Mathematical Induction

Step 1: Prove the result is true for $n = 1$ (or whatever the first term is)

Step 2: Assume the result is true for $n = k$, where k is a positive integer (or another condition that matches the question)

or using set notation;

Assume the result is true for $n = k$ where $k \in \mathbb{Z}^+$

Step 3: Prove the result is true for $n = k + 1$

NOTE: It is important to note in your conclusion that the result is true for $n = k + 1$ **if it is true for $\boldsymbol{n = k}$**

Step 4: Since the result is true for $n = 1$, then the result is true for all positive integral values of n by induction

or using set notation;

Since the result is true for $n = 1$, then it is true $\forall\ n \in \mathbb{Z}^+$ by induction

$$e.g.(i)\ 1^2+3^2+5^2+\ldots+(2n-1)^2=\frac{1}{3}n(2n-1)(2n+1)$$

Prove the result is true for $n = 1$

$$LHS = 1^2 = 1$$

$$RHS = \frac{1}{3}(1)(2-1)(2+1) = \frac{1}{3}(1)(1)(3) = 1$$

$$\therefore LHS = RHS$$

Hence the result is true for $n = 1$

Assume the result is true for $n = k$, where $k \in \mathbb{Z}^+$

$$i.e.\ 1^2+3^2+5^2+\ldots+(2k-1)^2=\frac{1}{3}k(2k-1)(2k+1)$$

Prove the result is true for $n = k + 1$

$$i.e. \text{ Prove}:\ 1^2+3^2+5^2+\ldots+(2k+1)^2=\frac{1}{3}(k+1)(2k+1)(2k+3)$$

Proof:

$$1^2+3^2+5^2+\ldots+(2k+1)^2$$

$$=1^2\underbrace{+3^2+5^2+\ldots+(2k-1)^2}_{S_k}+\underbrace{(2k+1)^2}_{T_{k+1}}$$

$$=\frac{1}{3}k(2k-1)(2k+1)+(2k+1)^2 \qquad \text{(by assumption)}$$

$$=(2k+1)\left[\frac{1}{3}k(2k-1)+(2k+1)\right]$$

$$=\frac{1}{3}(2k+1)[k(2k-1)+3(2k+1)]$$

$$=\frac{1}{3}(2k+1)(2k^2-k+6k+3)$$

$$=\frac{1}{3}(2k+1)(2k^2+5k+3)$$

$$=\frac{1}{3}(2k+1)(k+1)(2k+3)$$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 1$, then it is true $\forall\, n \in \mathbb{Z}^+$ by induction

$$(ii)\ \sum_{k=1}^{n}\frac{1}{(2k-1)(2k+1)}=\frac{n}{2n+1}$$

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2n-1)(2n+1)}=\frac{n}{2n+1}$$

Prove the result is true for $n = 1$

$$LHS=\frac{1}{1\times 3} \qquad\qquad RHS=\frac{1}{2+1}$$

$$=\frac{1}{3} \qquad\qquad =\frac{1}{3}$$

$$\therefore LHS = RHS$$

Hence the result is true for $n = 1$

Assume the result is true for $n = k$, where $k \in \mathbb{Z}^+$

$$i.e.\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2k-1)(2k+1)}=\frac{k}{2k+1}$$

Prove the result is true for $n = k + 1$

$$i.e.\ \text{Prove}: \frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2k+1)(2k+3)}=\frac{k+1}{2k+3}$$

Proof:

$$\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{1}{1\times 3}+\frac{1}{3\times 5}+\frac{1}{5\times 7}+\ldots+\frac{1}{(2k-1)(2k+1)}+\frac{1}{(2k+1)(2k+3)}$$

$$=\frac{k}{2k+1}+\frac{1}{(2k+1)(2k+3)} \qquad \text{(by assumption)}$$

$$=\frac{k(2k+3)+1}{(2k+1)(2k+3)}$$

$$=\frac{2k^2+3k+1}{(2k+1)(2k+3)}$$

$$=\frac{(2k+1)(k+1)}{(2k+1)(2k+3)}$$

$$= \frac{(k+1)}{(2k+3)}$$

Hence the result is true for $n = k + 1$ if it is also true for $n = k$

Since the result is true for $n = 1$, then it is true $\forall\, n \in \mathbb{Z}^+$ by induction

Exercise 2A; 2acfh, 3, 4b, 5c, 6, 7, 8, 9, 11ac, 12, 13