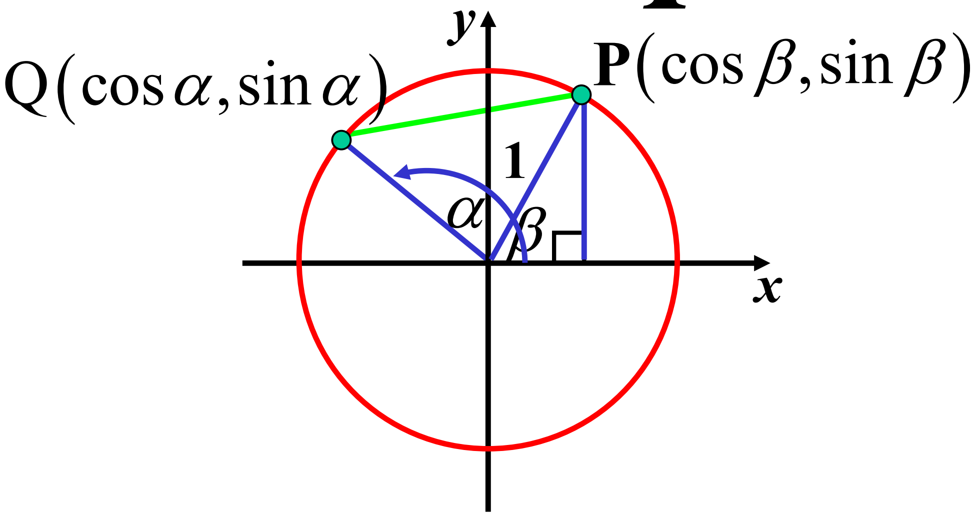
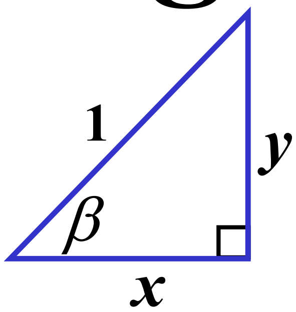
Compound Angles





$$\frac{x}{1} = \cos\beta \qquad \frac{y}{1} = \sin\beta$$

$$x = \cos\beta \qquad y = \sin\beta$$

By trigonometry

$$PQ^2 = 1^2 + 1^2 - 2\times1\times1\times\cos(\alpha-\beta)$$

$$PQ^2 = 2 - 2\cos(\alpha-\beta)$$

By coordinate geometry

$$PQ^2 = (\cos\alpha - \cos\beta)^2 + (\sin\alpha - \sin\beta)^2$$

$$PQ^2 = \cos^2\alpha - 2\cos\alpha\cos\beta + \cos^2\beta + \sin^2\alpha - 2\sin\alpha\sin\beta + \sin^2\beta$$

$$PQ^2 = 2 - 2\cos\alpha\cos\beta - 2\sin\alpha\sin\beta$$

$$\therefore 2-2\cos(\alpha-\beta)=2-2\cos\alpha\cos\beta-2\sin\alpha\sin\beta$$

$$\boxed{\cos(\alpha-\beta)=\cos\alpha\cos\beta+\sin\alpha\sin\beta}$$

Replace β with $-\beta$

$$\cos(\alpha+\beta)=\cos\alpha\cos(-\beta)+\sin\alpha\sin(-\beta)$$

$\cos(-\beta)=\cos\beta$ (even function i.e. $f(-x)=f(x)$)

$\sin(-\beta)=-\sin\beta$ (odd function i.e. $f(-x)=-f(x)$)

$$\boxed{\cos(\alpha+\beta)=\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

cos, cos, sin, sin

$$\cos(\alpha\pm\beta)=\cos\alpha\cos\beta\mp\sin\alpha\sin\beta$$

If it's not the sine,it's not the sign

Replace α with $90-\alpha$

$$\cos(90-\alpha-\beta)=\cos(90-\alpha)\cos\beta+\sin(90-\alpha)\sin\beta$$

$$\cos(90-(\alpha+\beta))=\cos(90-\alpha)\cos\beta+\sin(90-\alpha)\sin\beta$$

$$\boxed{\sin(\alpha+\beta)=\sin\alpha\cos\beta+\cos\alpha\sin\beta}$$

Replace β with $-\beta$

$$\sin(\alpha-\beta)=\sin\alpha\cos(-\beta)+\cos\alpha\sin(-\beta)$$

$$\boxed{\sin(\alpha-\beta)=\sin\alpha\cos\beta-\cos\alpha\sin\beta}$$

sin, cos, cos, sin

$$\sin(\alpha\pm\beta)=\sin\alpha\cos\beta\pm\cos\alpha\sin\beta$$

If it's the sine ,it's the sign

$\tan(\alpha+\beta)$

$$\tan(\alpha+\beta)=\frac{\sin(\alpha+\beta)}{\cos(\alpha+\beta)}$$

$$\tan(\alpha+\beta)=\frac{\sin\alpha\cos\beta+\cos\alpha\sin\beta}{\cos\alpha\cos\beta-\sin\alpha\sin\beta}$$

$$\tan(\alpha+\beta)=\frac{\dfrac{\sin\alpha\cos\beta}{\cos\alpha\cos\beta}+\dfrac{\cos\alpha\sin\beta}{\cos\alpha\cos\beta}}{\dfrac{\cos\alpha\cos\beta}{\cos\alpha\cos\beta}-\dfrac{\sin\alpha\sin\beta}{\cos\alpha\cos\beta}}$$

$$\boxed{\tan(\alpha+\beta)=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}}$$

Replace β with $-\beta$

$$\tan(\alpha-\beta)=\frac{\tan\alpha+\tan(-\beta)}{1-\tan\alpha\tan(-\beta)}$$

$$\tan(-\beta)=-\tan\beta$$

(odd function i.e. $f(-x)=-f(x)$)

$$\boxed{\tan(\alpha-\beta)=\frac{\tan\alpha-\tan\beta}{1+\tan\alpha\tan\beta}}$$

tan plus tan

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

on one minus tan, tan

Compound Angle Formulae

$$\sin(\alpha \pm \beta) = \sin\alpha\cos\beta \pm \cos\alpha\sin\beta$$

$$\cos(\alpha \pm \beta) = \cos\alpha\cos\beta \mp \sin\alpha\sin\beta$$

$$\tan(\alpha \pm \beta) = \frac{\tan\alpha \pm \tan\beta}{1 \mp \tan\alpha\tan\beta}$$

e.g. (i) Expand $\cos(2\alpha - 3\beta)$

$$\underline{\cos(2\alpha - 3\beta) = \cos 2\alpha \cos 3\beta + \sin 2\alpha \sin 3\beta}$$

(ii) Simplify $\dfrac{\tan 20^\circ + \tan 10^\circ}{1 - \tan 20^\circ \tan 10^\circ}$

$$\frac{\tan 20^\circ + \tan 10^\circ}{1 - \tan 20^\circ \tan 10^\circ} = \tan(20 + 10)$$

$$= \tan 30^\circ$$

$$\underline{= \frac{1}{\sqrt{3}}}$$

(iii) Find the exact value of $\sin 15^\circ$

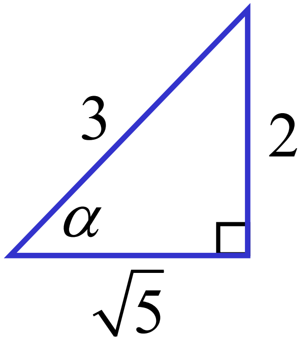
$$\sin 15^\circ = \sin(45 - 30)$$

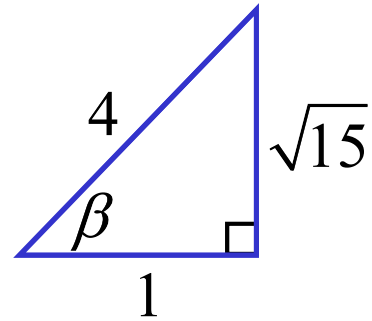
$$= \sin 45^\circ \cos 30^\circ - \cos 45^\circ \sin 30^\circ$$

$$= \left(\frac{1}{\sqrt{2}}\right)\left(\frac{\sqrt{3}}{2}\right) - \left(\frac{1}{\sqrt{2}}\right)\left(\frac{1}{2}\right)$$

$$\underline{= \frac{\sqrt{3} - 1}{2\sqrt{2}}}$$

(iv) If $\sin\alpha = \frac{2}{3}$ and $\cos\beta = \frac{1}{4}$, find $\sin(\alpha+\beta)$

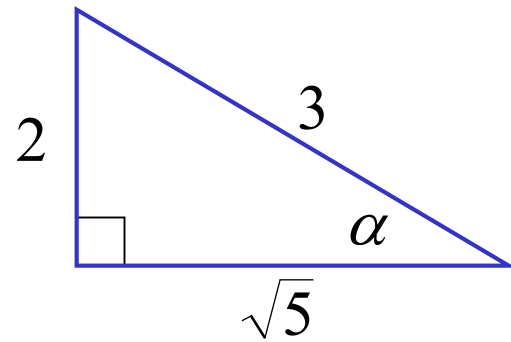


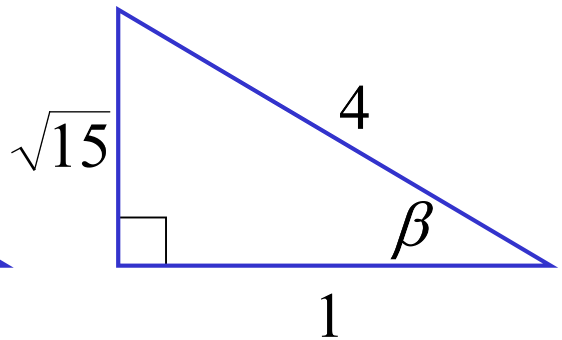


$$\begin{aligned}\sin(\alpha+\beta) &= \sin\alpha\cos\beta + \cos\alpha\sin\beta \\ &= \left(\frac{2}{3}\right)\left(\frac{1}{4}\right) + \left(\frac{\sqrt{5}}{3}\right)\left(\frac{\sqrt{15}}{4}\right) \\ &= \frac{2+5\sqrt{3}}{12}\end{aligned}$$

$$(vii)\ \tan\left(\sin^{-1}\frac{2}{3}+\cos^{-1}\frac{1}{4}\right)$$

$$\text{let } \alpha=\sin^{-1}\frac{2}{3} \qquad \text{let } \beta=\cos^{-1}\frac{1}{4}$$





$$=\frac{\tan\alpha+\tan\beta}{1-\tan\alpha\tan\beta}$$

$$=\frac{\frac{2}{\sqrt{5}}+\sqrt{15}}{1-\left(\frac{2}{\sqrt{5}}\right)\left(\sqrt{15}\right)}$$

$$=\frac{\frac{2+5\sqrt{3}}{\sqrt{5}}}{\frac{\sqrt{5}-2\sqrt{15}}{\sqrt{5}}}$$

$$=\frac{2+5\sqrt{3}}{\sqrt{5}-2\sqrt{15}}$$

Exercise 17D; 1ade, 2bce, 4ac, 5, 6ab, 7, 8, 9ac, 10ad, 11, 12b, 13, 14a, 15ab, 17, 18, 20