$

(A) $(n = k)$ $a_k < 2$

(P) $(n = k + 1)$ $a_{k+1} < 2$

Proof:

$$a_{k+1} = \sqrt{2 + a_k}$$
$$< \sqrt{2 + 2}$$
$$= \sqrt{4}$$
$$= 2$$
$$\therefore a_{k+1} < 2$$

Start with the recursive formula

(iii) The sequences x_n and y_n are defined by;

$$x_1 = 5, y_1 = 2 \qquad x_{n+1} = \frac{x_n + y_n}{2}, y_{n+1} = \frac{2x_n y_n}{x_n + y_n}$$

Prove $x_n y_n = 10$ for $n \geq 1$

Test: $n = 1$ $\quad x_1 y_1 = (5)(2)$

$$= 10$$

(A) $(n = k)$ $\quad x_k y_k = 10$

(P) $(n = k + 1)$ $\quad x_{k+1} y_{k+1} = 10$

Proof:

$$x_{k+1} y_{k+1} = \left(\frac{x_k + y_k}{2}\right)\left(\frac{2x_k y_k}{x_k + y_k}\right)$$

$$= x_k y_k$$

$$= 10$$

$$\therefore x_{k+1} y_{k+1} = 10$$

(iv) The Fibonacci sequence is defined by;

$$a_1 = a_2 = 1 \qquad a_{n+1} = a_n + a_{n-1} \text{ for } n > 1$$

Prove that $a_n < \left(\frac{1+\sqrt{5}}{2}\right)^n$ for $n \geq 1$

Test: n = 1 and n =2

$$L.H.S = a_1 \qquad R.H.S = \left(\frac{1+\sqrt{5}}{2}\right)^1$$

$$= 1 \qquad = 1.62\ldots$$

$$\therefore L.H.S \leq R.H.S$$

$$L.H.S = a_2 \qquad R.H.S = \left(\frac{1+\sqrt{5}}{2}\right)^2$$

$$= 1 \qquad = 2.62\ldots$$

$$\therefore L.H.S \leq R.H.S$$

(A) $(n = k-1 \,\&\, n = k)$ $a_{k-1} < \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}$ & $a_k < \left(\frac{1+\sqrt{5}}{2}\right)^k$

(P) $(n = k+1)$ $a_{k+1} < \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}$

Proof:

$$a_{k+1} = a_k + a_{k-1}$$

$$< \left(\frac{1+\sqrt{5}}{2}\right)^k + \left(\frac{1+\sqrt{5}}{2}\right)^{k-1}$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}\left[\left(\frac{1+\sqrt{5}}{2}\right)^{-1} + \left(\frac{1+\sqrt{5}}{2}\right)^{-2}\right]$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}\left[\frac{2}{1+\sqrt{5}} + \frac{4}{\left(1+\sqrt{5}\right)^2}\right]$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}\left[\frac{2+2\sqrt{5}+4}{\left(1+\sqrt{5}\right)^2}\right]$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}\left[\frac{6+2\sqrt{5}}{\left(1+\sqrt{5}\right)^2}\right]$$

$$= \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}$$

$$\therefore a_{k+1} < \left(\frac{1+\sqrt{5}}{2}\right)^{k+1}$$

Sheets

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Exercise 10E*