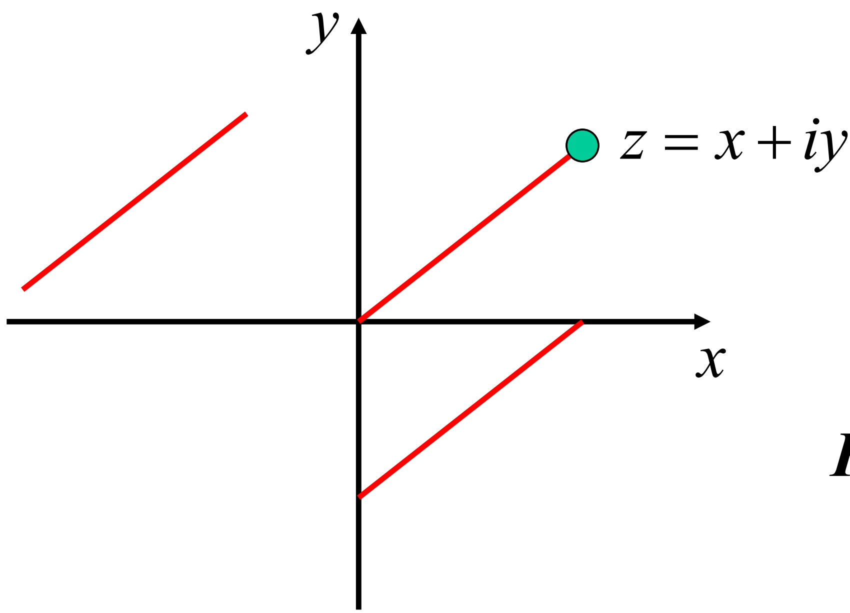
# ***Geometrical Representation of Complex Numbers***

Complex numbers can be represented on the Argand Diagram as vectors.

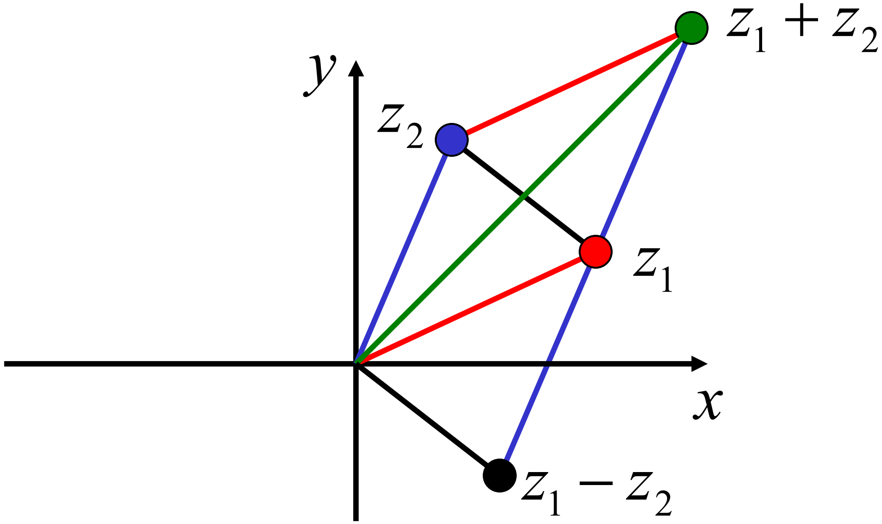



***A vector always represents HEAD minus TAIL***

The advantage of using vectors is that they can be moved around the Argand Diagram

No matter where the vector is placed its length (modulus) and the angle made with the *x* axis (argument) is constant

# Addition / Subtraction




*NOTE* :

the parallelogram formed by adding vectors has two diagonals;

$z_1 + z_2$ and $z_1 - z_2$

To add two complex numbers, place the vectors ***"head to tail"***

To subtract two complex numbers, place the vectors ***"head to head"*** (or add the negative vector)

**Trianglar Inequality**

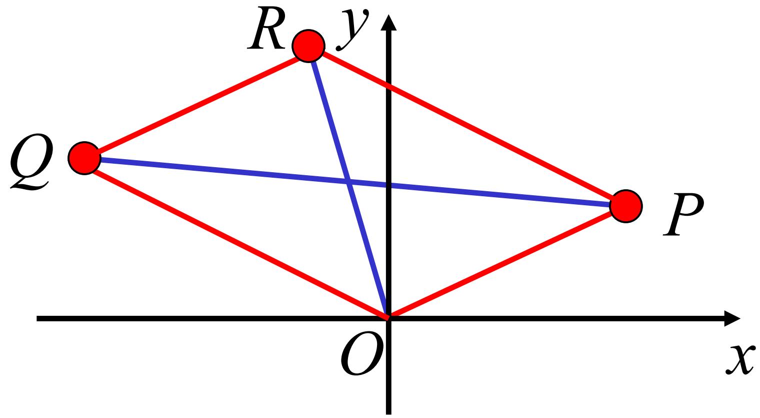
In any triangle a side will be shorter than the sum of the other two sides and bigger than the difference of the other two sides

In $\Delta ABC$; $AC \leq AB + BC$ and $AC - AB \leq BC$

*(equality occurs when ABC is a straight line)*

$$\boxed{\big||z_1| - |z_2|\big| \leq |z_1 + z_2| \leq |z_1| + |z_2|}$$

$e.g.(1995)$




The diagram shows a complex plane with origin $O$.
The points $P$ and $Q$ represent the complex numbers $z$ and $w$ respectively.
Thus the length of $PQ$ is $|z-w|$

$(i)$ Show that $|z-w| \leq |z|+|w|$

The length of $OP$ is $|z|$

The length of $OQ$ is $|w|$

The length of $PQ$ is $|z-w|$

Using the triangular inequality on $\Delta OPQ$

$$|z-w| \leq |z|+|w|$$

$(ii)$ Construct the point $R$ representing $z + w$, What can be said about the quadrilateral $OPRQ$?

$OPRQ$ is a parallelogram

$(iii)$ If $|z-w| = |z+w|$, what can be said about $\frac{w}{z}$?

$$|z-w| = |z+w| \quad \text{i.e. diagonals in } OPRQ \text{ are } =$$

$\therefore OPRQ$ is a rectangle

$$\arg w - \arg z = \frac{\pi}{2}$$

$$\arg \frac{w}{z} = \frac{\pi}{2} \qquad \therefore \frac{w}{z} \text{ is purely imaginary}$$

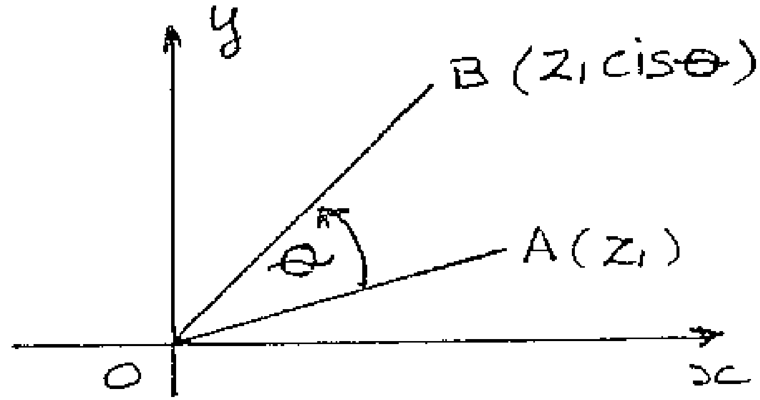
## Multiplication

$$|z_1 z_2| = |z_1||z_2| \qquad \arg z_1 z_2 = \arg z_1 + \arg z_2$$

$$r_1 cis\theta_1 \times r_2 cis\theta_2 = r_1 r_2 cis(\theta_1 + \theta_2)$$

*i.e.* if we multiply $z_1$ by $z_2$, the vector $z_1$ is rotated anticlockwise by $\theta_2$ and its length is multiplied by $r_2$

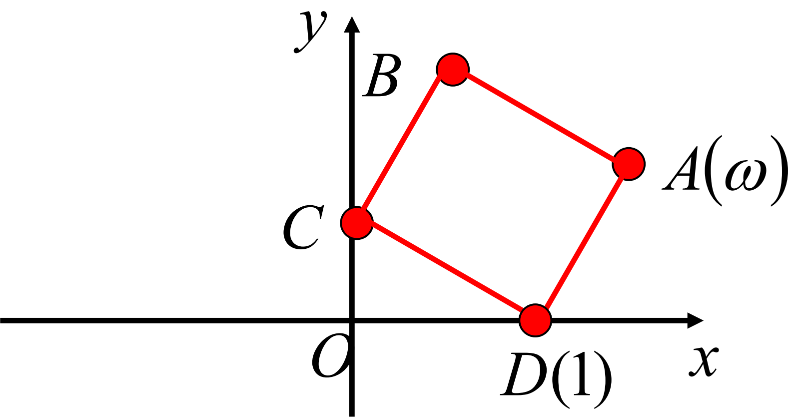
If we multiply $z_1$ by $cis\theta$ the vector $OA$ will rotate by an angle of $\theta$ in an anti-clockwise direction. If we multiply by $rcis\theta$ it will also multiply the length of $OA$ by a factor of $r$




*Note:* $\cos\frac{\pi}{2}+i\sin\frac{\pi}{2}=i \quad \therefore iz_1$ will rotate $OA$ anticlockwise 90 degrees.

> Multiplication by $i$ is a rotation anticlockwise by $\frac{\pi}{2}$

***REMEMBER: A vector is HEAD minus TAIL***




$$\overrightarrow{DC} = \overrightarrow{DA} \times i$$
$$C - 1 = (\omega - 1)i$$
$$C = 1 + (\omega - 1)i$$
$$= (1 - i) + i\omega$$

***OR***

$$B = A + \overrightarrow{DC}$$
$$B = \omega + C - 1$$
$$B = \omega + (\omega - 1)i$$
$$= -i + (1 + i)\omega$$

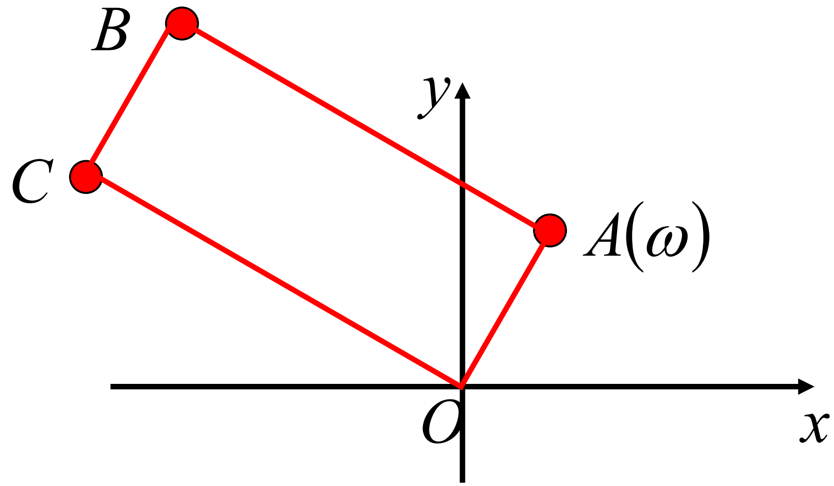
***OR***

$$B = C + \overrightarrow{DA}$$
$$B = (1 - i) + i\omega + (\omega - 1)$$
$$= -i + (1 + i)\omega$$

***OR***

$$\overrightarrow{DB} = \sqrt{2}cis\frac{\pi}{4} \times \overrightarrow{DA}$$
$$B - 1 = \sqrt{2}\left(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}\right)(\omega - 1)$$
$$B = (1 + i)(\omega - 1) + 1$$
$$= \omega - 1 + i\omega - i + 1$$
$$= -i + (1 + i)\omega$$

$e.g.(2000)$




In the Argand Diagram, $OABC$ is a rectangle, where $OC = 2OA$.
The vertex $A$ corresponds to the complex number $\omega$

$(i)$ What complex number corresponds to $C$?

$$\overrightarrow{OC} = \overrightarrow{OA} \times 2i$$

$$C = 2i\omega$$

$(ii)$ What complex number corresponds to the point of intersection $D$ of the diagonals $OB$ and $AC$?

diagonals bisect in a rectangle

$\therefore D =$ midpoint of $AC$

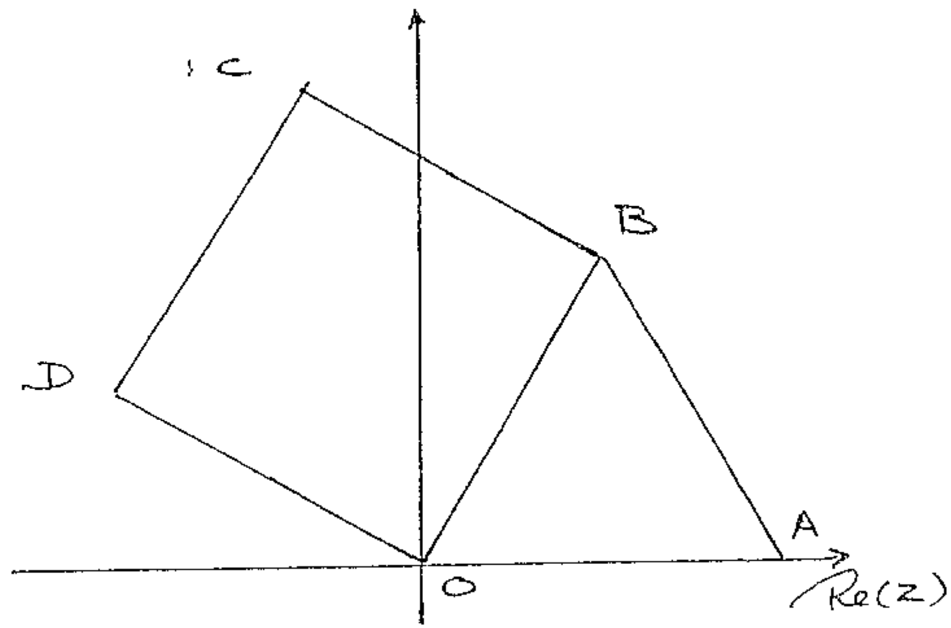
$$D = \frac{A+C}{2}$$

$$D = \frac{\omega + 2i\omega}{2}$$

$$\therefore D = \left(\frac{1}{2} + i\right)\omega$$

## Examples

1. $OBA$ is an equilateral triangle with sides of length 1 unit.
$OBCD$ is a square
Find the complex numbers represented by the points $B, D$ and $C$ (in exact terms in the form $x+iy$)




$$\overrightarrow{OB} = \overrightarrow{OA} \times cis\frac{\pi}{3}$$

$$B = 1cis\frac{\pi}{3}$$

$$\underline{B = \frac{1}{2} + \frac{\sqrt{3}}{2}i}$$

$$\overrightarrow{OD} = \overrightarrow{OB} \times i$$

$$D = iB$$

$$\underline{D = -\frac{\sqrt{3}}{2} + \frac{1}{2}i}$$

$$C = B + \overrightarrow{OD}$$

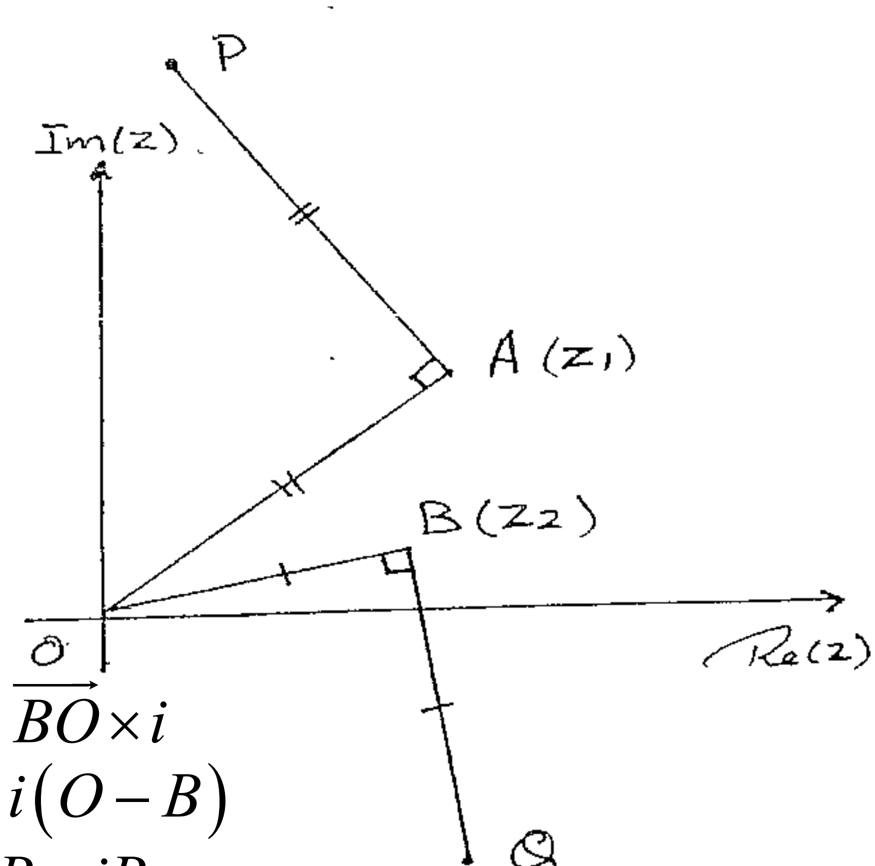
$$C = \left(\frac{1}{2} + \frac{\sqrt{3}}{2}i\right) + \left(-\frac{\sqrt{3}}{2} + \frac{1}{2}i\right)$$

$$\underline{C = \left(\frac{1-\sqrt{3}}{2}\right) + \left(\frac{1+\sqrt{3}}{2}\right)i}$$

2. The points $A$ and $B$ in the complex plane correspond to complex numbers $z_1$ and $z_2$ respectively. Both triangles $OAP$ and $OBQ$ are right-isosceles triangles.

(i) Explain why $P$ corresponds to the complex number $(1+i)z_1$.

(iii) Let $M$ be the midpoint of $PQ$. What complex number corresponds to $M$




(i) $\overrightarrow{AP} = \overrightarrow{AO} \times -i$

$$P - A = -i(O - A)$$

$$P - z_1 = -i(0 - z_1)$$

$$P = z_1 + iz_1$$

$$\underline{P = (1+i)z_1}$$

(ii) $\overrightarrow{QB} = \overrightarrow{BO} \times i$

$$Q - B = i(O - B)$$

$$Q = B - iB$$

$$Q = (1-i)z_2$$

$$M = \frac{P+Q}{2}$$

$$M = \frac{(1+i)z_1 + (1-i)z_2}{2}$$

$$\underline{M = \frac{(z_1 + z_2) + (z_1 - z_2)i}{2}}$$

**Exercise 1E; 2 to 9, 11, 12, 13, 16 to 19, 21, 23, 24**